Introduction to Lévy processes

Exercises

Alex Watson

8th September 2016

Exercise 1. Show that, if X is a stochastic process such that for every $n \in \mathbb{N}$, $\theta_1, \dots, \theta_n$ and $0 \le s_1 < t_1 \le s_2 < \dots \le s_n < t_n$,

$$\mathbb{E}\left[\prod_{i=1}^{n} e^{\mathrm{i}\theta_i (X_{t_i} - X_{s_i})}\right] = \prod_{i=1}^{n} \mathbb{E}\left[e^{\mathrm{i}\theta_i X_{t_i - s_i}}\right],$$

then X has stationary and independent increments.

Exercise 2. Suppose that X and Y are two *independent* Lévy processes with characteristic exponents Ψ and Φ. Show that the process $Z_t = X_t + Y_t$, $t \ge 0$, is a Lévy process with characteristic exponent $\Psi + \Phi$.

Exercise 3. Suppose that S is a random walk, i.e. for each n,

$$S_n = \sum_{i=0}^{n} \xi_i,$$

where (ξ_i) are i.i.d.. Let Γ be an independent, geometrically distributed random variable with parameter p, that is

$$\mathbb{P}(\Gamma = i) = (1-p)^{i-1} p, \qquad i \ge 1.$$

(i) Show that Γ has an infinitely divisible distribution. *[Hint: look at the negative binomial distribution.]*

(ii) Show that S_Γ has an infinitely divisible distribution.

(iii) Show that if X is independent of Γ and distributed Binomial(Γ, q), then X has an infinitely divisible distribution.

(The last part is interesting because a Binomial(n, p)-distributed random variable, with fixed n, is *not* infinitely divisible. We may prove this later.)

Exercise 4. We will consider alternative Lévy–Khintchine representations of the form

$$\Psi(\theta) = \mathrm{i}a_l\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta l(x))\,\Pi(\mathrm{d}x), \tag{E.1}$$

where $l\colon \mathbb{R} \to \mathbb{R}$ is called a *cutoff function.* The function $l(x) = x\mathbb{1}_{[-1,1]}(x)$ corresponds to the usual Lévy–Khintchine formula (1.4). It is clear that not all functions l can be used as cutoff functions; in particular, the integral in (E.1) may not converge absolutely if l is chosen poorly. (By 'converge absolutely', we mean that $\int_{\mathbb{R}} |e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta l(x)|\,\Pi(\mathrm{d}x) < \infty$.)

(i) Show that if $l(x) = 0$ for all x, and Π is the Lévy measure of a stable process (see §2.5) with $\alpha \in (1,2)$, then the integral in (E.1) does not converge absolutely.

(ii) Show that, if $l(x) = x\mathbb{1}_{[-\varepsilon,\varepsilon]}(x)$ or $l(x) = x/(1+x^2)$, then the integral in (E.1) converges absolutely for all Lévy measures Π. Give an expression for a_l in terms of l, a and Π (where a is the centre in the representation (1.4).)

(iii) Show that, if

$$l(x) = O(1), \qquad |x| \to \infty,$$

then the integral in (E.1) converges on $(-\infty,-K] \cup [K,\infty)$ for some $K > 0$. Give a similar asymptotic condition, as $x \to 0$, which ensures that the integral converges absolutely on a neighbourhood of zero.

Lemma (Frullani integral; Lemma 2.1 in the script)**.** *For all* $\alpha, \beta > 0$ *and* $z \in \mathbb{C}$ *such that* $\operatorname{Re} z \le 0$ *we have*

$$(1 - z/\alpha)^{-\beta} = \exp\left\{ -\int_0^\infty (1 - e^{zx})\beta x^{-1} e^{-\alpha x}\,\mathrm{d}x \right\}.$$

Exercise 5. We will prove the lemma above.

(i) First, suppose that $f\colon [0,\infty) \to \mathbb{R}$ is a function such that f' exists and is continuous, and $f(0)$ and $f(\infty)$ exist and are finite. Show that, for $0 < a < b$,

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = (f(0) - f(\infty))\log(b/a).$$

(ii) By taking $f(x) = e^{-x}$, $a = \alpha$ and $b = \alpha - z$, prove the lemma for $z < 0$.

(iii) Show, using analytic extension, that it also holds for any z such that $\operatorname{Re} z \le 0$.

Exercise 6. Let $\alpha, \beta > 0$ and define the probability measure

$$\mu_{\alpha,\beta}(\mathrm{d}x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}\,\mathrm{d}x$$

supported on $[0,\infty)$.

(i) Show that the characteristic function of $\mu_{\alpha,\beta}$ is given by

$$h(\theta) = \frac{1}{(1 - \mathrm{i}\theta/\alpha)^{\beta}}, \qquad \theta \in \mathbb{R},$$

and hence show that the Gamma(α, β) distribution is infinitely divisible.

(ii) The Lévy process associated with this distribution is called a gamma process, or a gamma subordinator. Using the Frullani integral from Exercise 5, show that its Lévy–Khintchine representation is given by $\sigma = 0$; $\Pi(\mathrm{d}x) = \beta x^{-1} e^{-\alpha x}\mathrm{d}x$, supported on $[0, \infty)$; and $a = -\int_0^1 x\Pi(\mathrm{d}x)$.

Exercise 7. Let U be a probability distribution on $\mathbb{R}^d$. Just as in the one-dimensional case, U is said to be infinitely divisible for each $n = 1, 2, \dots$ there exists a collection of i.i.d. random variables $U_{1,n}, \dots, U_{n,n}$ such that

$$U \stackrel{d}{=} U_{1,n} + \cdots + U_{n,n},$$

where $\stackrel{d}{=}$ represents equality in distribution.

(i) Let $a \in \mathbb{R}^d$; Π be a measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} 1 \wedge |x|^2\, \Pi(\mathrm{d}x) < \infty$; and Q be a symmetric, non-negative definite matrix. Define the function $\Psi\colon \mathbb{R}^d \to \mathbb{C}$ by

$$\Psi(\theta) = \mathrm{i}\langle a, \theta\rangle - \frac{1}{2}\langle \theta, Q\theta\rangle + \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle\theta,x\rangle} - 1 - \mathrm{i}\langle\theta, x\rangle \mathbb{1}_{\{|x|\le 1\}})\, \Pi(\mathrm{d}x), \qquad \text{(E.2)}$$

where $\langle\cdot,\cdot\rangle$ represents the Euclidean inner product, and $|\cdot|$ the Euclidean norm. Show that Ψ is continuous, and outline an argument to demonstrate that there exists an infinitely divisible distribution U such that

$$\mathbb{E}[e^{\mathrm{i}\langle\theta,U\rangle}] = e^{\Psi(\theta)}, \qquad \theta \in \mathbb{R}^d.$$

[Hint: such a matrix Q is the covariance matrix of a possibly-degenerate Gaussian distribution.]

(ii) Clearly, if $U_1, \dots, U_d$ are independent infinitely divisible distributions, then $U = (U_1, \dots, U_d)$ is an $\mathbb{R}^d$-valued infinitely divisible distribution. We will consider the converse: give a criterion, in terms of a, Q and Π, for the $\mathbb{R}^d$-valued infinitely divisible distribution U with characteristic exponent (E.2) to possess independent components.

Exercise 8. Let S be the random walk from Exercise 3, and let $\tilde\Gamma$ be the $\{0, 1, 2, \dots\}$-valued variant of the Geometric distribution, i.e.

$$\mathbb{P}(\tilde\Gamma = i) = (1-p)^i p, \qquad i \ge 0.$$

It follows from Exercise 3 that $(\tilde\Gamma, S_{\tilde\Gamma})$ possesses an infinitely divisible distribution. By examining the characteristic function

$$\mathbb{E}[e^{\mathrm{i}\theta S_{\tilde\Gamma} + \mathrm{i}\phi\tilde\Gamma}], \qquad \theta, \phi \in \mathbb{R},$$

compute its Lévy measure. *[Hint: Use the Taylor series* $\log(1-z) = -\sum_{n\ge1} \frac{1}{n} z^n$. *This is the inspiration for the Frullani integral.]*

Exercise 9. If μ is a probability measure on (the Borel sets of) a separable metric space S, we define the *support* of μ as

$$\operatorname{supp}\mu = \{x \in S : \text{if } G \text{ is open and } x \in G, \text{ then } \mu(G) > 0\}.$$

Note that $\operatorname{supp}\mu$ is closed in S, and indeed, it is the smallest closed set F such that $\mu(F) = 1$. When U is a random variable in S, we will define $\operatorname{supp} U := \operatorname{supp}\mu_U$, where the measure μ_U is given by $\mu_U(A) = \mathbb{P}(U \in A)$, i.e. it is the law of U.

(i) Show that if U and V are independent random variables in $\mathbb{R}$, then

$$\operatorname{supp}(U + V) = \overline{\operatorname{supp} U + \operatorname{supp} V},$$

where the 'sum' of sets A and B is defined to be $A + B = \{x + y : x \in A,\, y \in B\}$ and $\overline{A}$ indicates the closure of A.

(ii) Suppose that X is a Lévy process in $\mathbb{R}$ which is not a pure drift δt; i.e., the Lévy measure Π or the Gaussian coefficient σ of X is non-zero. Show that, for any $t \geq 0$, the support of X_t is unbounded.

(iii) If there exists some $k \geq 0$ such that $\operatorname{supp} X_t \subset k\mathbb{Z}$ for every $t \geq 0$, we say that X is *lattice-valued*. Give a criterion for this to hold in terms of the characteristics (i.e., a, σ and Π) of X.

A corollary of (ii) is that, as we intimated in Exercise 3, the distributions Uniform$[a, b]$ and Binomial(n, p) are never infinitely divisible.

Exercise 10. Let N be a Poisson random measure on $[0, \infty) \times S$ with intensity Leb $\times \eta$, and A a measurable subset of S. Let $(\mathscr{F}_t)_{t\geq 0}$ represent the filtration as in Lemma 3.5. Define

$$T_A = \inf\{t \geq 0 : N([0, t] \times A) > 0\},$$

which can be thought of as the hitting time of A.

(i) Show that T_A is an $(\mathscr{F}_t)_{t\geq 0}$-stopping time and has an exponential distribution with rate $\eta(A)$.

(ii) Deduce that the *counting process* $\tilde{N}_t = N([0, t] \times A)$ is a Poisson process in the sense of §2.1. *[Hint: the proof of Lemma 3.5(i) is similar.]*

Exercise 11. Show that a Lévy process of bounded variation may be written as the difference of two independent subordinators.

Exercise 12. If the Lévy measure Π of a Lévy process X satisfies $\Pi(0, \infty) = 0$ and X does not have decreasing paths, we say that it is a *spectrally negative Lévy process.* Define the function $\phi\colon [0, \infty) \to \mathbb{R}$ by

$$\mathbb{E}[e^{\lambda X_t}] = e^{t\phi(\lambda)}.$$

Observe that, by Theorem 4.9, ϕ is finite everywhere.

(i) Show that ϕ is strictly convex, in the sense that, for any $\lambda, \mu \geq 0$ with $\lambda \neq \mu$ and $t \in (0,1)$,

$$\phi(t\lambda + (1-t)\mu) < t\phi(\lambda) + (1-t)\phi(\mu).$$

(ii) Show that $\lim_{\lambda\to\infty} \phi(\lambda) = \infty$. *[Hint: notice that $\mathbb{P}(X_1 > 0) > 0$.]*

(iii) Show that, for any $\lambda \geq 0$, the stochastic process

$$M_t := e^{\lambda X_t - t\phi(\lambda)}, \qquad t \geq 0,$$

is a martingale in the filtration of X.

Exercise 13. Let X be a spectrally negative Lévy process, and define

$$\tau_a^+ = \inf\{t \geq 0 : X_t > a\}, \qquad a \geq 0.$$

Suppose that $\tau_a^+ < \infty$ for all $a \geq 0$. This implies that $\phi'(0+) \geq 0$.

Show that $(\tau_a^+)_{a\geq 0}$ is a subordinator, and find its Laplace exponent in terms of the function ϕ defined in the previous exercise. *[Hint: Use the fact that $X_{\tau_a^+} = a$. For the second part, you may want to use the previous exercise and the Optional Stopping Theorem. You may use without proof the fact that ϕ has a right-inverse.]*

Exercise 14. Let Z be a subordinator with Laplace exponent κ and potential measure U. Show that

$$\int_{[0,\infty)} e^{-\lambda x}\, U(\mathrm{d}x) = 1/\kappa(\lambda).$$

Exercise 15. Pick $\alpha \in (0,1)$ and let Z be a strictly α-stable subordinator, which is to say, the Laplace exponent of Z is $\kappa(\lambda) = k\lambda^\alpha$, for $\lambda \geq 0$.

(i) Use the previous exercise to compute its potential measure U. *[Hint: consider the integral expression for the Γ function.]*

(ii) Recall that the characteristic exponent Ψ of a stable process is given by (2.5). Determine parameters c, β, η such that $-\kappa(-\mathrm{i}\theta) = \Psi(\theta)$ for $\theta \in \mathbb{R}$.

(iii) Use Proposition 4.11 to compute the measure $\mathbb{P}(Z_{\tau_x^+} - x \in \mathrm{d}v, x - Z_{\tau_x^+ -} \in \mathrm{d}z)$.

Exercise 16. (i) If X is a Lévy process with characteristic exponent Ψ, show that $\Psi(\mathbb{R}) \subset \mathbb{C}_\ell = \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$.

(ii) Suppose that Z is a subordinator with Laplace exponent κ, and X is an independent Lévy process with characteristic exponent Ψ. Show that $Y_t = X_{Z_t}$ forms a Lévy process with characteristic exponent $\theta \mapsto -\kappa(-\Psi(\theta))$.

(iii) Show that, if Z is a strictly $\frac{\alpha}{2}$-stable subordinator, with $\alpha \in (0,2)$, and X is an independent Brownian motion, then $Y_t = X_{Z_t}$ is a symmetric strictly α-stable Lévy process.

Exercise 17. Let X be a Lévy process with associated filtration $(\mathscr{F}_t)_{t\geq 0}$. Recall the definition of the supremum process $\bar{X}_t = \sup\{0 \vee X_s : s \leq t\}$. Show that the reflected process $\bar{X} - X$ is a strong Markov process in $(\mathscr{F}_t)_{t\geq 0}$. (Warning: for $x \geq 0$, the reflected process starts at point x if and only if the Lévy process starts from point $-x$.)

Exercise 18. Let X be a Lévy process and define the stopping time

$$T = \inf\{t > 0 : X_t < 0\}.$$

Fix $t \geq 0$. Show that

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{\{X_s = \bar{X}_s\}}\, \mathrm{d}s > 0\right) > 0$$

if and only if $T > 0$ almost surely. *[Hint: Use Fubini's theorem and Lemma 4.13.]*