Stable and extended hypergeometric Lévy processes

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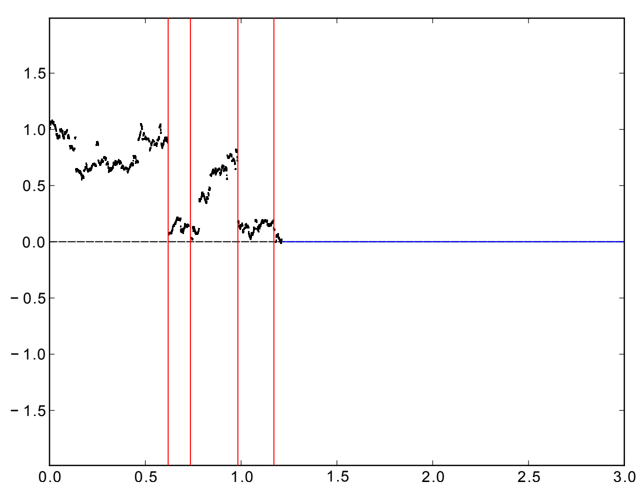
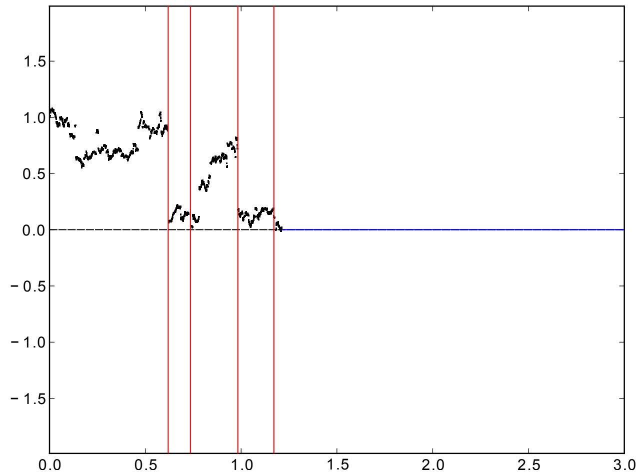
A process, and a problem

- Let X be a *stable process*
  - a Lévy process satisfying the scaling property that $cX_{\cdot c^{-\alpha}} \overset{d}{=} X$
- Erase the negative sections of path
- Make zero absorbing
- This is the path-censored stable process, Y



A process, and a problem



How does Y attain new maxima and minima?
When does it hit zero?

Lévy processes

Let ξ be a Lévy process:

- Càdlàg paths
- Stationary, independent increments
- Characterised by the Laplace exponent ψ:

$$\mathbb{E}\left[e^{z\xi_1}\right] = e^{\psi(z)}.$$

Examples:

- Brownian motion with drift
- Compound Poisson process
- Variance gamma process
- Stable process

Lévy processes: the Wiener–Hopf factorisation

If ψ is the Laplace exponent of a Lévy process ξ,

$$\psi(z) = -\kappa(-z)\hat{\kappa}(z);$$

κ and $\hat{\kappa}$ are Laplace exponents of *subordinators* (increasing Lévy processes):

$$\mathbb{E}\left[e^{-zH_1}\right] = e^{-\kappa(z)}$$

—the *ladder height processes*.

Positive, self-similar Markov processes

α-pssMp

$[0,\infty)$-valued Markov process,
equipped with initial measures P_x, $x > 0$,
with 0 an absorbing state,
satisfying the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \overset{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x, c > 0$$

pssMps: examples

Begin with a stable process X: a Lévy process satisfying the scaling property. Necessarily $\alpha \in (0, 2]$.

X is parameterised by (α, ρ), where $\rho = \mathrm{P}_0(X_t > 0)$.

We can then manufacture pssMps:

- The path-censored stable process Y
- The stable process killed upon exiting $[0, \infty)$

$$X_t^* = X_t \mathbb{1}_{\{t < \tau_0^-\}}$$

- The stable process conditioned to stay positive, $X^{\uparrow}$

$$h^{\uparrow}(x) = x^{\alpha(1-\rho)}$$

- The stable process conditioned to hit zero continuously, $X^{\downarrow}$

$$h^{\downarrow}(x) = x^{\alpha(1-\rho)-1}$$

Lamperti transform

$(X, \mathrm{P}_x)_{x>0}$ α-pssMp	$\leftrightarrow$	$(\xi, \mathbb{P}_y)_{y \in \mathbb{R}}$ killed Lévy
$X_t = \exp(\xi_{S(t)})$,		$\xi_s = \log(X_{T(s)})$,
S a random time-change		T a random time-change

$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

Lamperti transform: examples

- X^*, the killed stable process: the Lévy process ξ^* has Laplace exponent
$$-\frac{\Gamma(\alpha - z)}{\Gamma(\alpha(1-\rho) - z)} \frac{\Gamma(1+z)}{\Gamma(1-\alpha(1-\rho)+z)}$$
and is killed.
- $X^\uparrow$, the stable process conditioned to stay positive: here $\xi^\uparrow$ has Laplace exponent
$$-\frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)} \frac{\Gamma(1+\alpha(1-\rho)+z)}{\Gamma(1+z)}$$
and drifts to $+\infty$.
- $X^\downarrow$, the stable process conditioned to hit zero continuously: here $\xi^\downarrow$ has Laplace exponent
$$-\frac{\Gamma(1+\alpha\rho - z)}{\Gamma(1-z)} \frac{\Gamma(\alpha(1-\rho)+z)}{\Gamma(z)}$$
and drifts to $-\infty$.

Hypergeometric Lévy processes (Kuznetsov and Pardo)

Laplace exponent
$$-\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)} \frac{\Gamma(\hat\beta+\hat\gamma+z)}{\Gamma(\hat\beta+z)}$$
with parameter set
$$\{\beta \le 1;\ \hat\beta \ge 0;\ \gamma, \hat\gamma \in (0,1)\}$$

Explicit Wiener–Hopf factorisation:
$$\kappa(z) = \frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(1-\beta+z)} \qquad \hat\kappa(z) = \frac{\Gamma(\hat\beta+\hat\gamma+z)}{\Gamma(\hat\beta+z)}$$

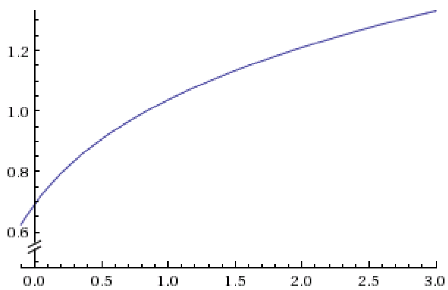
The path-censored process and its Lamperti transform

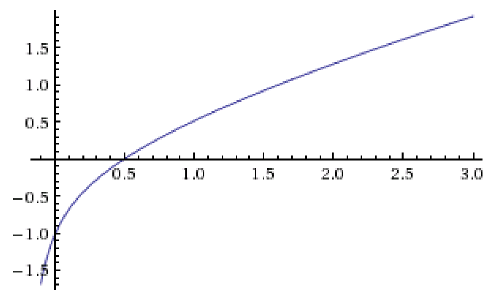
Recall the path-censored Y – write ξ^Y for its Lamperti transform. It has Laplace exponent

$$-\frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)} \frac{\Gamma(1-\alpha\rho+z)}{\Gamma(1-\alpha+z)},$$

and when $\alpha \leq 1$ it is a hypergeometric Lévy process.

When $\alpha > 1$, things go wrong:





Introducing: the extended hypergeometric class

Laplace exponent:

$$-\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)} \qquad \text{(the same!)}$$

and parameter space:

$$\{\beta \in [1,2];\ \hat{\beta} \in [-1,0];\ \gamma, \hat{\gamma} \in (0,1);\ \text{etc.}\} \qquad \text{(different!)}$$

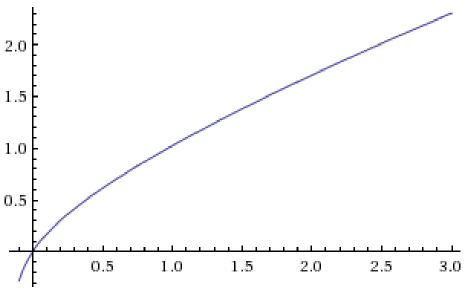
The Wiener–Hopf factorisation looks like this:

$$\kappa(z) = (-\hat{\beta}+z)\frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(2-\beta+z)} \qquad \hat{\kappa}(z) = (\beta-1+z)\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(1+\hat{\beta}+z)}$$

The path-censored stable process, $\alpha > 1$

Now we know:

$$\kappa(z) = (\alpha - 1 + z)\frac{\Gamma(\alpha\rho + z)}{\Gamma(1+z)}, \qquad \hat{\kappa}(z) = z\frac{\Gamma(1 - \alpha\rho + z)}{\Gamma(2 - \alpha + z)}.$$



(much better!)

Can compute first passage identities for Y (and hence X).

EHG class: more details

Large time behaviour:

- ξ is killed if $\beta \in (1, 2)$, $\hat{\beta} \in (-1, 0)$; otherwise:
- drifts to $+\infty$ if $\beta > 1$,
- drifts to $-\infty$ if $\hat{\beta} < 0$,
- oscillates if $\beta = 1$, $\hat{\beta} = 0$.

Lévy density:

$$\begin{cases} \mathbf{c}_+ e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta - \hat{\gamma}; e^{-x}), & x > 0, \\ \mathbf{c}_- e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta - \gamma; e^{x}), & x < 0, \end{cases}$$

where $\eta = 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, and ${}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!}$.

Exponential functionals

We are interested in

$$I(\xi/\delta) = \int_0^\infty e^{-\xi_u/\delta}\,\mathrm{d}u,$$

the exponential functional of ξ.

To find the law, we compute

$$\mathcal{M}(s) = \mathbb{E}\big[\big(I(\xi/\delta)\big)^{s-1}\big],$$

the Mellin transform.

We use the functional equation

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s/\delta)}\mathcal{M}(s).$$

EHG class: exponential functional

Let ξ be an extended hypergeometric Lévy process with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$.

Theorem

For $\operatorname{Re} s \in (0, 1 + \delta(\beta - 1))$

$$\mathcal{M}(s) = c\widetilde{\mathcal{M}}(s)\frac{\Gamma(\delta(1-\beta+\gamma)+s)}{\Gamma(-\delta\hat{\beta}+s)}\frac{\Gamma(\delta(\beta-1)+1-s)}{\Gamma(\delta(\hat{\beta}+\hat{\gamma})+1-s)},$$

where

$$\widetilde{\mathcal{M}}(s) = \Gamma(s)\frac{G((2-\beta)\delta+s;\delta)}{G((2-\beta+\gamma)\delta+s;\delta)}\frac{G((1+\hat{\beta}+\hat{\gamma})\delta+1-s;\delta)}{G((1+\hat{\beta})\delta+1-s;\delta)}$$

is the Mellin transform associated to a hypergeometric Lévy process with parameters $(\beta-1, \gamma, \hat{\beta}+1, \hat{\gamma})$.

Path-censored stable process: exponential functional

Let $\mathcal{I} = \int_0^{T_0} \mathbb{1}_{\{X_t>0\}}\, \mathrm{d}t$. Then $\mathcal{I} = I(-\alpha\xi^Y)$, and we have:

Corollary

For $\operatorname{Re} s \in (0, 2-1/\alpha)$,

$$\mathcal{M}(s) = c\,\frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)}\,\frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \times \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)}\Gamma(2 - 1/\alpha - s),$$

Example: symmetric stable process

Take X to be the *symmetric* stable process, killed upon hitting zero.

Let $R = \frac{1}{2}|X|$: a pssMp, the radial part of X.

ξ^R is hypergeometric when $\alpha \leq 1$, and extended hypergeometric when $\alpha > 1$, parameters

$$(1, \alpha/2, (1-\alpha)/2, \alpha/2).$$

Example: symmetric stable process

Let $T_0 = \inf\{t \geq 0 : X_t = 0\}$. Then $T_0 = 2^{-\alpha} I(-\alpha \xi^R)$, and:

Corollary

For $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathrm{E}_1\left[T_0^{s-1}\right] = 2^{-\alpha(s-1)} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})} \times \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\tfrac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}.$$

cf. Yano, Yano, Yor (2009); Cordero (2010);
Kuznetsov, Kyprianou, Pardo, W. (2013).

Example: symmetric stable process

Let $\sigma_{-1}^{1} = \inf\{t \geq 0 : X_t \notin (-1,1)\}$.

Proposition

For $|x| < 1$, $y > 1$,

$$\begin{aligned} &\mathrm{P}_x\left(\left|X_{\sigma_{-1}^1}\right| \in \mathrm{d}y;\ \sigma_{-1}^1 < T_0\right) \\ &= \frac{\sin(\pi\alpha/2)}{\pi} |x| (1-|x|)^{\alpha/2} y^{-1} (y-1)^{-\alpha/2} (y-|x|)^{-1} \\ &\quad + \frac{1}{2}\frac{\sin(\pi\alpha/2)}{\pi} y^{-1} (y-1)^{-\alpha/2} |x|^{(\alpha-1)/2} \\ &\quad \times \int_0^{1-|x|} t^{\alpha/2-1} (1-t)^{-(\alpha-1)/2} \, \mathrm{d}t. \end{aligned}$$

Corollary

For $|x| < 1$,

Example: a conditioned symmetric stable process

Take X to be a symmetric stable process, with $\alpha > 1$, killed upon hitting zero. The Doob h-transform using

$$h^{\updownarrow}(x) = |x|^{\alpha-1}$$

gives the symmetric stable process conditioned to avoid zero, $X^{\updownarrow}$.

Let $R^{\updownarrow} = \frac{1}{2}|X^{\updownarrow}|$, a pssMp.

$\xi^{R^{\updownarrow}}$ is an extended hypergeometric Lévy process with parameters

$$((\alpha+1)/2, \alpha/2, 0, \alpha/2).$$

Further reading

A. E. Kyprianou, J. C. Pardo, A. Watson
The extended hypergeometric class of Lévy processes.
arXiv:1310.1135 [math.PR]

Thank you!