

# Stable and extended hypergeometric Lévy processes

Andreas Kyprianou<sup>1</sup>   Juan-Carlos Pardo<sup>2</sup>   Alex Watson<sup>2</sup>

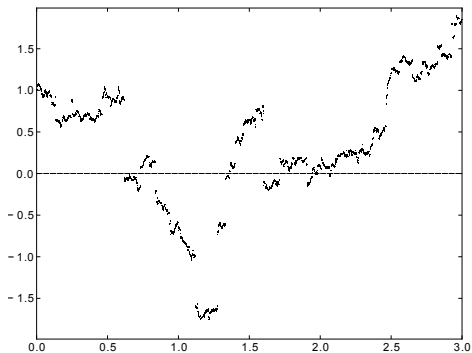
<sup>1</sup>University of Bath

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as given at CIMAT, 23 October 2013

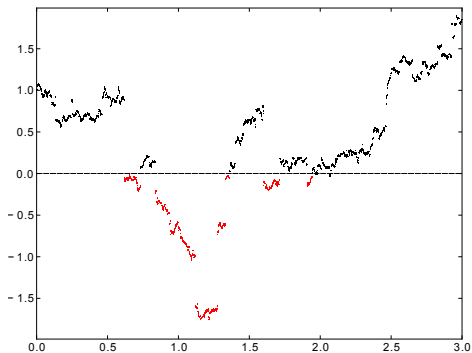
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- Let  $X$  be a *stable process*
  - a Lévy process satisfying the scaling property that  $cX_{c^{-\alpha}} \stackrel{d}{=} X$



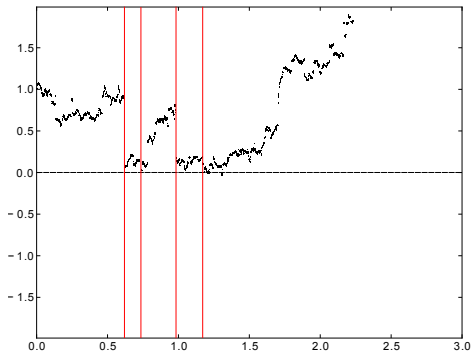
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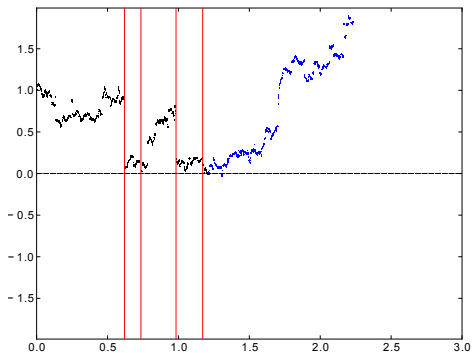
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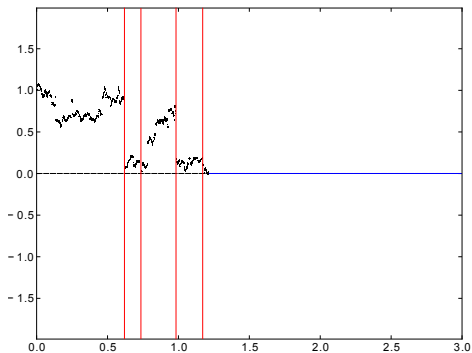
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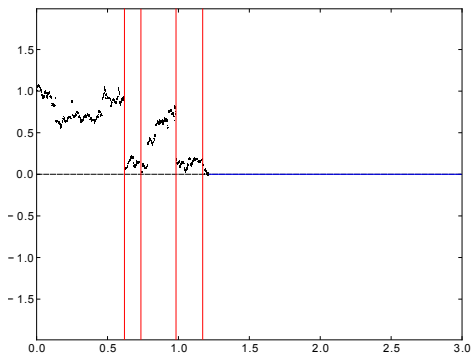


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  - a Lévy process satisfying the scaling property that  $cX_{c^{-\alpha}} \stackrel{d}{=} X$
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- Make zero absorbing
- This is the **path-censored stable process**,  $Y$



# A process, and a problem



How does  $Y$  attain new maxima and minima?  
When does it hit zero?

# Lévy processes

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Examples:

- Brownian motion with drift
- Compound Poisson process
- Variance gamma process
- Stable process

# Lévy processes: the Wiener–Hopf factorisation

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—the *ladder height processes*.

# Positive, self-similar Markov processes

## $\alpha$ -pssMp

$[0, \infty)$ -valued Markov process,  
equipped with initial measures  $P_x$ ,  $x > 0$ ,  
with 0 an absorbing state,  
satisfying the scaling property

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## pssMps: examples

Begin with a stable process  $X$ : a Lévy process satisfying the scaling property. Necessarily  $\alpha \in (0, 2]$ .

$X$  is parameterised by  $(\alpha, \rho)$ , where  $\rho = P_0(X_t > 0)$ .

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$$h^\uparrow(x) = x^{\alpha(1-\rho)}$$

- The stable process conditioned to hit zero continuously,  $X^\downarrow$

$$h^\downarrow(x) = x^{\alpha(1-\rho)-1}$$

# Lamperti transform

$(X, P_x)_{x>0}$   $\alpha$ -pssMp

$\leftrightarrow$

$(\xi, \mathbb{P}_y)_{y \in \mathbb{R}}$  killed Lévy

$$X_t = \exp(\xi_{S(t)}),$$

$$\xi_s = \log(X_{T(s)}),$$

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$$\xi_s = \log(X_{T(s)}),$$

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$\left. \begin{array}{l} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array} \right\}$

$\leftrightarrow$

$\left\{ \begin{array}{l} \xi \rightarrow \infty \text{ or } \xi \text{ oscillates} \\ \xi \rightarrow -\infty \\ \xi \text{ is killed} \end{array} \right.$



# Lamperti transform: examples

- $X^*$ , the killed stable process: the Lévy process  $\xi^*$  has Laplace exponent

$$-\frac{\Gamma(\alpha - z)}{\Gamma(\alpha(1 - \rho) - z)} \frac{\Gamma(1 + z)}{\Gamma(1 - \alpha(1 - \rho) + z)}$$

and is killed.

# Lamperti transform: examples

- $X^\uparrow$ , the stable process conditioned to stay positive: here  $\xi^\uparrow$  has Laplace exponent

$$-\frac{\Gamma(\alpha\rho - z) \Gamma(1 + \alpha(1 - \rho) + z)}{\Gamma(-z) \Gamma(1 + z)}$$

and drifts to  $+\infty$ .

# Lamperti transform: examples

- $X^\downarrow$ , the stable process conditioned to hit zero continuously:  
here  $\xi^\downarrow$  has Laplace exponent

$$-\frac{\Gamma(1 + \alpha\rho - z)}{\Gamma(1 - z)} \frac{\Gamma(\alpha(1 - \rho) + z)}{\Gamma(z)}$$

and drifts to  $-\infty$ .

# Hypergeometric Lévy processes (Kuznetsov and Pardo)

Laplace exponent

$$-\frac{\Gamma(1 - \beta + \gamma - z) \Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 - \beta - z) \Gamma(\hat{\beta} + z)}$$

with parameter set

$$\{\beta \leq 1; \hat{\beta} \geq 0; \gamma, \hat{\gamma} \in (0, 1)\}$$

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Explicit Wiener–Hopf factorisation:

$$\kappa(z) = \frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(1 - \beta + z)} \quad \hat{\kappa}(z) = \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(\hat{\beta} + z)}$$

# The path-censored process and its Lamperti transform

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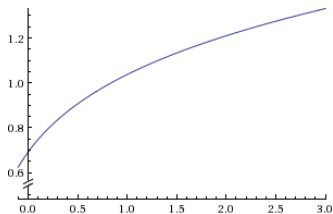
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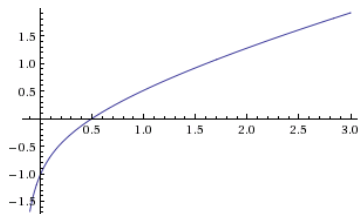
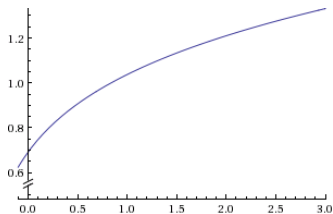
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and when  $\alpha \leq 1$  it is a hypergeometric Lévy process.

When  $\alpha > 1$ , things go wrong:



# Introducing: the extended hypergeometric class

Laplace exponent:

$$-\frac{\Gamma(1 - \beta + \gamma - z) \Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 - \beta - z) \Gamma(\hat{\beta} + z)} \quad (\text{the same!})$$

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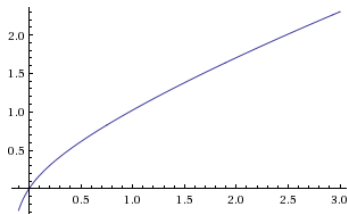
The Wiener–Hopf factorisation looks like this:

$$\kappa(z) = (-\hat{\beta} + z) \frac{\Gamma(1 - \beta + \gamma + z)}{\Gamma(2 - \beta + z)} \quad \hat{\kappa}(z) = (\beta - 1 + z) \frac{\Gamma(\hat{\beta} + \hat{\gamma} + z)}{\Gamma(1 + \hat{\beta} + z)}$$

# The path-censored stable process, $\alpha > 1$

Now we know:

$$\kappa(z) = (\alpha - 1 + z) \frac{\Gamma(\alpha\rho + z)}{\Gamma(1 + z)}, \quad \hat{\kappa}(z) = z \frac{\Gamma(1 - \alpha\rho + z)}{\Gamma(2 - \alpha + z)}.$$

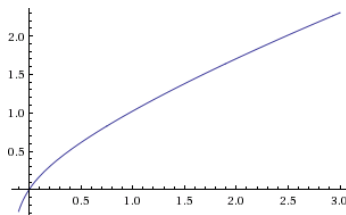


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(much better!)

Can compute first passage identities for  $Y$  (and hence  $X$ ).

# EHG class: more details

Large time behaviour:

- $\xi$  is killed if  $\beta \in (1, 2)$ ,  $\hat{\beta} \in (-1, 0)$ ; otherwise:
- drifts to  $+\infty$  if  $\beta > 1$ ,
- drifts to  $-\infty$  if  $\hat{\beta} < 0$ ,
- oscillates if  $\beta = 1$ ,  $\hat{\beta} = 0$ .

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Lévy density:

$$\begin{cases} \mathbf{c}_+ e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta - \hat{\gamma}; e^{-x}), & x > 0, \\ \mathbf{c}_- e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta - \gamma; e^x), & x < 0, \end{cases}$$

where  $\eta = 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$ , and  ${}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!}$ .



# Exponential functionals

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the Mellin transform.

We use the **functional equation**

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s/\delta)} \mathcal{M}(s).$$

# EHG class: exponential functional

Let  $\xi$  be an extended hypergeometric Lévy process with parameters  $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ .

## Theorem

For  $\operatorname{Re} s \in (0, 1 + \delta(\beta - 1))$

$$\mathcal{M}(s) = c \widetilde{\mathcal{M}}(s) \frac{\Gamma(\delta(1 - \beta + \gamma) + s)}{\Gamma(-\delta\hat{\beta} + s)} \frac{\Gamma(\delta(\beta - 1) + 1 - s)}{\Gamma(\delta(\hat{\beta} + \hat{\gamma}) + 1 - s)},$$

where

$$\widetilde{\mathcal{M}}(s) = \Gamma(s) \frac{G((2 - \beta)\delta + s; \delta)}{G((2 - \beta + \gamma)\delta + s; \delta)} \frac{G((1 + \hat{\beta} + \hat{\gamma})\delta + 1 - s; \delta)}{G((1 + \hat{\beta})\delta + 1 - s; \delta)}$$

is the Mellin transform associated to a hypergeometric Lévy process with parameters  $(\beta - 1, \gamma, \hat{\beta} + 1, \hat{\gamma})$ .

# Path-censored stable process: exponential functional

Let  $\mathcal{I} = \int_0^{T_0} \mathbb{1}_{\{X_t > 0\}} dt$ . Then  $\mathcal{I} = I(-\alpha\xi^Y)$

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Let  $\mathcal{I} = \int_0^{T_0} \mathbb{1}_{\{X_t > 0\}} dt$ . Then  $\mathcal{I} = I(-\alpha\xi^Y)$ , and we have:

## Corollary

For  $\operatorname{Re} s \in (0, 2 - 1/\alpha)$ ,

$$\begin{aligned} \mathcal{M}(s) = c & \frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)} \frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \\ & \times \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)} \Gamma(2 - 1/\alpha - s), \end{aligned}$$

## Example: symmetric stable process

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## Example: symmetric stable process

Take  $X$  to be the *symmetric* stable process, killed upon hitting zero.

Let  $R = \frac{1}{2}|X|$ : a pssMp, the radial part of  $X$ .

$\xi^R$  is hypergeometric when  $\alpha \leq 1$ , and extended hypergeometric when  $\alpha > 1$ , parameters

$$(1, \alpha/2, (1 - \alpha)/2, \alpha/2).$$

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### Corollary

For  $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$ ,

$$\begin{aligned} E_1 [T_0^{s-1}] &= 2^{-\alpha(s-1)} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1 - \frac{1}{\alpha})} \\ &\quad \times \frac{\Gamma(1 + \frac{\alpha}{2} - \frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2} + \frac{\alpha s}{2})} \Gamma(\frac{1}{\alpha} - 1 + s) \frac{\Gamma(2 - \frac{1}{\alpha} - s)}{\Gamma(2 - s)}. \end{aligned}$$

cf. Yano, Yano, Yor (2009); Cordero (2010);  
Kuznetsov, Kyprianou, Pardo, W. (2013).

## Example: symmetric stable process

Let  $\sigma_{-1}^1 = \inf\{t \geq 0 : X_t \notin (-1, 1)\}$ .

### Proposition

For  $|x| < 1$ ,  $y > 1$ ,

$$\begin{aligned} & P_x(|X_{\sigma_{-1}^1}| \in dy; \sigma_{-1}^1 < T_0) \\ &= \frac{\sin(\pi\alpha/2)}{\pi} |x|(1-|x|)^{\alpha/2} y^{-1} (y-1)^{-\alpha/2} (y-|x|)^{-1} \\ &\quad + \frac{1}{2} \frac{\sin(\pi\alpha/2)}{\pi} y^{-1} (y-1)^{-\alpha/2} |x|^{(\alpha-1)/2} \\ &\quad \times \int_0^{1-|x|} t^{\alpha/2-1} (1-t)^{-(\alpha-1)/2} dt. \end{aligned}$$

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### Corollary

For  $|x| < 1$ ,

$$P_x(T_0 < \sigma_{-1}^1) = (1 - |x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2} \int_0^{1-|x|} t^{\alpha/2-1} (1-t)^{-(\alpha-1)/2} dt.$$

## Example: a conditioned symmetric stable process

Take  $X$  to be a symmetric stable process, with  $\alpha > 1$ , killed upon hitting zero. The Doob  $h$ -transform using

$$h^\uparrow(x) = |x|^{\alpha-1}$$

gives the symmetric stable process conditioned to avoid zero,  $X^\uparrow$ .

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$\xi^{R^\uparrow}$  is an extended hypergeometric Lévy process with parameters

$$((\alpha + 1)/2, \alpha/2, 0, \alpha/2).$$



## Further reading



A. E. Kyprianou, J. C. Pardo, A. R. Watson

The extended hypergeometric class of Lévy processes.

[arXiv:1310.1135](https://arxiv.org/abs/1310.1135) [math.PR]

Thank you!