Stable and extended hypergeometric Lévy processes

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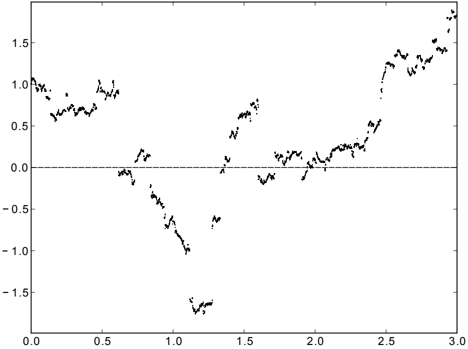
as given at CIMAT, 23 October 2013

A process, and a problem

- Let X be a *stable process*
  - a Lévy process satisfying the scaling property that $cX_{\cdot c^{-\alpha}} \overset{d}{=} X$



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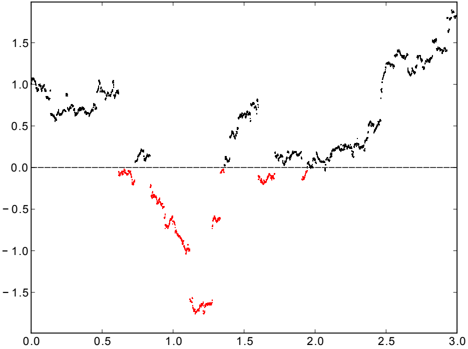

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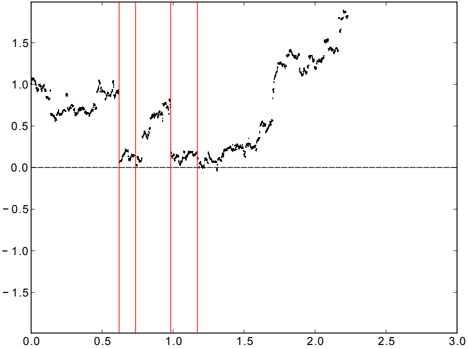

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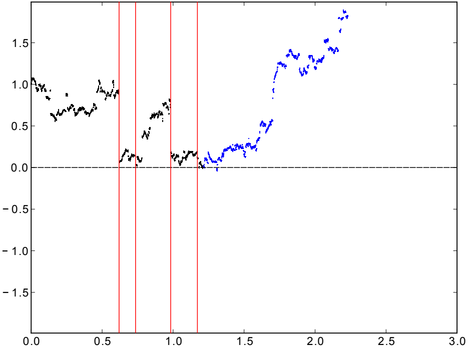

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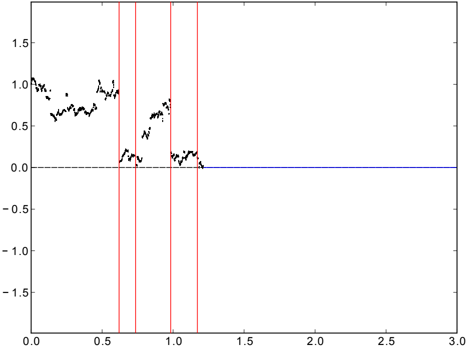
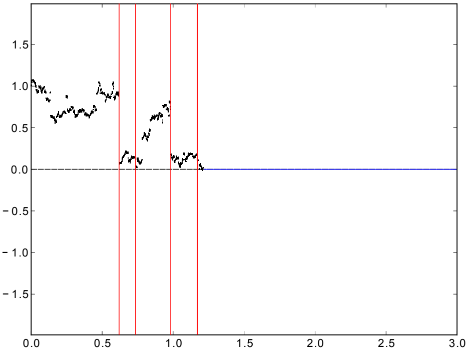
- This is the path-censored stable process, Y





How does Y attain new maxima and minima?
When does it hit zero?

Lévy processes

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Examples:

- Brownian motion with drift
- Compound Poisson process
- Variance gamma process
- Stable process

Lévy processes: the Wiener–Hopf factorisation

If ψ is the Laplace exponent of a Lévy process ξ,

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—the *ladder height processes*.

Positive, self-similar Markov processes

α-pssMp

$[0, \infty)$-valued Markov process,
equipped with initial measures P_x, $x > 0$,
with 0 an absorbing state,
satisfying the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{P_x} \stackrel{d}{=} X|_{P_{cx}}, \qquad x, c > 0$$

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Begin with a stable process X: a Lévy process satisfying the scaling property. Necessarily $\alpha \in (0, 2]$.

X is parameterised by (α, ρ), where $\rho = \mathrm{P}_0(X_t > 0)$.

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- The stable process conditioned to hit zero continuously, $X^\downarrow$

$$h^\downarrow(x) = x^{\alpha(1-\rho)-1}$$

Lamperti transform

$(X, \mathrm{P}_x)_{x>0}$ α-pssMp $\leftrightarrow$ $(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy

$$X_t = \exp(\xi_{S(t)}),$$

S a random time-change

$$\xi_s = \log(X_{T(s)}),$$

T a random time-change

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$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

Lamperti transform: examples

- X^*, the killed stable process: the Lévy process ξ^* has Laplace exponent

$$-\frac{\Gamma(\alpha - z)}{\Gamma(\alpha(1-\rho) - z)} \frac{\Gamma(1+z)}{\Gamma(1-\alpha(1-\rho)+z)}$$

and is killed.

Lamperti transform: examples

- $X^{\uparrow}$, the stable process conditioned to stay positive: here $\xi^{\uparrow}$ has Laplace exponent

$$-\frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)}\frac{\Gamma(1+\alpha(1-\rho)+z)}{\Gamma(1+z)}$$

and drifts to $+\infty$.

Lamperti transform: examples

- $X^{\downarrow}$, the stable process conditioned to hit zero continuously: here $\xi^{\downarrow}$ has Laplace exponent

$$-\frac{\Gamma(1+\alpha\rho-z)}{\Gamma(1-z)}\frac{\Gamma(\alpha(1-\rho)+z)}{\Gamma(z)}$$

and drifts to $-\infty$.

Hypergeometric Lévy processes (Kuznetsov and Pardo)

Laplace exponent

$$-\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}$$

with parameter set

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Explicit Wiener–Hopf factorisation:

$$\kappa(z) = \frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(1-\beta+z)} \qquad \hat{\kappa}(z) = \frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}$$

The path-censored process and its Lamperti transform

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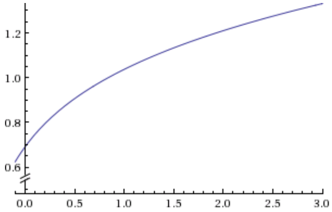

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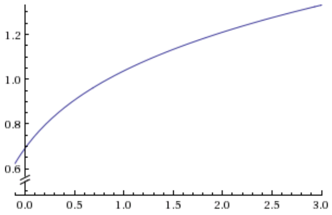
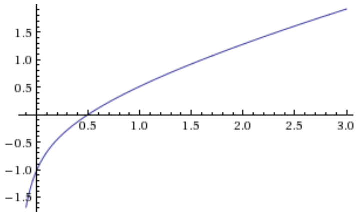


When $\alpha > 1$, things go wrong:



Introducing: the extended hypergeometric class

Laplace exponent:

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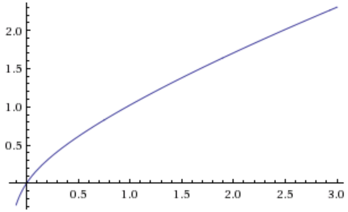
The Wiener–Hopf factorisation looks like this:

$$\kappa(z) = (-\hat{\beta}+z)\frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(2-\beta+z)} \qquad \hat{\kappa}(z) = (\beta-1+z)\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(1+\hat{\beta}+z)}$$

The path-censored stable process, $\alpha > 1$

Now we know:

$$\kappa(z) = (\alpha - 1 + z)\frac{\Gamma(\alpha\rho + z)}{\Gamma(1+z)}, \qquad \hat{\kappa}(z) = z\frac{\Gamma(1 - \alpha\rho + z)}{\Gamma(2 - \alpha + z)}.$$



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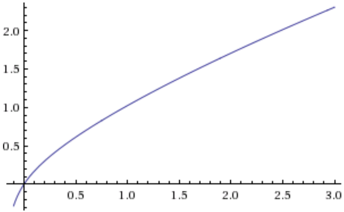

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Can compute first passage identities for Y (and hence X).

EHG class: more details

Large time behaviour:

- ξ is killed if $\beta \in (1,2)$, $\hat{\beta} \in (-1,0)$; otherwise:
- drifts to $+\infty$ if $\beta > 1$,
- drifts to $-\infty$ if $\hat{\beta} < 0$,
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Lévy density:

$$\begin{cases} \mathbf{c}_+ e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat{\gamma}; e^{-x}), & x > 0, \\ \mathbf{c}_- e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta-\gamma; e^{x}), & x < 0, \end{cases}$$

where $\eta = 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, and ${}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!}$.

Exponential functionals

We are interested in

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We use the functional equation

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s/\delta)}\mathcal{M}(s).$$

EHG class: exponential functional

Let ξ be an extended hypergeometric Lévy process with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$.

Theorem

For $\mathrm{Re}\, s \in (0, 1 + \delta(\beta - 1))$

$$\mathcal{M}(s) = c\widetilde{\mathcal{M}}(s) \frac{\Gamma(\delta(1-\beta+\gamma)+s)}{\Gamma(-\delta\hat{\beta}+s)} \frac{\Gamma(\delta(\beta-1)+1-s)}{\Gamma(\delta(\hat{\beta}+\hat{\gamma})+1-s)},$$

where

$$\widetilde{\mathcal{M}}(s) = \Gamma(s) \frac{G((2-\beta)\delta+s;\delta)}{G((2-\beta+\gamma)\delta+s;\delta)} \frac{G((1+\hat{\beta}+\hat{\gamma})\delta+1-s;\delta)}{G((1+\hat{\beta})\delta+1-s;\delta)}$$

is the Mellin transform associated to a hypergeometric Lévy process with parameters $(\beta - 1, \gamma, \hat{\beta} + 1, \hat{\gamma})$.

Path-censored stable process: exponential functional

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Corollary

For $\operatorname{Re} s \in (0, 2 - 1/\alpha)$,

$$\mathcal{M}(s) = c \frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)} \frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \times \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)} \Gamma(2 - 1/\alpha - s),$$

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ξ^R is hypergeometric when $\alpha \leq 1$, and extended hypergeometric when $\alpha > 1$, parameters

$$(1, \alpha/2, (1-\alpha)/2, \alpha/2).$$

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Corollary

For $\mathrm{Re}\, s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathrm{E}_1\left[T_0^{s-1}\right] = 2^{-\alpha(s-1)} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})} \times \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\tfrac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}.$$

cf. Yano, Yano, Yor (2009); Cordero (2010);
Kuznetsov, Kyprianou, Pardo, W. (2013).

Example: symmetric stable process

Let $\sigma_{-1}^{1} = \inf\{t \geq 0 : X_t \notin (-1,1)\}$.

Proposition

For $|x| < 1$, $y > 1$,

$$\begin{aligned}
&\mathrm{P}_x\left(\left|X_{\sigma_{-1}^{1}}\right| \in \mathrm{d}y;\ \sigma_{-1}^{1} < T_0\right) \\
&= \frac{\sin(\pi\alpha/2)}{\pi}|x|(1-|x|)^{\alpha/2}y^{-1}(y-1)^{-\alpha/2}(y-|x|)^{-1} \\
&\quad + \frac{1}{2}\frac{\sin(\pi\alpha/2)}{\pi}y^{-1}(y-1)^{-\alpha/2}|x|^{(\alpha-1)/2} \\
&\qquad \times \int_0^{1-|x|} t^{\alpha/2-1}(1-t)^{-(\alpha-1)/2}\,\mathrm{d}t.
\end{aligned}$$

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Let $\sigma_{-1}^{1} = \inf\{t \geq 0 : X_t \notin (-1, 1)\}$.

Corollary

For $|x| < 1$,

$$\mathrm{P}_x(T_0 < \sigma_{-1}^{1}) = (1 - |x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2} \int_0^{1-|x|} t^{\alpha/2-1}(1-t)^{-(\alpha-1)/2}\,\mathrm{d}t.$$

Example: a conditioned symmetric stable process

Take X to be a symmetric stable process, with $\alpha > 1$, killed upon hitting zero. The Doob h-transform using

$$h^{\updownarrow}(x) = |x|^{\alpha-1}$$

gives the symmetric stable process conditioned to avoid zero, $X^{\updownarrow}$.

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$\xi^{R^{\updownarrow}}$ is an extended hypergeometric Lévy process with parameters

$$((\alpha+1)/2, \alpha/2, 0, \alpha/2).$$

Further reading

A. E. Kyprianou, J. C. Pardo, A. Watson
The extended hypergeometric class of Lévy processes.
arXiv:1310.1135 [math.PR]

Thank you!