Stable and extended hypergeometric Lévy processes

A. R. Watson*

as given at CIMAT, 23 October 2013

Abstract. The Lamperti representation gives a connection between positive self-similar Markov processes on the one hand, and Lévy processes on the other. It has been a great help in computing identities for stable processes. However, to use it successfully, we require tractable families of Lévy processes 'on the other side'. One such family is given by the hypergeometric Lévy processes, defined by Kuznetsov and Pardo in 2010. They have a simple structure which makes their analysis relatively straightforward, but they are deeply connected with killed and conditioned stable processes.

In a number of recent works we have seen new path deformations of stable processes which do not always fit into the hypergeometric family. We therefore present a new family of Lévy processes, called the 'extended hypergeometric class', which encompasses these new cases and whose analysis remain feasible. We will discuss the connection between the classes, the relationship with stable processes, and some identities which may be derived.

Based on [12], which is joint work with A. E. Kyprianou and J. C. Pardo.

About these notes. This document is generated from my `beamer` presentation based on notes left in the source. My hope is that the reader who wishes they had attended my talk, but found it necessary instead to file some expenses/write a grant proposal/play Minecraft, will be able to read this and obtain a similar experience in the comfort of their own home. It should read somewhere between an informal talk and a formal article, and as such there are likely to be some inaccuracies due to the informal language and the fact that I wrote most of it from memory.

This document is probably best read along with the accompanying slides, but 'flattened' (and somewhat mangled) versions are included for convenience, demarcated by horizonal rules.

Are you sitting comfortably? Then I'll begin.

*CIMAT, Mexico. `alexander.watson@cimat.mx`

I'm going to talk about the extended hypergeometric class of Lévy processes, which is simply a class of Lévy processes with a particularly nice characteristic exponent.

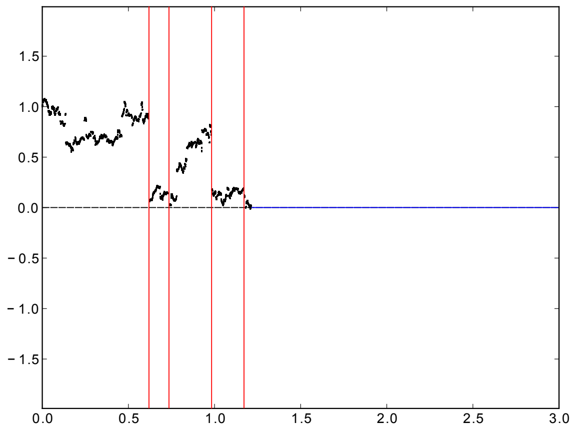
I could give the definition right away, but I think it's much more helpful to begin with a problem, which you can bear in mind as an example throughout the talk. The problem is related to a process called the 'path-censored stable process', which we came up with during my Ph.D. and is therefore dear to my heart.

We begin with a stable process X, which is a Lévy process satisfying a certain scaling property—we will discuss this more thoroughly very soon. We then erase the time during which X is negative, and we erase any part of X after it hits zero.

(I have simplified the figure, and in fact the stable process most certainly does not hit zero in the way pictured – in fact there would be an accumulation of red 'cutting times' leading up to the blue section.)

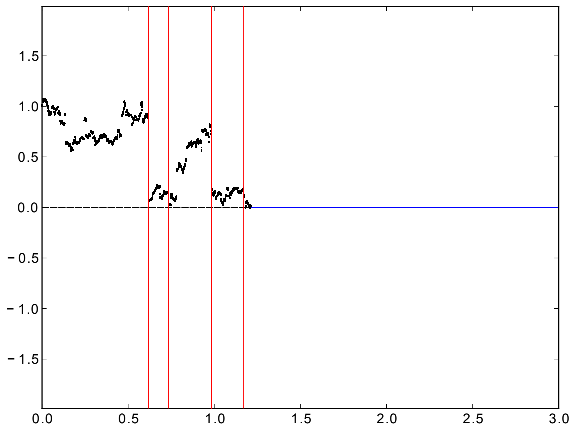
A process, and a problem

- Let X be a *stable process*
  - a Lévy process satisfying the scaling property that $cX_{\cdot c^{-\alpha}} \overset{d}{=} X$
- Erase the negative sections of path
- Make zero absorbing
- This is the *path-censored stable process*, Y



The problem is essentially to describe Y: in particular, how does it attain new maxima and minima, and when does it hit zero?

A process, and a problem



How does Y attain new maxima and minima?

When does it hit zero?

Our tools for this are going come from the theory of Lévy processes, so let's make a short foray into the general theory for a moment.

We begin with the definition of a Lévy process, ξ: it is a stochastic process with càdlàg paths, possessing stationary, independent increments. This property means that the law of the entire process is characterised by its one-dimensional distributions; in particular, by the law of ξ_1, say. Hence, the law of the ξ is characterised by the Laplace exponent ψ of ξ_1.

Note that we do not specify the domain of ψ. It will contain at least the imaginary axis $\{z \in \mathbb{C} : \operatorname{Re} z = 0\}$, because the characteristic function of a probability measure is always defined; however, in our later examples it will even be an entire vertical strip in the complex plane.

Many well-known processes are in fact Lévy processes, most notably Brownian motion. We give only a few examples.

Lévy processes

Let ξ be a Lévy process:

- Càdlàg paths
- Stationary, independent increments
- Characterised by the Laplace exponent ψ:

$$\mathbb{E}\big[e^{z\xi_1}\big] = e^{\psi(z)}.$$

Examples:

- Brownian motion with drift
- Compound Poisson process
- Variance gamma process
- Stable process

The definition of a Lévy process is quite general, but there is still an astonishing amount of 'general theory' about them. One of the most useful aspects is the Wiener–Hopf factorisation. This decomposes a Lévy process into two *subordinators*, or increasing Lévy processes.

The factorisation as we've presented it appears to be purely analytic, but in fact the subordinator corresponding to κ is given by a time-change of the running maximum $\bar{\xi} = \big(\bar{\xi}_t = \sup\{\xi_s, s \le t\}\big)_t$; and the negative of the subordinator corresponding to $\hat{\kappa}$ is given by a time-change of the running minimum.

It's clear that the Wiener–Hopf factorisation is a very useful tool. It has been around for a long time, but until recently there have been relatively few classes of Lévy processes for which the Wiener–Hopf factorisation is known explicitly. This has changed recently; for instance, one can explicitly describe spectrally negative Lévy processes [10], β- and θ-processes [4, 5] and hypergeometric Lévy processes [6] (which we will look at in a moment) in terms of their Wiener–Hopf factorisation; and semi-explicit expressions are available for the large class of meromorphic Lévy processes [8].

The 'extended hypergeometric class' which we are about to discuss will be a new contribution to this field, with explicit Wiener–Hopf factorisation and relatively simple factors.

Lévy processes: the Wiener–Hopf factorisation

If ψ is the Laplace exponent of a Lévy process ξ,

$$\psi(z) = -\kappa(-z)\hat{\kappa}(z);$$

κ and $\hat{\kappa}$ are Laplace exponents of *subordinators*

(increasing Lévy processes):

$$\mathbb{E}\big[e^{-zH_1}\big] = e^{-\kappa(z)}$$

—the *ladder height processes*.

Recall that we wanted to describe the process Y. If we had a way to relate it to a Lévy process, and the Lévy process had a known Wiener–Hopf factor, we would be away.

The crucial property of Y which will allow us to do this is the fact that it is a positive, self-similar Markov process (pssMp).

Positive, self-similar Markov processes

α-pssMp

$[0,\infty)$-valued Markov process, equipped with initial measures P_x, $x > 0$, with 0 an *absorbing state*, satisfying the *scaling property*

$$(cX_{c^{-\alpha}t})_{t\geq 0}\Big|_{\mathrm{P}_x} \overset{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x, c > 0$$

Before proceeding, let us give some more examples of pssMps. Of course, the foremost is the path-censored stable process Y which we are carrying with us. However, there are others, and the principal three are the 'killed and conditioned stable processes'.

By now, we have covered enough ground that we can rigorously define the stable process – it is simply a Lévy process which satisfies exactly the same scaling property as a pssMp. A consequence of this is that $\alpha \in (0,2]$, and $\alpha = 2$ corresponds to the case of Brownian motion; we will largely ignore this case from now on.

Furthermore, apart from a normalisation, the set of stable processes is governed by two parameters, the scaling parameter α and the so-called *positivity parameter* ρ. We will take as the domain of (α,ρ) the following set:

$$\big\{(\alpha,\rho) : \alpha \in (0,1),\ \rho \in (0,1)\big\} \\ \cup \big\{(\alpha,\rho) : \alpha \in (1,2),\ \rho \in (1/\alpha, 1-1/\alpha)\big\} \cup \big\{(\alpha,\rho) = (1,1/2)\big\}.$$

We are excluding Brownian motion; stable processes with one-sided jumps; and symmetric Cauchy processes with non-zero drift.

The process X^* is simply the stable process sent to zero, or killed, when it attempts to jump below zero. A stable process will always fall below zero, in finite time, by a jump; it can never approach zero continuously from above. The processes $X^\uparrow$ and $X^\downarrow$ are essentially Doob h-transforms of the sub-Markov process X^*. $X^\uparrow$ is transient and approaches $+\infty$, while $X^\downarrow$ reaches the point zero continuously in finite time.

---

pssMps: examples

Begin with a stable process X: a Lévy process satisfying the scaling property. Necessarily $\alpha \in (0, 2]$.

X is parameterised by (α, ρ), where $\rho = \mathrm{P}_0(X_t > 0)$.

We can then manufacture pssMps:

- The path-censored stable process Y
- The stable process killed upon exiting $[0, \infty)$

$$X_t^* = X_t \mathbb{1}_{\{t < \tau_0^-\}}$$

- The stable process conditioned to stay positive, $X^\uparrow$

$$h^\uparrow(x) = x^{\alpha(1-\rho)}$$

- The stable process conditioned to hit zero continuously, $X^\downarrow$

$$h^\downarrow(x) = x^{\alpha(1-\rho)-1}$$

Having a stable of examples, we will now discuss how to relate pssMps to Lévy processes. There is in fact a bijection, named after Lamperti. It involves a change of space and time.

Lamperti transform

$$(X, \mathrm{P}_x)_{x>0} \ \alpha\text{-pssMp}$$

$$X_t = \exp(\xi_{S(t)}),$$

S a random time-change

$$\leftrightarrow$$

$$(\xi, \mathbb{P}_y)_{y\in\mathbb{R}} \text{ killed Lévy}$$

$$\xi_s = \log(X_{T(s)}),$$

T a random time-change

$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array}\right\}$$

$$\leftrightarrow$$

$$\left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

Recall our killed and conditioned stable processes – we now apply the Lamperti transform to these examples. We will return to the example of Y shortly.

Lamperti transform: examples

- X^*, the killed stable process: the Lévy process ξ^* has Laplace exponent

$$-\frac{\Gamma(\alpha - z)}{\Gamma(\alpha(1-\rho) - z)} \frac{\Gamma(1+z)}{\Gamma(1 - \alpha(1-\rho) + z)}$$

  and is killed.

- $X^{\uparrow}$, the stable process conditioned to stay positive: here $\xi^{\uparrow}$ has Laplace exponent

$$-\frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)} \frac{\Gamma(1 + \alpha(1-\rho) + z)}{\Gamma(1+z)}$$

  and drifts to $+\infty$.

- $X^{\downarrow}$, the stable process conditioned to hit zero continuously: here $\xi^{\downarrow}$ has Laplace exponent
$$-\frac{\Gamma(1+\alpha\rho-z)}{\Gamma(1-z)}\frac{\Gamma(\alpha(1-\rho)+z)}{\Gamma(z)}$$
and drifts to $-\infty$.

You will notice the similarity of the Laplace exponents. In fact, all of these processes belong to the *hypergeometric class* of Lévy processes. We present the definition of [6], but see also [13] and [7].

These Lévy processes have a Laplace exponent again consisting of ratios of gamma functions, and in fact, one can find the Wiener–Hopf factorisation explicitly simply by splitting the two ratios apart. The ladder height processes are so-called 'Lamperti-stable subordinators', a particularly nice class of processes in which one has, for instance, the Lévy measures and renewal measures explicitly.

Hypergeometric Lévy processes (Kuznetsov and Pardo)

Laplace exponent
$$-\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}$$
with parameter set
$$\{\beta \leq 1;\ \hat{\beta} \geq 0;\ \gamma, \hat{\gamma} \in (0,1)\}$$

Explicit Wiener–Hopf factorisation:
$$\kappa(z) = \frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(1-\beta+z)} \qquad \hat{\kappa}(z) = \frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}$$

We return at last to the path-censored stable process Y, and look at its Lamperti transform ξ^Y. When $\alpha \leq 1$, it is simple to see from ψ that it is a hypergeometric Lévy process (although deriving ψ itself is not easy).

However, when $\alpha > 1$, things go wrong. The process falls outside of the hypergeometric class – in particular, the Wiener–Hopf factorisation is clearly not the same, and this can be seen from the figures. The left-hand plot is the descending Wiener–Hopf factor $\hat{\kappa}$ of the hypergeometric class when $\alpha < 1$, and the right-hand plot is the same quantity when $\alpha > 1$. It cannot be correct, since Laplace exponents of subordinators must be positive.

The path-censored process and its Lamperti transform

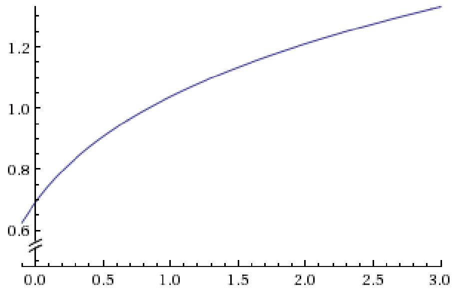
Recall the path-censored Y – write ξ^Y for its Lamperti transform.

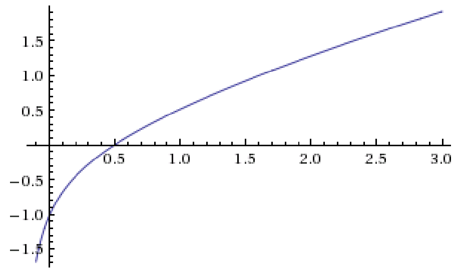
It has Laplace exponent

$$-\frac{\Gamma(\alpha\rho - z)}{\Gamma(-z)}\frac{\Gamma(1-\alpha\rho+z)}{\Gamma(1-\alpha+z)},$$

and when $\alpha \leq 1$ it is a hypergeometric Lévy process.

When $\alpha > 1$, things go wrong:





The solution to this problem is the extended hypergeometric family. (We are now 12 slides into the talk, so it is about time!) The Laplace exponent of such a process is just the same as for a hypergeometric Lévy process, but the parameter space is different: we have moved β to the right and $\hat{\beta}$ to the left. (In fact, the parameter space should include the conditions '$1 - \beta + \hat{\beta} + \gamma \geq 0$, $1 - \beta + \hat{\beta} + \hat{\gamma} \geq 0$', which I omitted from the slides.)

The key difference with the extended hypergeometric class is the Wiener–Hopf factorisation: each factor has 'borrowed' a linear term from the other. This means that the ladder height processes are no longer Lamperti-stable subordinators, but so-called $\mathcal{T}_\beta$-transformations of Lamperti-stable subordinators. This makes them a little less nice to work with, but one can still calculate their Lévy measures and potential measures.

We should pause at this point to say something about our methods: the technique for proving the existence relies heavily on the theory of philanthropy, developed by Vigon [16], which essentially allows one to manufacture a Lévy process from a pair of desired Wiener–Hopf factors. On the other hand, the idea for what the Wiener–Hopf factors should be came mainly from the theory of 'meromorphic Lévy processes' [8], in which Lévy processes are described via the poles and zeroes of their Laplace exponents. From this perspective, the extended hypergeometric class is derived from the hypergeometric class by switching two of these zeroes.

Introducing: the extended hypergeometric class

Laplace exponent:

$$-\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat\beta+\hat\gamma+z)}{\Gamma(\hat\beta+z)} \qquad \text{(the same!)}$$

and parameter space:

$$\{\beta \in [1,2];\ \hat\beta \in [-1,0];\ \gamma,\hat\gamma \in (0,1);\ \text{etc.}\} \qquad \text{(different!)}$$

The Wiener–Hopf factorisation looks like this:

$$\kappa(z) = (-\hat\beta + z)\frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(2-\beta+z)} \qquad \hat\kappa(z) = (\beta-1+z)\frac{\Gamma(\hat\beta+\hat\gamma+z)}{\Gamma(1+\hat\beta+z)}$$

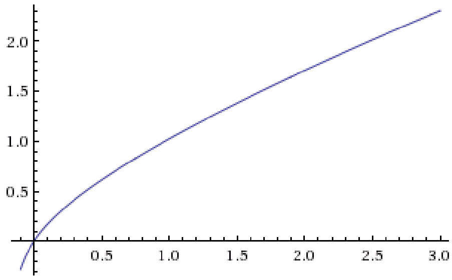
Let's return very briefly to the process Y, our motivation for this definition. We can now see the correct Wiener–Hopf factorisation. Moreover, with this factorisation one can compute several quantities for the process Y, and hence for the process X; this is the method employed in Kyprianou et al. [11]. This is not the focus of this talk, but as a sample, one may find the hitting distribution of the finite interval $(-1,1)$ for a stable process; and the hitting distribution of $(1,\infty)$ on the event that this occurs before hitting $\{0\}$.

Incidentally, what is the difference between $\alpha \le 1$ and $\alpha > 1$? I believe that the answer is that in the former case, the process will never hit the point zero, and therefore the process Y encapsulates the entire time that the stable process is is $(0,\infty)$; whereas in the latter case, the process X, and hence Y, hits zero in finite time with probability one, and Y is truncated here.

The path-censored stable process, $\alpha > 1$

Now we know:

$$\kappa(z) = (\alpha-1+z)\frac{\Gamma(\alpha\rho+z)}{\Gamma(1+z)}, \qquad \hat\kappa(z) = z\frac{\Gamma(1-\alpha\rho+z)}{\Gamma(2-\alpha+z)}.$$



(much better!)

Can compute first passage identities for Y (and hence X).

We will give some additional examples towards the end; for now, let us collect some results on the extended hypergeometric class.

One observes killing, drifting or oscillating according to the choice of parameters.

The Lévy measure is explicit, and in terms of a Gauss hypergeometric function. Note that at zero, the Lévy density looks like $x^{-(1+\gamma+\hat{\gamma})}$, if my back-of-the-envelope calculation is correct; and at $\pm\infty$, it has exponential decay.

EHG class: more details

Large time behaviour:

- ξ is killed if $\beta \in (1,2)$, $\hat{\beta} \in (-1,0)$; *otherwise*:
- drifts to $+\infty$ if $\beta > 1$,
- drifts to $-\infty$ if $\hat{\beta} < 0$,
- oscillates if $\beta = 1$, $\hat{\beta} = 0$.

Lévy density:

$$\begin{cases} \mathbf{c}_+ e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat{\gamma}; e^{-x}), & x > 0, \\ \mathbf{c}_- e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta-\gamma; e^{x}), & x < 0, \end{cases}$$

where $\eta = 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, and ${}_2F_1(a,b;c;z) = \sum_{n\geq 0} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!}$.

Another quantity which one can obtain for extended hypergeometric Lévy processes is the exponential functional. The exponential functional of a Lévy process has been studied extensively in general; the paper of Bertoin and Yor [2] gives a survey of the literature, and mentions, among other aspects, applications to diffusions in random environments, mathematical finance and fragmentation theory. In the context of self-similar Markov processes, the exponential functional appears in the entrance law of a pssMp started at zero (see, for example, Bertoin and Yor [1]), and Pardo [15] relates the exponential functional of a Lévy process to envelopes of its associated pssMp; furthermore, it is related to the hitting time of points for pssMps – we will see an example shortly, using the process Y.

Exponential functionals

We are interested in

$$I(\xi/\delta) = \int_0^\infty e^{-\xi_u/\delta}\,\mathrm{d}u,$$

the *exponential functional* of ξ.

To find the law, we compute

$$\mathcal{M}(s) = \mathbb{E}\big[\big(I(\xi/\delta)\big)^{s-1}\big],$$

the *Mellin transform*.

We use the *functional equation*

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s/\delta)}\mathcal{M}(s).$$

We give an explicit characterisation of the exponential functional of a process in the EHG class. The function G is a double-gamma function, as defined by Barnes and Alexeiewsky; this function has recently been exploited in several papers by A. Kuznetsov, and also appears in the literature of analytic number theory.

Note that one may give an expression for the density of I in terms of (Laurent) series whose coefficients are defined iteratively – this approach was taken in Kuznetsov and Pardo [6] and we will not pursue it here.

EHG class: exponential functional

Let ξ be an extended hypergeometric Lévy process with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$.
Theorem 1. *For* $\operatorname{Re} s \in (0, 1 + \delta(\beta - 1))$

$$\mathcal{M}(s) = c\widetilde{\mathcal{M}}(s)\frac{\Gamma(\delta(1-\beta+\gamma)+s)}{\Gamma(-\delta\hat{\beta}+s)}\frac{\Gamma(\delta(\beta-1)+1-s)}{\Gamma(\delta(\hat{\beta}+\hat{\gamma})+1-s)},$$

where

$$\widetilde{\mathcal{M}}(s) = \Gamma(s)\frac{G((2-\beta)\delta+s;\delta)}{G((2-\beta+\gamma)\delta+s;\delta)}\frac{G((1+\hat{\beta}+\hat{\gamma})\delta+1-s;\delta)}{G((1+\hat{\beta})\delta+1-s;\delta)}$$

is the Mellin transform associated to a hypergeometric Lévy process with parameters $(\beta-1, \gamma, \hat{\beta}+1, \hat{\gamma})$.

We give a simple application of this theorem to the process Y. Here we also bring in the connection from the Lamperti transform: the exponential functional of ξ^Y is precisely the *occupation time of the* $(0,\infty)$ *for* X *before hitting zero.*

Incidentally, when the process X is in the Doney class for which

$$\rho + k = l/\alpha, \qquad k,\, l \in \mathbb{Z},$$

the expression given for $\mathcal{M}$ can be rewritten as a product of gamma functions and trigonometric functions.

Path-censored stable process: exponential functional

Let $\mathcal{I} = \int_0^{T_0} \mathbb{1}_{\{X_t>0\}}\,\mathrm{d}t$. Then $\mathcal{I} = I(-\alpha\xi^Y)$, and we have:
Corollary 2. *For* $\operatorname{Re} s \in (0, 2-1/\alpha)$,

$$\mathcal{M}(s) = c\frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)} \frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \times \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)}\Gamma(2 - 1/\alpha - s),$$

Finally we turn to additional examples of EHG processes. We consider the symmetric stable process and look at its radial part – the Lamperti transform gives us another process which is hypergeometric for small α and EHG for large α.

Example: symmetric stable process

Take X to be the *symmetric* stable process, killed upon hitting zero.

Let $R = \frac{1}{2}|X|$: a pssMp, the *radial part of* X.

ξ^R is hypergeometric when $\alpha \le 1$, and extended hypergeometric when $\alpha > 1$, parameters

$$(1, \alpha/2, (1-\alpha)/2, \alpha/2).$$

The exponential functional of ξ^R has the same law as the first hitting time of zero for a symmetric stable process; we characterise its law here. This distribution has previously been calculated by Yano et al. [17] and Cordero [3]. The distribution for the asymmetric stable process has been calculated by the authors and A. Kuznetsov in [9], using a very similar approach.

Example: symmetric stable process

Let $T_0 = \inf\{t \geq 0 : X_t = 0\}$. Then $T_0 = 2^{-\alpha} I(-\alpha\xi^R)$, and:
Corollary 3. *For* $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathrm{E}_1[T_0^{s-1}] = 2^{-\alpha(s-1)} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})} \times \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\tfrac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}.$$

cf. Yano, Yano, Yor (2009); Cordero (2010);

Kuznetsov, Kyprianou, Pardo, W. (2013).

The Wiener–Hopf factorisation can be exploited to give fluctuation identities for the symmetric stable process; here we consider first exit from an interval on the event of exiting before hitting zero. Integrating this gives the hitting probability.

Example: symmetric stable process

Let $\sigma_{-1}^{1} = \inf\{t \geq 0 : X_t \notin (-1,1)\}$.
Proposition. *For* $|x| < 1$, $y > 1$,

$$\begin{aligned} &\mathrm{P}_x(|X_{\sigma_{-1}^1}| \in \mathrm{d}y;\ \sigma_{-1}^1 < T_0) \\ &= \frac{\sin(\pi\alpha/2)}{\pi} |x|(1-|x|)^{\alpha/2} y^{-1}(y-1)^{-\alpha/2}(y-|x|)^{-1} \\ &\quad + \frac{1}{2}\frac{\sin(\pi\alpha/2)}{\pi} y^{-1}(y-1)^{-\alpha/2}|x|^{(\alpha-1)/2} \\ &\quad\quad \times \int_0^{1-|x|} t^{\alpha/2-1}(1-t)^{-(\alpha-1)/2}\,\mathrm{d}t. \end{aligned}$$

Corollary 4. *For* $|x| < 1$,

$$\mathrm{P}_x(T_0 < \sigma_{-1}^1) = (1-|x|)^{\alpha/2} - \frac{1}{2}|x|^{(\alpha-1)/2} \int_0^{1-|x|} t^{\alpha/2-1}(1-t)^{-(\alpha-1)/2}\,\mathrm{d}t.$$

We give one last example of an extended hypergeometric process; this is again derived from the symmetric stable process, but here we condition the process, in the sense of Doob h-transform, to avoid zero. This conditioning comes from Pantí [14].

Example: a conditioned symmetric stable process

Take X to be a symmetric stable process, with $\alpha > 1$, killed upon hitting zero. The Doob h-transform using

$$h^{\updownarrow}(x) = |x|^{\alpha-1}$$

gives the *symmetric stable process conditioned to avoid zero*, $X^{\updownarrow}$.

Let $R^{\updownarrow} = \frac{1}{2}|X^{\updownarrow}|$, a pssMp.

$\xi^{R^{\updownarrow}}$ is an extended hypergeometric Lévy process with parameters

$$((\alpha+1)/2, \alpha/2, 0, \alpha/2).$$

References

[1] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002. ISSN 0926-2601. doi:10.1023/A:1016377720516.

[2] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005. ISSN 1549-5787. doi:10.1214/154957805100000122.

[3] F. Cordero. *On the excursion theory for the symmetric stable Lévy processes with index* $\alpha \in]1,2]$ *and some applications*. PhD thesis, Université Pierre et Marie Curie – Paris VI, 2010.

[4] A. Kuznetsov. Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.*, 20(5):1801–1830, 2010. ISSN 1050-5164. doi:10.1214/09-AAP673.

[5] A. Kuznetsov. Wiener-Hopf factorization for a family of Lévy processes related to theta functions. *J. Appl. Probab.*, 47(4):1023–1033, 2010. ISSN 0021-9002.

[6] A. Kuznetsov and J. C. Pardo. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. arXiv:1012.0817v1 [math.PR], 2010.

[7] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, and K. van Schaik. A Wiener-Hopf Monte Carlo simulation technique for Lévy processes. *Ann. Appl. Probab.*, 21(6): 2171–2190, 2011. ISSN 1050-5164. doi:10.1214/10-AAP746.

[8] A. Kuznetsov, A. E. Kyprianou, and J. C. Pardo. Meromorphic Lévy processes and their fluctuation identities. *Ann. Appl. Probab.*, 22(3):1101–1135, 2012. doi:10.1214/11-AAP787.

[9] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, and A. R. Watson. The hitting time of zero for a stable process. arXiv:1212.5153v1 [math.PR], 2012.

[10] A. E. Kyprianou and V. Rivero. Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.*, 13:1672–1701, 2008. ISSN 1083-6489.

[11] A. E. Kyprianou, J. C. Pardo, and A. R. Watson. Hitting distributions of α-stable processes via path censoring and self-similarity. arXiv:1112.3690v2 [math.PR], 2012. To appear, *Ann. Probab.*, .

[12] A. E. Kyprianou, J. C. Pardo, and A. R. Watson. The extended hypergeometric class of Lévy processes. arXiv:1310.1135v1 [math.PR], 2013., .

[13] A. E. Kyprianou, J. C. Pardo, and V. Rivero. Exact and asymptotic n-tuple laws at first and last passage. *Ann. Appl. Probab.*, 20(2):522–564, 2010. ISSN 1050-5164. doi:10.1214/09-AAP626.

[14] H. Pantí. On Lévy processes conditioned to avoid zero. arXiv:1304.3191v1 [math.PR], 2012.

[15] J. C. Pardo. The upper envelope of positive self-similar Markov processes. *J. Theoret. Probab.*, 22(2):514–542, 2009. ISSN 0894-9840. doi:10.1007/s10959-008-0152-z.

[16] V. Vigon. *Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf.* PhD thesis, INSA de Rouen, 2002.

[17] K. Yano, Y. Yano, and M. Yor. On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes. In *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 187–227. Springer, Berlin, 2009. doi:10.1007/978-3-642-01763-6_8.