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Adaptive Estimation and Rejection of Unknown Sinusoidal Disturbances in A Class of Non-minimum-phase Nonlinear Systems

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Abstract

This paper deals with adaptive estimation of unknown disturbances in a class of non-minimum phase nonlinear systems, and the stabilization and disturbance rejection based on the estimated disturbances. The unknown disturbances are combination of sinusoidal disturbances with unknown frequencies, unknown phases and amplitudes. The only information of the unknown disturbances is the number of distinctive frequencies inside. The class of nonlinear systems considered in this paper consists of nonlinear systems in the output feedback form and the systems may be nonminimum phase, ie, with unstable zero dynamics. An adaptive estimation algorithm is developed to give exponentially convergent estimates of the unknown disturbance and the system states. The asymptotic convergent estimates of unknown frequencies are also obtained. The proposed estimation algorithm works for both minimum phase and nonminimum phase nonlinear systems in output feedback form. Disturbance rejection with stabilisation is achieved by combining the control designed for the stabilisation of the disturbance-free system and the exponentially convergent estimate of the disturbance. Under the conditions specified for the control design of the system with no disturbance, the overall stability and complete rejection of the unknown disturbance are guaranteed by the proposed control algorithm.

1. Introduction

In engineering systems, there are deterministic disturbances, apart from random disturbances. Among the various types of deterministic disturbances, sinusoidal disturbances have attracted a large amount of research interests, from the estimation of the disturbance frequencies to the compensation or rejection of disturbances. Estimation and reconstruction of unknown disturbances have their importance for detection and monitoring, apart from the stabilization of a system and disturbance rejection. It was until fairly recently that a global convergent estimation algorithm was proposed for estimation of a single frequency of the stand alone sinusoidal signal [1], and more recently an algorithm was proposed to estimate multiple frequencies from a sinusoidal signal using adaptive observers [2]. On the other hand, a series of results have been published for rejecting disturbances of unknown frequencies [3, 4, 5]. Two algorithms, a direct and an indirect one, are presented in [4] for disturbance compensation for stable linear time invariant systems. The indirect one estimates the disturbance frequency first and then to compensate it. Only the direct one ensures the complete compensation or asymptotic rejection of disturbances with unknown frequencies. The algorithm shown in [5] ensures robust compensation of unknown disturbances for linear systems. For nonlinear systems, a result for strict feedback nonlinear system is shown in [6] based on full state feedback. For nonlinear systems using output feedback, semiglobal output regulation is achieved in [7] using adaptive internal model, and global rejection with stabilization is reported in [8] for nonlinear systems in output feedback form. Both results for nonlinear systems require that the nonlinear systems are minimum phase.

In this paper, we consider estimation of unknown sinusoidal disturbances and their complete rejection for nonlinear systems in the output feedback form. We allow the system to be nonminimum phase, and we also consider the stabilization of the system together with the disturbance rejection. Our approach is indirect design, with a separate estimation of disturbances. In the estimation stage, a new set of filters are designed to extract the contribution of the disturbance to the states and to estimate disturbance and the frequencies. The estimation starts from the contribution to the output of the system, from which the disturbance characterization such as frequencies can be obtained. Based on this estimation, the contributions to other states can then be calculated and finally the unknown disturbance is reconstructed. The proposed estimation algorithm imposes no restriction on the number or the range of disturbance frequencies, and no restriction such as projection used in [5] for the adaptive law for parameter estimation. The estimated disturbance and frequencies asymptotically converge to their ideal values. Control design of nonminimum phase nonlinear systems is a very challenging problem itself. Although there are progresses recently such as results on design of nonlinear systems with only one nonminimum phase zero reported in [9] and the semi-global stabilisation in [10], a general control design method does not exist for achieving the global stabilisation of the systems considered in this paper even when the systems are disturbance-free. In the absence of a general control design method for the global stability, we will identify the conditions for the control design of the disturbance-free system to ensure that the estimate of disturbance obtained by the proposed algorithm can be used effectively in disturbance rejection. We will show that once the conditions are satisfied, the complete rejection of disturbance with stabilisation can be achieved by adding the disturbance estimate directly to the control input designed for the stabilisation of the disturbance-free system. As there are a quite number of filters involved in the estimation and control design, an example is included in the paper to demonstrate the actual filters and observers used in the estimation and control. The simulation results for the demonstrated example are also included.

2. Problem Formulation

We consider a single-input-single-output nonlinear system which can be transformed into the output feedback form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_c\mathbf{x} + \boldsymbol{\phi}(y) + \mathbf{b}(u-\mu) \\ y &= \mathbf{C}\mathbf{x} \end{aligned} \tag{1}$$

with

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

where $\mathbf{x} \in R^n$ is the state vector, $u \in R$ is the control, $\boldsymbol{\phi}$, is a known nonlinear smooth vector field in R^n with $\boldsymbol{\phi}(0) = 0$, $\mu \in R$ is a matched disturbance which is generated from an unknown exosystem

$$\dot{\mathbf{w}} = \mathbf{S}\mathbf{w}, \mathbf{w}(0) = \mathbf{w}_0$$

$$\mu = \mathbf{l}^T \mathbf{w} \quad (2)$$

with $\mathbf{w} \in R^s$.

Remark 1. The coordinate-free geometric conditions for the existence of state transform for transforming a nonlinear system into (1) are specified in [11]. $b_\rho \neq 0$ indicates the nonlinear system before the transformation has a constant relative degree of ρ.

The problem we will solve in this paper is to propose estimation algorithms which provide convergent estimates of unknown disturbances and disturbance frequencies from the system output, and a control algorithm that asymptotically rejects the disturbances based on the estimates of disturbances. To solve the problem, we need the following assumptions.

Assumption 1. The zeros of polynomial $\mathbf{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ have non-zero real parts.

Remark 2. Assumption 1 only requires that $\mathbf{B}(s)$ has not zeros on the imaginary axis. It allows $\mathbf{B}(s)$ to have positive real parts. In fact, the zeros of polynomial $\mathbf{B}(s)$ are the invariant zeros of the triple $\{\mathbf{A}_c, \mathbf{b}, \mathbf{C}\}$, and hence Assumption 1 implies that the triple $\{\mathbf{A}_c, \mathbf{b}, \mathbf{C}\}$ might be of nonminimum phase.

Assumption 2. The eigenvalues of $\mathbf{S}$ are with zero real parts and are distinct, and the initial state $\mathbf{w}_0$ is such that all the frequency components in the disturbance system are fully excited. Furthermore, the state $\mathbf{w}$ of the exosystem is observable from the output y.

Remark 3. Assumption 2 ensures that the disturbances generated from the exosystem (2) are combination of sinusoidal signals including a possible constant bias. The dimension of $\mathbf{S}$ decides the number of independent frequencies in the disturbances. It follows the assumption made on unknown exosystems in [6, 12, 8]. Unlike the neutral stable assumptions on exosystems in [13, 14, 15], the dynamics are not assumed to be known. In case there is a degeneration of independent frequencies in the disturbance due to the initial state $\mathbf{w}_0$, the exosystem model can be reduced in dimension such that the disturbance is fully excited in the reduced oder. Therefore, for a disturbance with the known number of independent frequencies, Assumption 2 does not impose a restriction on the initial state of the exosystem.

Remark 4. As shown in [8], the unmatched disturbances in the nonlinear systems in the output feedback form can be transformed to the matched case of (1), if Assumption 2 is satisfied. In this paper, we only consider the matched disturbance for the convenience of presentation.

Remark 5. With Assumption 2, we could relax Assumption 1 to the case that the invariant zeros of the triple $\{\mathbf{A}_c, \mathbf{b}, \mathbf{C}\}$ and the eigenvalues of $\mathbf{S}$ are exclusive. Since $\mathbf{S}$ is unknown, it is convenient to use Assumption 1 that rules out any possibility of overlapping of the invariant zeros and the eigenvalues of $\mathbf{S}$.

In this paper, we adopt an indirect approach to disturbance rejection with stabilisation. First, we estimate the disturbance μ, the state $\mathbf{x}$ and the unknown disturbance frequencies characterized by the eigenvalues of $\mathbf{S}$. Then, we use the estimates obtained in the estimation stage to design a feedback control which ensures the overall stability of the feedback control system and that the output converges to zero. Control design of nonminimum phase nonlinear systems is a challenging problem itself. To concentrate on the disturbance estimation and rejection, we specify the conditions for the control design of the nonlinear systems when there is no disturbance.

Assumption 3. Consider the dynamic system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_c\mathbf{x} + \boldsymbol{\phi}(y) + \mathbf{b}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned} \tag{3}$$

There exists an output feedback controller

$$\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v}, y) \tag{4}$$

$$u = h(\mathbf{v}, y) \tag{5}$$

such that the closed-loop control described under the state $\bar{\mathbf{x}} = [\mathbf{x}^T, \mathbf{v}^T]^T$ is asymptotically stable. Furthermore, there exists a Lyapunov functions $V(\bar{\mathbf{x}})$ such that

$$\alpha_1(\|\bar{\mathbf{x}}\|) \leq V(\bar{\mathbf{x}}) \leq \alpha_2(\|\bar{\mathbf{x}}\|) \tag{6}$$

$$\frac{\partial V(\bar{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{A}_c\mathbf{x} + \boldsymbol{\phi}(y) + \mathbf{b}h(\mathbf{v}, y)) + \frac{\partial V(\bar{\mathbf{x}})}{\partial \mathbf{v}}\mathbf{f}(\mathbf{v}, y) \leq -\alpha_3(\|\bar{\mathbf{x}}\|) \tag{7}$$

$$\alpha_3(\|\bar{\mathbf{x}}\|) \geq c_1\|\frac{\partial V(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\|^{c_2} \tag{8}$$

where $\alpha_i, i = 1, 2, 3$, are K_∞ functions and $c_i, i = 1, 2$, are positive reals with $c_2 > 1$.

Remark 6. The conditions specified in (6) and (7) are automatically satisfied if the closed-loop system is asymptotically stable. The condition (8) is always satisfied if the closed-loop system is exponentially stable. Note that there are systems such that the conditions in Assumption 3 are all satisfied, but the systems are not exponentially stable [16].

3. Preliminary Design and Analysis

If the disturbance does not exist in (1), the system (1) is in the linear-observer-error format [14]. In that case, we can design a state observer as

$$\dot{\mathbf{p}} \;=\; (\mathbf{A}_c + \mathbf{k}\mathbf{C})\mathbf{p} + \boldsymbol{\phi}(y) + \mathbf{b}u - \mathbf{k}y \tag{9}$$

where $\mathbf{p} \in R^n$, $\mathbf{k} \in R^n$ is chosen so that $\mathbf{A}_c + \mathbf{k}\mathbf{C}$ is Hurwitz. The difficulty in the state estimation is due to the unknown disturbance μ. Based on Assumption 2 and the design of $\mathbf{k}$, $\mathbf{A}_c + \mathbf{k}\mathbf{C}$, and $\mathbf{S}$ have exclusive eigenvalues, and therefore there exists a solution $\mathbf{Q} \in R^{n\times s}$ of the following matrix equation for a given $\mathbf{S}$

$$\mathbf{Q}\mathbf{S} = (\mathbf{A}_c + \mathbf{k}\mathbf{C})\mathbf{Q} + \mathbf{b}\mathbf{l}^T \tag{10}$$

Define

$$\mathbf{q}(\mathbf{w}) = \mathbf{Q}\mathbf{w} \tag{11}$$

then (10) guarantees

$$\dot{\mathbf{q}} \;=\; (\mathbf{A}_c + \mathbf{k}\mathbf{C})\mathbf{q} + \mathbf{b}\mu \tag{12}$$

Since $\mathbf{S}$ is unknown, we do not have the solution $\mathbf{Q}$ from (10) and the filter (12) cannot be implemented due to the unknown disturbance μ. But the two equations (10) and (12) are important in the reformulation of the estimation problem through the property stated in the following lemma [8].

Lemma 3.1 The state variable $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \mathbf{p} - \mathbf{q} + \boldsymbol{\epsilon} \tag{13}$$

where $\mathbf{p}$ is generated from (9) with $\mathbf{q}$ satisfying (12) and $\boldsymbol{\epsilon}$ satisfying

$$\dot{\boldsymbol{\epsilon}} = (\mathbf{A}_c + \mathbf{k}\mathbf{C})\boldsymbol{\epsilon} \tag{14}$$

The state estimation is solved if an estimate of $\mathbf{q}$ is provided. Referring to (12), the problem we are going to solve is to estimate both the state and the unknown input to a nonminimum phase linear dynamic system. The solution depends on the characteristics of the matched disturbance μ.

For the convenience of filter design for adaptive estimation, we introduce a reformulation of the exosystem (2). Choose a controllable pair $\{\mathbf{F}, \mathbf{G}\}$ with $\mathbf{F} \in R^{s\times s}$ Hurwitz and $\mathbf{G} \in R^s$. For a matrix $\mathbf{S}$ satisfying Assumption 2 which also implies $\{\mathbf{S}, \mathbf{Q}_{(1)}\}$ observable, there exists a nonsingular $\mathbf{M} \in R^{s\times s}$ satisfying the following equation

$$\mathbf{MS} - \mathbf{FM} = \mathbf{GQ}_{(1)} \tag{15}$$

where $\mathbf{Q}_{(i)}$ denotes the ith row of $\mathbf{Q}$. Introduce a state transform of the exosystem

$$\boldsymbol{\eta} = \mathbf{Mw} \tag{16}$$

we have

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= (\mathbf{FM} + \mathbf{GQ}_{(1)})\mathbf{w} \\ &= (\mathbf{F} + \mathbf{G}\boldsymbol{\psi}_1^T)\boldsymbol{\eta} \\ &:= \mathbf{F}_o\boldsymbol{\eta} \\ q_1 &= \boldsymbol{\psi}_1^T\boldsymbol{\eta} \end{aligned} \tag{17}$$

where $\boldsymbol{\psi}_1^T = \mathbf{Q}_{(1)}\mathbf{M}^{-1}$. In the new coordinate $\boldsymbol{\eta}$, we then express $\mathbf{q}$ as

$$\mathbf{q} = \mathbf{QM}^{-1}\boldsymbol{\eta} := [\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_s]^T\boldsymbol{\eta} \tag{18}$$

and

$$\mu = \mathbf{l}^T\mathbf{M}^{-1}\boldsymbol{\eta} := \boldsymbol{\psi}_u^T\boldsymbol{\eta} \tag{19}$$

Relating $\mathbf{q}$ and μ expressed in (18) and (19) to the dynamics shown in (12), we have

$$\boldsymbol{\psi}_i^T\mathbf{F}_o = \boldsymbol{\psi}_{i+1}^T + k_i\boldsymbol{\psi}_1^T, \text{ for } i = 1, \dots, \rho - 1 \tag{20}$$

and

$$\begin{aligned} \boldsymbol{\psi}_i^T\mathbf{F}_o &= \boldsymbol{\psi}_{i+1}^T + k_i\boldsymbol{\psi}_1^T + b_i\boldsymbol{\psi}_u^T, \text{ for } i = \rho, \dots, n-1 \\ \boldsymbol{\psi}_n^T\mathbf{F}_o &= k_n\boldsymbol{\psi}_1^T + b_n\boldsymbol{\psi}_u^T \end{aligned} \tag{21}$$

Define

$$\boldsymbol{\psi}_z^T := [\boldsymbol{\psi}_{\rho+1}, \ldots, \boldsymbol{\psi}_n]^T - \sum_{i=1}^{\rho} \mathbf{B}^{\rho-i}\bar{\mathbf{b}}\boldsymbol{\psi}_i^T \tag{22}$$

where $\mathbf{B}$ and $\bar{\mathbf{b}}$ are given by

$$\mathbf{B} = \begin{bmatrix} -b_{\rho+1}/b_\rho & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n-1}/b_\rho & 0 & \ldots & 1 \\ -b_n/b_\rho & 0 & \ldots & 0 \end{bmatrix}, \quad \bar{\mathbf{b}} = \begin{bmatrix} b_{\rho+1}/b_\rho \\ \vdots \\ b_n/b_\rho \end{bmatrix}$$

It can be shown from (21) that

$$\boldsymbol{\psi}_z^T \mathbf{F}_o = \mathbf{B}\boldsymbol{\psi}_z^T + \mathbf{k}_z \boldsymbol{\psi}_1^T \tag{23}$$

where

$$\mathbf{k}_z = [k_{\rho+1}, \ldots, k_n]^T - \sum_{i=1}^{\rho} \mathbf{B}^{\rho-i}\bar{\mathbf{b}}k_i + \mathbf{B}^\rho \bar{\mathbf{b}}$$

Using the notation $\otimes$ for the Kronecker product of matrices and $\text{vec}(\cdot)$ for the vector obtained by rolling out the column vectors of a matrix, and using the identity $\text{vec}(\mathbf{ABC}^T) = (\mathbf{A} \otimes \mathbf{C})\text{vec}(\mathbf{B})$ [17], from (23), we obtain

$$[\mathbf{F}_o^T \otimes \mathbf{I}_{(n-\rho)} - \mathbf{I}_s \otimes \mathbf{B}]\text{vec}(\boldsymbol{\psi}_z) = \text{vec}(\boldsymbol{\psi}_1 \mathbf{k}_z^T) \tag{24}$$

and

$$\text{vec}(\boldsymbol{\psi}_z) = \boldsymbol{\Sigma}^{-1}\text{vec}(\boldsymbol{\psi}_1 \mathbf{k}_z^T) \tag{25}$$

where

$$\boldsymbol{\Sigma} = \mathbf{F}_o^T \otimes \mathbf{I}_{(n-\rho)} - \mathbf{I}_s \otimes \mathbf{B} \tag{26}$$

Note that $\boldsymbol{\Sigma}$ is nonsingular, as the eigenvalues of $\mathbf{F}_o$ are the same as those of $\mathbf{S}$, which are exclusively different from the eigenvalues of $\mathbf{B}$.

4. Filter Design and Disturbance Estimation

Based on the analysis of the influence of disturbance in the state variables through $\mathbf{Q}$, we propose a few filters and estimation algorithms for state variables and the input disturbance. From the analysis in the previous section, it is clear that $\mathbf{q}$ and

μ can be estimated or evaluated if $\boldsymbol{\eta}$ and $\boldsymbol{\psi}_1$ are available. For the estimation of $\boldsymbol{\eta}$ and $\boldsymbol{\psi}_1$, we design the following filters and adaptive law

$$\dot{\boldsymbol{\xi}} = \mathbf{F}\boldsymbol{\xi} + \mathbf{G}(p_1 - y) \tag{27}$$

$$\dot{\boldsymbol{\zeta}} = \mathbf{F}\boldsymbol{\zeta} + \mathbf{G}\hat{\boldsymbol{\psi}}_1^T\boldsymbol{\xi} \tag{28}$$

$$\dot{\hat{\boldsymbol{\psi}}}_1 = \boldsymbol{\Gamma}\boldsymbol{\xi}(\boldsymbol{\xi} - \boldsymbol{\zeta})^T\mathbf{P}\mathbf{G} \tag{29}$$

where $\boldsymbol{\Gamma}$ is a positive definite matrix, and $\mathbf{P}$ is the positive definite matrix satisfying

$$\mathbf{P}\mathbf{F} + \mathbf{F}^T\mathbf{P} = -2\mathbf{I}_s \tag{30}$$

The following lemma describes the properties of the estimates.

Lemma 4.1 The estimates $\boldsymbol{\xi}$ and $\hat{\boldsymbol{\psi}}_1$ converge to $\boldsymbol{\eta}$ and $\boldsymbol{\psi}_1$ respectively. Furthermore the errors of the estimates are bounded by exponentially decaying functions, ie, there exist some positive real constants d_ξ, d_ψ, λ_ξ and λ_ψ such that

$$\|\boldsymbol{\eta}(t) - \boldsymbol{\xi}(t)\| < d_\xi e^{-\lambda_\xi t} \tag{31}$$

$$\|\boldsymbol{\psi}_1 - \hat{\boldsymbol{\psi}}_1(t)\| < d_\psi e^{-\lambda_\psi t} \tag{32}$$

Proof. Let us define $\mathbf{e}_\xi = \boldsymbol{\eta} - \boldsymbol{\xi}$. It can be obtained from (17) and (27) that

$$\dot{\mathbf{e}}_\xi = \mathbf{F}\mathbf{e}_\xi + \mathbf{G}\mathbf{C}\boldsymbol{\epsilon} \tag{33}$$

From (14), it can be seen that ϵ is exponentially decaying. In fact, to put $\mathbf{e}_\xi$ and $\boldsymbol{\epsilon}$ together, we have,

$$\begin{bmatrix} \dot{\mathbf{e}}_\xi \\ \dot{\boldsymbol{\epsilon}} \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{G}\mathbf{C} \\ 0 & (\mathbf{A}_c + \mathbf{k}\mathbf{C}) \end{bmatrix} \begin{bmatrix} \mathbf{e}_\xi \\ \boldsymbol{\epsilon} \end{bmatrix} \tag{34}$$

Since $\mathbf{F}$ and $(\mathbf{A}_c + \mathbf{k}\mathbf{C})$ are Hurwitz, the system (34) is exponentially stable, and therefor there exist positive reals d_1 and λ_1 such that (31) is satisfied.

To establish the convergence of $\hat{\boldsymbol{\psi}}_1$, we define

$$\mathbf{e} = \boldsymbol{\xi} - \boldsymbol{\zeta} \tag{35}$$

From (27) and (28), we obtain

$$\dot{\mathbf{e}} = \mathbf{F}\mathbf{e} + \mathbf{G}\boldsymbol{\psi}_1^T\mathbf{e}_\xi - \mathbf{G}\mathbf{C}\boldsymbol{\epsilon} + \mathbf{G}\boldsymbol{\xi}^T\tilde{\boldsymbol{\psi}}_1 \tag{36}$$

where $\tilde{\boldsymbol{\psi}}_1 = \boldsymbol{\psi} - \hat{\boldsymbol{\psi}}_1$. Define

$$\bar{\mathbf{e}} = \begin{bmatrix} \mathbf{e} \\ \mathbf{e}_\xi \\ \boldsymbol{\epsilon} \end{bmatrix} \tag{37}$$

Based on (14), (34) and (29), we can arrange the adaptive systems in the following format,

$$\begin{aligned} \dot{\bar{\mathbf{e}}} &= \bar{\mathbf{A}}\bar{\mathbf{e}} + \mathbf{\Omega}(t)^T\tilde{\boldsymbol{\psi}}_1 \\ \dot{\tilde{\boldsymbol{\psi}}}_1 &= -\mathbf{\Gamma}\mathbf{\Omega}(t)\bar{\mathbf{P}}\bar{\mathbf{e}} \end{aligned} \tag{38}$$

where $\tilde{\boldsymbol{\psi}}_1 = \boldsymbol{\psi}_1 - \hat{\boldsymbol{\psi}}_1$,

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{F} & \mathbf{G}\boldsymbol{\psi}_1^T & -\mathbf{GC} \\ \mathbf{0} & \mathbf{F} & \mathbf{GC} \\ \mathbf{0} & \mathbf{0} & (\mathbf{A}_c + \mathbf{kC}) \end{bmatrix},$$

$$\mathbf{\Omega}(t) = \begin{bmatrix} \boldsymbol{\xi}\mathbf{G}^T & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\bar{\mathbf{P}} = \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_1\mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma_2\mathbf{P}_\epsilon \end{bmatrix},$$

with γ_1 and γ_2 being positive reals and $\mathbf{P}_\epsilon$ being the positive definite matrix satisfying

$$\mathbf{P}_\epsilon(\mathbf{A}_c + \mathbf{kC}) + (\mathbf{A}_c + \mathbf{kC})^T\mathbf{P}_\epsilon = -2\mathbf{I} \tag{39}$$

If we let

$$\bar{\mathbf{P}}\bar{\mathbf{A}} + \bar{\mathbf{A}}\bar{\mathbf{P}} = -\bar{\mathbf{Q}} \tag{40}$$

A direct evaluation gives

$$\bar{\mathbf{Q}} = \begin{bmatrix} 2\mathbf{I}_s & -\mathbf{PG}\boldsymbol{\psi}_1^T & \mathbf{PGC} \\ -\boldsymbol{\psi}_1\mathbf{G}^T\mathbf{P} & 2\gamma_1\mathbf{I}_s & -\gamma_1\mathbf{PGC} \\ \mathbf{C}^T\mathbf{G}^T\mathbf{P} & -\gamma_1\mathbf{C}^T\mathbf{G}^T\mathbf{P} & 2\gamma_2\mathbf{I}_n \end{bmatrix} \tag{41}$$

From the structure of $\bar{\mathbf{Q}}$, we make $\bar{\mathbf{Q}}$ positive definite by choosing a sufficient large γ_1 and then a sufficient large γ_2.

Define

$$V = \bar{\mathbf{e}}^T\bar{\mathbf{P}}\bar{\mathbf{e}} + \tilde{\boldsymbol{\psi}}_1^T\mathbf{\Gamma}^{-1}\tilde{\boldsymbol{\psi}}_1 \tag{42}$$

Then from (38), we have

$$\dot{V} = -\bar{\mathbf{e}}^T\bar{\mathbf{Q}}\bar{\mathbf{e}} \tag{43}$$

Therefore $\bar{\mathbf{e}}$ and $\tilde{\boldsymbol{\psi}}_1$ are bounded and the invariant set theorem [16] ensures that $\lim_{t\to\infty}\bar{\mathbf{e}}(t) = \mathbf{0}$.

To establish the convergence of $\hat{\boldsymbol{\psi}}_1$, we need the consistent excitation condition of $\boldsymbol{\Omega}(t)$. From the definition of $\boldsymbol{\eta}$ in the previous section, it can be seen that $\boldsymbol{\eta}$ is persistently excited, ie, there exist two positive reals T and γ_3 such that

$$\int_t^{t+T} \boldsymbol{\eta}(\tau)\boldsymbol{\eta}(\tau)^T d\tau \geq \gamma_3 \mathbf{I}_s > \mathbf{0}, \forall t \geq 0 \tag{44}$$

With

$$\begin{aligned} & \int_t^{t+T} \boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}(\tau)^T d\tau \\ = & \|\mathbf{G}\|^2 \int_t^{t+T} \boldsymbol{\xi}(\tau)\boldsymbol{\xi}(\tau)^T d\tau \\ = & \|\mathbf{G}\|^2 \int_t^{t+T} (\boldsymbol{\eta}(\tau) - \mathbf{e}_\xi(\tau))(\boldsymbol{\eta}(\tau) - \mathbf{e}_\xi(\tau))^T d\tau \end{aligned} \tag{45}$$

and the fact that $\boldsymbol{\eta}$ is bounded and $\mathbf{e}_\xi$ converges to $\mathbf{0}$ exponentially, we can conclude that there exist a $t_o > 0$ and a γ_4 with $0 < \gamma_4 < \gamma_3\|\mathbf{G}\|^2$ such that

$$\int_t^{t+T} \boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}(\tau)^T d\tau \geq \gamma_4 \mathbf{I}_s > 0, \forall t \geq t_0 > 0 \tag{46}$$

Since $\bar{\mathbf{e}}(t_0)$ and $\tilde{\boldsymbol{\psi}}_1(t_0)$ are bounded, we apply Lemma B.2.3 [18] to obtain that $(\bar{\mathbf{e}}, \tilde{\boldsymbol{\psi}}_1) = \mathbf{0}$ is a globally exponentially stable equilibrium point for (38), which implies (32).

Remark 7. In the proof, we have argued that the positive real numbers γ_1 and γ_2 can be set to large enough values, for the convenience of establishing the positive definite matrix $\bar{\mathbf{Q}}$. It should be noted that even though they appear in $\bar{\mathbf{P}}$, they do not affect the choice of $\boldsymbol{\Gamma}$ in the adaptive law in (29).

With the estimates $\hat{\boldsymbol{\psi}}_1$ and $\boldsymbol{\xi}$ for $\boldsymbol{\psi}_1$ and $\boldsymbol{\eta}$ respectively, we propose the following algorithms for estimation of $\boldsymbol{\psi}_i$, $i = 2, \ldots, n$, and finally for $\mathbf{q}$ and μ. For $i = 2, \ldots, \rho$,

$$\hat{\boldsymbol{\psi}}_i^T = \hat{\boldsymbol{\psi}}_{i-1}^T(\mathbf{F} + \mathbf{G}\hat{\boldsymbol{\psi}}_1^T) + k_{i-1}\boldsymbol{\psi}_1^T, \tag{47}$$

and

$$\begin{bmatrix} \hat{\boldsymbol{\psi}}_{\rho+1}^T \\ \vdots \\ \hat{\boldsymbol{\psi}}_n^T \end{bmatrix} = \hat{\boldsymbol{\psi}}_z^T - \sum_{i=1}^{\rho} \mathbf{B}^{\rho-i}\bar{\mathbf{b}}\hat{\boldsymbol{\psi}}_i^T \tag{48}$$

where

$$\mathrm{vec}(\hat{\boldsymbol{\psi}}_z) = \frac{|\hat{\boldsymbol{\Sigma}}|}{\sigma + |\hat{\boldsymbol{\Sigma}}|^2}\mathrm{adj}(\hat{\boldsymbol{\Sigma}})\mathrm{vec}(\hat{\boldsymbol{\psi}}_1\mathbf{k}_z^T) \tag{49}$$

with

$$\hat{\mathbf{\Sigma}} = (\mathbf{F}+\mathbf{G}\hat{\boldsymbol{\psi}}_1^T)^T \otimes \mathbf{I}_{(n-\rho)} - \mathbf{I}_s \otimes \mathbf{B} \tag{50}$$

$$\dot{\sigma} = -\lambda_\sigma \sigma, \quad \sigma(0) = \sigma_0 \tag{51}$$

for some positive reals λ_σ and σ_0. For notations, we use $|\cdot|$ and adj$(\cdot)$ to denote the determinant and the adjoint matrix of a matrix respectively. The following theorem summarize the results of the disturbance and state estimation.

Theorem 4.2 Based on the filters (27), (28), (29) and estimates shown in (47) and (49), estimates of the state and the disturbance of (1) are given by

$$\hat{\mathbf{x}} = \mathbf{p} + \hat{\boldsymbol{\psi}}^T \boldsymbol{\xi} \tag{52}$$

$$\hat{\mu} = \hat{\boldsymbol{\psi}}_u^T \boldsymbol{\xi} \tag{53}$$

where

$$\hat{\boldsymbol{\psi}}_u^T = \frac{1}{b_\rho}[\hat{\boldsymbol{\psi}}_{\rho+1}^T - \hat{\boldsymbol{\psi}}_\rho^T(\mathbf{F}+\mathbf{G}\hat{\boldsymbol{\psi}}_1^T) - k_\rho \boldsymbol{\psi}_1^T] \tag{54}$$

and an estimate of exosystem matrix $\mathbf{F}+\mathbf{G}\boldsymbol{\psi}_1^T$ is given by

$$\hat{\mathbf{F}}_o = \mathbf{F}+\mathbf{G}\hat{\boldsymbol{\psi}}_1^T \tag{55}$$

There exist positive real constants λ_x, d_x, λ_μ, d_μ, λ_F, and d_F such that

$$\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\| \le d_x e^{-\lambda_x t} \tag{56}$$

$$\|\mu(t) - \hat{\mu}(t)\| \le d_\mu e^{-\lambda_\mu t} \tag{57}$$

$$\|\mathbf{F}_o - \hat{\mathbf{F}}_o(t)\| \le d_F e^{-\lambda_F t} \tag{58}$$

Proof. For the convenience of expression, we define that an estimate is an exponentially convergent estimate if the estimation error is bounded by a decaying exponential function. We need to establish that the estimates for the state and for the disturbance are exponentially convergent estimates. Let $\tilde{\mathbf{F}}_o = \mathbf{F}_o - \hat{\mathbf{F}}_o$. From (17) and (55), we have

$$\begin{aligned} \|\tilde{\mathbf{F}}_o\| &= \|\mathbf{G}\tilde{\boldsymbol{\psi}}_1^T\| \\ &\le \|\mathbf{G}\|\|\tilde{\boldsymbol{\psi}}_1\| \\ &\le \|\mathbf{G}\| d_2 e^{-\lambda_2 t} \end{aligned} \tag{59}$$

Hence, (58) is established. Let $\tilde{\boldsymbol{\psi}}_i = \boldsymbol{\psi}_i - \hat{\boldsymbol{\psi}}_i$. From (20) and (47), we have, for $i = 2, \ldots, \rho$,

$$\begin{aligned} \|\tilde{\boldsymbol{\psi}}_i\| &= \| -\tilde{\boldsymbol{\psi}}_{i-1}^T \mathbf{F}_o - \hat{\boldsymbol{\psi}}_{i-1}^T \tilde{\mathbf{F}}_o + k_{i-1}\tilde{\boldsymbol{\psi}}_1^T\| \\ &\leq \|\tilde{\boldsymbol{\psi}}_{i-1}\|\|\mathbf{F}_o\| + \|\hat{\boldsymbol{\psi}}_{i-1}\|\|\tilde{\mathbf{F}}_o\| + |k_{i-1}|\|\tilde{\boldsymbol{\psi}}_1\| \end{aligned} \tag{60}$$

Since $\hat{\boldsymbol{\psi}}_1$ and $\hat{\mathbf{F}}_o$ are exponentially convergent estimates, and $\hat{\boldsymbol{\psi}}_1$ is bounded, we conclude from (60) that $\hat{\boldsymbol{\psi}}_2$ is an exponentially convergent estimate. To use (60) recursively, we can conclude that $\hat{\boldsymbol{\psi}}_i$, for $i = 2, \ldots, \rho$ are exponentially convergent.

It can be shown that $|\hat{\boldsymbol{\Sigma}}|$ and $\mathrm{adj}(\hat{\boldsymbol{\Sigma}})\mathrm{vec}(\hat{\boldsymbol{\psi}}_1\mathbf{k}_z^T)$ are exponentially convergent estimates of $|\boldsymbol{\Sigma}|$ and $\mathrm{adj}(\boldsymbol{\Sigma})\mathrm{vec}(\boldsymbol{\psi}_1\mathbf{k}_z^T)$ respectively, as they are functions of the elements of $\hat{\boldsymbol{\psi}}_1$ obtained by multiplications and additions. From (25) and (49), we have

$$\begin{aligned} \mathrm{vec}(\boldsymbol{\psi}_z) - \mathrm{vec}(\hat{\boldsymbol{\psi}}_z) &= \frac{1}{|\boldsymbol{\Sigma}|}\mathrm{adj}(\boldsymbol{\Sigma})\mathrm{vec}(\boldsymbol{\psi}_1\mathbf{k}_z^T) - \frac{|\hat{\boldsymbol{\Sigma}}|}{\sigma + |\hat{\boldsymbol{\Sigma}}|^2}\mathrm{adj}(\hat{\boldsymbol{\Sigma}})\mathrm{vec}(\hat{\boldsymbol{\psi}}_1\mathbf{k}_z^T) \\ &= \frac{\sigma \mathrm{adj}(\boldsymbol{\Sigma})\mathrm{vec}(\boldsymbol{\psi}_1\mathbf{k}_z^T)}{|\boldsymbol{\Sigma}|(\sigma + |\hat{\boldsymbol{\Sigma}}|^2)} + \frac{(|\hat{\boldsymbol{\Sigma}}| - |\boldsymbol{\Sigma}|)\mathrm{adj}(\hat{\boldsymbol{\Sigma}})\mathrm{vec}(\hat{\boldsymbol{\psi}}_1\mathbf{k}_z^T)}{|\boldsymbol{\Sigma}|(\sigma + |\hat{\boldsymbol{\Sigma}}|^2)} \\ &\quad + \frac{|\hat{\boldsymbol{\Sigma}}|[(\mathrm{adj}(\boldsymbol{\Sigma})\mathrm{vec}(\boldsymbol{\psi}_1\mathbf{k}_z^T)\mathrm{adj}(\hat{\boldsymbol{\Sigma}})\mathrm{vec}(\hat{\boldsymbol{\psi}}_1\mathbf{k}_z^T)]}{|\boldsymbol{\Sigma}|(\sigma + |\hat{\boldsymbol{\Sigma}}|^2)} \end{aligned} \tag{61}$$

It can be shown that each of the three terms in (61) is bounded by a decaying exponential function, as σ is a decaying exponential function. Therefore we can conclude from (48) that $\hat{\boldsymbol{\psi}}_i$, $i = \rho + 1, \ldots, n$ are exponentially convergent estimates, and hence

$$\hat{\boldsymbol{\Psi}} := [\hat{\boldsymbol{\psi}}_1, \ldots, \hat{\boldsymbol{\psi}}_n] \tag{62}$$

is exponentially convergent. Finally from

$$\begin{aligned} \|\mathbf{x} - \hat{\mathbf{x}}\| &= \|\boldsymbol{\epsilon} - \boldsymbol{\Psi}^T\boldsymbol{\eta} + \hat{\boldsymbol{\Psi}}^T\boldsymbol{\xi}\| \\ &= \|\boldsymbol{\epsilon} - \boldsymbol{\Psi}^T(\boldsymbol{\eta} - \boldsymbol{\xi}) + (\hat{\boldsymbol{\Psi}}^T - \boldsymbol{\Psi}^T)\boldsymbol{\xi}\| \\ &\leq \|\boldsymbol{\epsilon}\| + \|\boldsymbol{\Psi}\|\|\boldsymbol{\eta} - \boldsymbol{\xi}\| + \|\hat{\boldsymbol{\Psi}} - \boldsymbol{\Psi}\|\|\boldsymbol{\xi}\| \end{aligned} \tag{63}$$

$$\begin{aligned} \|\mu - \hat{\mu}\| &= \|\boldsymbol{\psi}_u^T\boldsymbol{\eta} - \hat{\boldsymbol{\psi}}_u^T\boldsymbol{\xi}\| \\ &\leq \|\boldsymbol{\psi}_u\|\|\boldsymbol{\eta} - \boldsymbol{\xi}\| + \|\boldsymbol{\psi}_u - \hat{\boldsymbol{\psi}}_u\|\|\boldsymbol{\xi}\| \end{aligned} \tag{64}$$

we conclude that $\hat{\mathbf{x}}$ and $\hat{\mu}$ are exponentially convergent estimates of $\mathbf{x}$ and μ respecitvely.

Remark 8. If the system is minimum phase, ie., $\mathbf{B}$ is Hurwitz, an exponentially convergent estimate of disturbance can be obtained in an easier way as described in [8]. The operations in (47) and (48) are introduced because $\mathbf{B}$ may not be Hurwitz.

5. Stabilization with disturbance rejection of non-linear systems

We shall show that disturbance rejection can be achieved by combining the feedback control designed in the case when there is absence of disturbance and the estimate of the disturbance.

Theorem 5.1 If the system (1) satisfies Assumptions 1 to 3, then the control input defined as

$$u = h(\mathbf{v}, y) + \hat{\mu} \tag{65}$$

completely ejects the unknown disturbances and asymptotically stabilise the system.

Proof: Define $\tilde{\mu} = \mu - \hat{\mu}$. Since $\tilde{\mu}$ is exponentially convergent signal, we construct a first order system

$$\frac{d\bar{\mu}}{dt} = -\lambda_\mu \bar{\mu}, \bar{\mu}(0) = \bar{\mu}_0 \tag{66}$$

such that $\bar{\mu}(t) \geq \tilde{\mu}(t)$. Define a Lyapunov function candidate

$$W(\bar{\mathbf{x}}, \bar{\mu}) = V(\bar{\mathbf{x}}) + c_3|\bar{\mu}|^{c_4} \tag{67}$$

where c_3 and c_4 are positive real constants with $c_4 = \frac{c_2}{c_2-1}$, and $|\cdot|$ is used here to denote the absolute value of a scaler. It can be obtained that

$$\begin{aligned}
\frac{dW}{dt} &= \frac{\partial V(\bar{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{A}_c\mathbf{x} + \boldsymbol{\phi}(y) + \mathbf{b}(h(\mathbf{v}, y) + \hat{\mu} - \mu) \\
&\quad + \frac{\partial V(\bar{\mathbf{x}})}{\partial \mathbf{v}}\mathbf{f}(\mathbf{v}, y) - c_3 c_4 \lambda_\mu |\bar{\mu}|^{c_4} \\
&\leq -\alpha_3(\|\bar{\mathbf{x}}\|) + \|\frac{\partial V(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\|\|\mathbf{b}\||\tilde{\mu}| - c_3 c_4 \lambda_\mu |\bar{\mu}|^{c_4}
\end{aligned} \tag{68}$$

Applying Young's inequality to the second term in the right hand side of (68) gives

$$\|\frac{\partial V(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\|\|\mathbf{b}\||\tilde{\mu}| \leq \frac{c_5^{c_2}}{c_2}\|\frac{\partial V(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\|^{c_2} + \frac{1}{c_4 c_5^{c_4}}\|\mathbf{b}\|^{c_4}|\tilde{\mu}|^{c_4} \tag{69}$$

where c_5 is any positive real constant. We set $c_5 = (\frac{c_1 c_2}{2})^{1/c_2}$ and $c_3 = \frac{2\|b\|^{c_4}}{\lambda_\mu c_2^2 c_5^{c_4}}$, which results in

$$\|\frac{\partial V(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\|\|\mathbf{b}\||\tilde{\mu}| \leq \frac{c_1}{2}\|\frac{\partial V(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}}\|^{c_2} + \frac{1}{2}\lambda_\mu c_3 c_4 |\tilde{\mu}|^{c_4} \tag{70}$$

Substituting (70) into (68), we have

$$\frac{dW}{dt} \leq -\frac{1}{2}\alpha_3(\|\bar{\mathbf{x}}\|) - \frac{1}{2}c_3 c_4 \lambda_\mu |\bar{\mu}|^{c_4} \tag{71}$$

Therefore we conclude that the extended system with state $(\bar{\mathbf{x}}, \bar{\mu})$ is asymptotically stable, which implies that $\lim_{t\to\infty} \mathbf{x}(t) = 0$.

6. An example

Consider a nonlinear system in output feedback form

$$
\begin{aligned}
\dot{x}_1 &= x_2 - y^3 + (u - \mu) \\
\dot{x}_2 &= -(u - \mu) \\
y &= x_1
\end{aligned} \tag{72}
$$

where μ is a sinusoidal disturbance generated by

$$
\begin{aligned}
\dot{\mathbf{w}} &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \mathbf{w}, \quad \mathbf{w}(0) = \mathbf{w}_0 \\
\mu &= \mathbf{l}^T \mathbf{w}
\end{aligned} \tag{73}
$$

with ω, $\mathbf{l}$ and $\mathbf{w}_0$ unknown. It is easy to see that the system (72) are in the format of (1) with $\boldsymbol{\phi}(y) = [-y^3 \ 0]^T$ and $\mathbf{b} = [1 \ -1]^T$. The system is nonminimum phase with the nonminimum phase zero at $s = 1$.

The control design can be carried out for the example when there is no disturbances. In a similar way to the method presented in [9] for control design with one non-minimum-phase zero, we have the control design, with reference to (4) and (5), with $\mathbf{v} \in R^2$,

$$
\mathbf{f}(\mathbf{v}, y) = \begin{bmatrix} -k_{r1} & k_{r1} + 1 \\ -k_{r2} & k_{r2} \end{bmatrix} \mathbf{v} + \begin{bmatrix} y^3 \\ 0 \end{bmatrix} - \begin{bmatrix} k_{r1} \\ k_{r2} \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} h(\mathbf{v}, y) \tag{74}
$$

$$
\begin{aligned}
h(\mathbf{v}, y) &= (1 + 3y^2)^{-1}[-v_1 - (d_3 + d_4(1 + 9y^4))(v_2 + d_1 v_1 + d_2 v_1 + y^3) \\
&\quad -(d_1 + d_2)(-k_{r1} v_1 + (k_{r2} + 1) v_2 + y^3) + 3y^2(v_2 + y^3)]
\end{aligned} \tag{75}
$$

where d_i, $i = 1$ to 4, are positive real design parameters, k_{r1} and k_{r2} the design parameters such that $\begin{bmatrix} -k_{r1} & k_{r1} + 1 \\ -k_{r2} & k_{r2} \end{bmatrix}$ is Hurwitz. The feedback control based on the pair $\{\mathbf{f}, h\}$ introduced above renders the closed loop system exponentially stable for the disturbance-free case, and therefore Assumption 3 is satisfied.

The filters for disturbance estimation are designed as

$$
\dot{\mathbf{p}} = \begin{bmatrix} k_1 & 1 \\ k_2 & 0 \end{bmatrix} \mathbf{p} + \begin{bmatrix} -y^3 \\ 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} y \tag{76}
$$

$$
\dot{\boldsymbol{\xi}} = \begin{bmatrix} -f_1 & 1 \\ -f_2 & 0 \end{bmatrix} \boldsymbol{\xi} + \begin{bmatrix} 0 \\ g \end{bmatrix} (p_1 - y) \tag{77}
$$

$$
\dot{\boldsymbol{\zeta}} = \begin{bmatrix} -f_1 & 1 \\ -f_2 & 0 \end{bmatrix} \boldsymbol{\zeta} + \begin{bmatrix} 0 \\ g \end{bmatrix} \hat{\boldsymbol{\psi}}_1^T \boldsymbol{\xi} \tag{78}
$$

$$\dot{\hat{\boldsymbol{\psi}}}_1 = \boldsymbol{\Gamma}\boldsymbol{\xi}(\boldsymbol{\xi}-\boldsymbol{\zeta})^T \begin{bmatrix} \frac{1}{f_1}+\frac{f_2}{f_1} & -1 \\ -1 & \frac{1}{f_1}+\frac{f_1}{f_2}+\frac{1}{f_1 f_2} \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} \quad (79)$$

With $\hat{\boldsymbol{\psi}}_1$, $\hat{\boldsymbol{\psi}}_z$ is calculated by

$$\hat{\boldsymbol{\psi}}_z = (k_1+k_2-1)\frac{|\hat{\boldsymbol{\Sigma}}|}{|\hat{\boldsymbol{\Sigma}}|^2+\sigma}\mathrm{adj}(\hat{\boldsymbol{\Sigma}})\hat{\boldsymbol{\psi}}_1 \quad (80)$$

where

$$\hat{\boldsymbol{\Sigma}} = \mathbf{F}^T + [0\ g]^T\hat{\boldsymbol{\psi}}_1 - \mathbf{I} \quad (81)$$

Finally, $\hat{\boldsymbol{\psi}}_u$ is given by

$$\hat{\boldsymbol{\psi}}_u = -\hat{\boldsymbol{\psi}}_2 - k_1\hat{\boldsymbol{\psi}}_1 + \mathbf{F}^T\hat{\boldsymbol{\psi}}_1 + \hat{\boldsymbol{\psi}}_1^T\mathbf{G}\hat{\boldsymbol{\psi}}_1 \quad (82)$$

with $\hat{\boldsymbol{\psi}}_2 = \hat{\boldsymbol{\psi}}_z - \hat{\boldsymbol{\psi}}_1$. The final control design is given by

$$u = h(\mathbf{v}, y) + \hat{\boldsymbol{\psi}}_u^T\boldsymbol{\xi} \quad (83)$$

The simulation study has been carried out for the estimation and control design shown in this example. The simulation results shown below are for the settings $k_1 = -3$, $k_2 = -2$, $f_1 = 3$, $f_2 = 2$, $g = 1$, $\boldsymbol{\Gamma} = 1000\mathbf{I}$, $k_{r1} = 5$, $k_{r2} = 2$, $d_1 = d_2 = d_3 = d_4 = 1$. The settings for the disturbance are $\omega = 1$, $\mathbf{w}_0 = [0, 1]^T$, i.e., the disturbance is set as $\mu(t) = \sin t$. The estimate for $\boldsymbol{\psi}_1$ is shown in Figure 1, where $\hat{\boldsymbol{\psi}}_1$ converges to $[-8, 3]^T$, the correct value for $\boldsymbol{\psi}_1$. In fact, it is easy to check that the eigenvalues of $(\mathbf{F} + \mathbf{G}[-8, 3]^T)$ are $\pm 1j$. The estimate of the disturbance is shown in Figure 2, with a clear convergence to the disturbance. An enlargement of a section of Figure 2 is shown in Figure 3. The control input and the system output are shown in Figure 4, in which the output converges to zero with the input to asymptotically cancel the disturbance. Comparing the results with the results of a system with minimum phase shown in [8], we find the settling time of the nonminimum-phase system is much longer. This is due to the delay in the disturbance and state estimation caused by the nonminimum phase.

7. Conclusions

We have presented separate estimation and disturbance rejection algorithms for rejecting unknown sinusoidal disturbances and stabilization of nonlinear systems in the output feedback form. The major difference between the proposed methods and the methods in the literature for nonlinear systems is that our algorithms work for the nonminimum phase nonlinear systems, and the others do not. The nonminimum phase makes the estimation and control design much more difficult. One can compare the results shown in this paper and the one shown in [8] where the disturbance rejection is achieved for the same system, but with minimum phase. The nonminimum phase causes the involvement of vector and matrix manipulation for the estimation of disturbances, and the re-estimation of the states for the control design. The proposed algorithms work for both the minimum phase and nonminimum phase systems.

Despite the difficulty of nonminimum phase, the proposed algorithms achieve exponentially convergent estimates of the disturbance and its characteristic matrix, from which the disturbance frequencies can be calculated. Conditions are specified for the control design of the disturbance-free system such that the estimate of disturbance can be added to the control input directly to ensure the asymptotic rejection or complete compensation of unknown sinusoidal disturbances.

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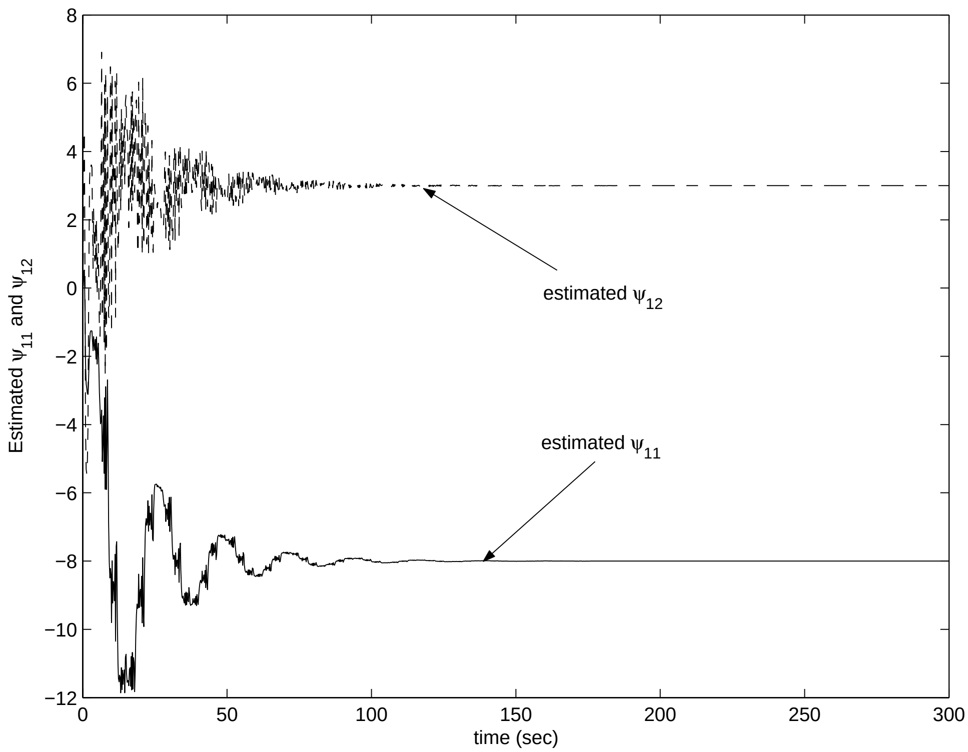

Figure 1: Estimate of ψ_1

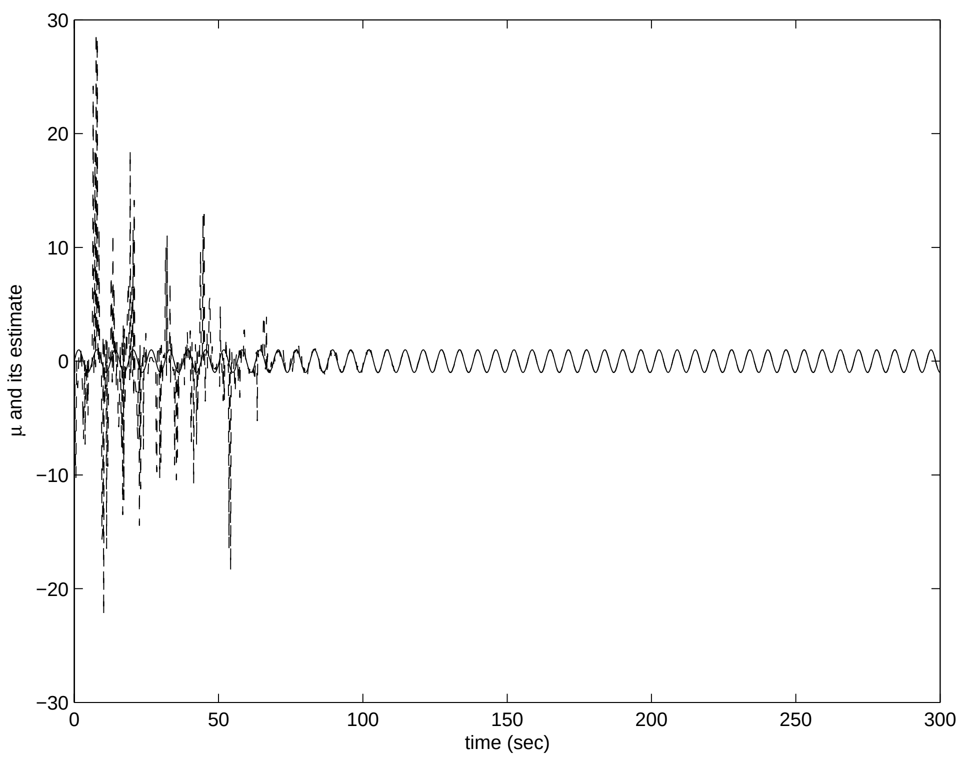

Figure 2: Estimate of the disturbance

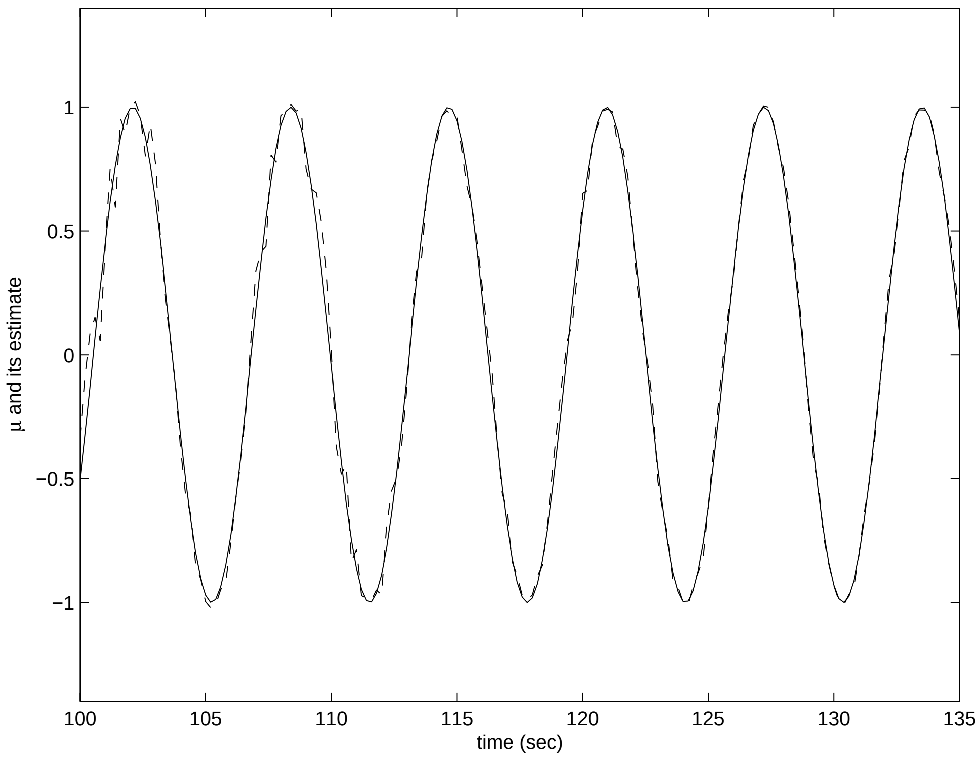

Figure 3: An enlargement of a part of Figure 2

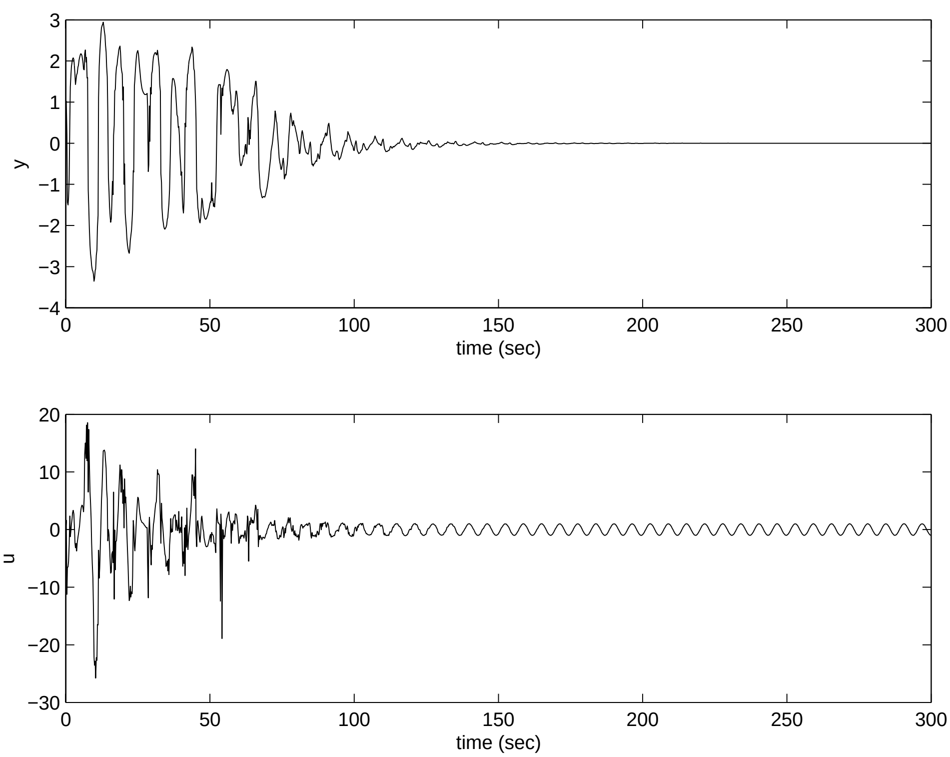

Figure 4: Control input and system output