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Asymptotic Rejection of Asymmetric Periodic Disturbances in Output-feedback Nonlinear Systems ⋆

Zhengtao Ding

Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, PO Box 88, Manchester M60 1QD, United Kingdom

Abstract

This paper deals with asymptotic rejection of periodic disturbances which may have asymmetric basic wave patterns. This class of disturbances covers asymmetric wave forms in the half period such as alternating sawtooth wave form, some disturbances which are generated from nonlinear oscillation such as Van de Pol oscillators, as well as disturbances with symmetric half-period wave forms such as sinusoidal disturbances and triangular disturbances etc. The systems considered in this paper can be transformed to the nonlinear output feedback form. The amplitude and phase of the disturbances are unknown. The novel concept of integral phase shift is introduced together with the newly introduced half-period integration operator to investigate the invariant properties of asymmetric periodic disturbances. They are used for the estimation of unknown disturbances in the systems, together with observer design techniques to deal with nonlinearity. The proposed control design with the disturbance estimation asymptotically rejects the unknown disturbance, and ensures the overall stability of the system.

Key words: Disturbance rejection, Nonlinear systems, Periodic disturbances, Disturbance estimation

1 Introduction

One of the common deterministic disturbances considered for asymptotic rejection in dynamic systems is sinusoidal disturbance (Bodson, Sacks and Khosla, 1994; Bodson and Douglas, 1997; Marino, Santosuosso and Tomei, 2003; Ding, 2003), and very often the internal model principle is used to generate the desired feedforward control input to reject the unknown disturbances. A related problem is formulated as output regulation, where the output measurement contains the disturbance (Isidori and Byrnes, 1990; Huang and Rugh, 1990; Isidori, 1995; Pavlov, van de Wouw and Nijmeijer, 2004; Chen and Huang, 2005*a*). For sinusoidal disturbances, if the disturbance frequencies are available, the disturbance can be easily modelled as an output of a known linear dynamic model which is often referred to as the exosystem, and therefore a corresponding internal model can be designed (Isidori and Byrnes, 1990; Huang and Rugh, 1990; Isidori, 1995; Serrani and Isidori, 2000; Ding, 2001). If the frequencies are unknown, adaptive internal model can be used for disturbance rejection and output regulation (Serrani, Isidori and Marconi, 2001; Ding, 2003).

Many periodic signals are not sinusoidal, and therefore can not be modeled as an output of a linear exosystem. If we really force ourself to find a model for its generation, then a model can often be infinite dimensional, or nonlinear finite dimensional for a limited class nonlinear disturbances. Limited results are available on output regulation with nonlinear exosystems (Priscoli, 2004; Chen and Huang, 2005*b*; Ding, 2006*b*), of which the periodic solutions can be viewed as smooth disturbances. Recently, a half-period integration method is proposed to characterize general periodic disturbances, and applied for asymptotic rejection of a class of general disturbances which have symmetric wave form in the half of the period, such as symmetric triangular waves and square waves. The half-period integration based disturbance rejection is demonstrated in a class of nonlinear output feedback systems which can be transformed to the output feedback form (Ding, 2006*a*).

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Email address: `zhengtao.ding@manchester.ac.uk` (Zhengtao Ding).

In this paper, we deal with asymptotic rejection of more general disturbances than the disturbances with odd-function and symmetric wave form. In particular, we consider a class of periodic disturbances whose basic half-period wave forms may be asymmetric, and the second half-period form follows the first half-period with opposite sign. This class of periodic disturbances includes non-smooth disturbances such as alternating sawtooth wave forms, some disturbances which are generated from nonlinear oscillation such as Van de Pol oscillators, as well as disturbances with symmetric half-period wave forms such as sinusoidal disturbances and triangular disturbances etc. A new concept, integral phase shift, is introduced to tackle asymmetric wave patterns. The integral phase shift reflects the phase change of the basic wave pattern after half-period integration, and we introduce a new delay operator with the delay that depends on the integral phase shift. The half-period integration operator together with the integral phase shift is used to establish the invariant properties of the asymmetric periodic disturbances. A set of results for the class of disturbances are obtained and they are applied in control design for asymptotic disturbance rejection in nonlinear output feedback systems. With the information of the basic wave form, the phase and amplitude can be estimated by the proposed design. With the estimated disturbance, control design is then proposed for disturbance rejection with stability. The nice property of the estimate ensures the asymptotic rejection of general periodical disturbances under the proposed control for nonlinear systems in the output feedback form. A simpler control algorithm is proposed for linear systems. An example is included to demonstrate the proposed estimation and control algorithm for rejection of an alternating sawtooth disturbance.

2 Problem Formulation

Consider a single-input-single-output nonlinear system which can be transformed into the output feedback form

$$\begin{aligned} \dot{x} &= A_c x + \psi(y) + b(u - w) \\ y &= Cx \end{aligned} \tag{1}$$

with

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

where $x \in R^n$ is the state vector, $u \in R$ is the control, ψ, is a known nonlinear smooth vector field in R^n with $\psi(0) = 0$, $w \in R$ is a periodical disturbance.

Assumption 1. The disturbance can be expressed as

$$w(t) = a w_b(t + \phi) \tag{2}$$

where the unknown constants a and ϕ are referred to as amplitude and phase, and $w_b(t)$ is a known function satisfying the following

A1.1 $w_b(t + T) = w_b(t)$ with T, the known period.
A1.2 $w_b(t + \frac{T}{2}) = -w_b(t)$.
A1.3 There exists a δ, $0 < \delta < \frac{T}{4}$, such that for $t \in (0, \delta)$, $w_b(t) > K_b t^l$, and for $t \in (\frac{T}{2} - \delta, \frac{T}{2})$, $w_b(t) > K_b(\frac{T}{2} - t)^l$ and with K_b and l are positive reals, and $w_b(t) \geq K_b \delta^l$ for $t \in [\delta, \frac{T}{2} - \delta]$.
A1.4 For $t \in [0, T)$, the function $w_b(t)$ is bounded, and has bounded derivatives except at a finite number of discontinuous points, where the left and right derivatives exist and are bounded.

From A1.1 and A1.2, we have $w_b(\frac{T}{2}) = w_b(\frac{T}{2} - T) = w_b(-\frac{T}{2}) = -w_b(\frac{T}{2})$. Hence we can conclude $w_b(\frac{T}{2}) = 0$.

Remark 1. Assumption 1 specifies the class of disturbances considered in this paper, and it is different from the assumption 1 in (Ding, 2006*a*) which requires the wave pattern to be an odd function and the pattern for half of the period to be symmetric. A number of limit cycles produced from nonlinear systems satisfy Assumption 1, for example, the output of a Van de Pol oscillator, but not the assumption 1 in (Ding, 2006*a*). Furthermore, Assumption 1 allows the disturbances to have finite number of discontinuous points in one period, provided that the left and right derivatives exist at the discontinuous points. The class of disturbances considered in here includes discontinuous periodic disturbances such as alternating sawtooth waves that cannot be dealt by any existing methods in literature.

The problem considered in this paper is to design a dynamic feedback control law u so that the overall system is stable and the unknown disturbance $w(t)$ is asymptotically rejected in the sense that $\lim_{t\to\infty} y(t) = 0$. The disturbance is first estimated and then the estimated disturbance is used for control design for disturbance rejection.

Assumption 2. The system is minimum phase, ie, the zeros of polynomial $\mathbf{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ have negative real parts.

3 Integral Phase Shift and Half-Period Integration

Since the basic disturbance pattern is described by the function $w_b(t)$, the disturbance can be reproduced if the amplitude a and phase ϕ can be estimated. In this section, the periodic property and wave pattern properties described in Assumption 1 will be exploited to design estimation algorithms for a and ϕ.

Define the half-period integration operator $\mathcal{I}$ and the delay operator $\mathcal{D}(d)$ as

$$\begin{aligned} \mathcal{I} \circ f(t) &:= \mathcal{I}(f(t)) = \int_{t-\frac{T}{2}}^{t} f(s)ds \\ \mathcal{D}(d) \circ f(t) &:= \mathcal{D}(d, f(t)) = f(t-d) \end{aligned} \tag{3}$$

where $0 \le d < T$. For the convenience of notations, we often write $\mathcal{I} \circ f$ and $\mathcal{D} \circ f$ as $\mathcal{I}f$ and $\mathcal{D}f$ when no confusions are caused. It is easy to see the following properties of the introduced operators such as

$$\mathcal{I}\frac{df(t)}{dt} = f(t) - f(t - \frac{T}{2}) \tag{4}$$

for a C^1 function f, and

$$\mathcal{D}(d_1) \circ \mathcal{D}(d_2) \circ w(t) = \mathcal{D}(d_1 + d_2) \circ w(t) \tag{5}$$

$$\mathcal{D}(d) \circ w(t) = \mathcal{D}(\bar{d}) \circ w(t) \tag{6}$$

with $\bar{d} = \mod (d, T)$, for a periodic function $w(t)$ with period T. The operations of $\mathcal{D}$ and $\mathcal{I}$ can be swapped in sequence, ie, $\mathcal{D} \circ \mathcal{I} \circ f = \mathcal{I} \circ \mathcal{D} \circ f$. Consider the half-period integration, we notice that there is a phase shift of the resultant function, similar as the phase shift after integration of sinusoidal functions. For this property, we have the following lemma.

Lemma 3.1 If a function $f(t)$ satisfies the conditions specified in Assumption 1, then there exists a unique positive real constant $d \in (\frac{T}{2}, T)$ such that $(\mathcal{I} \circ f)(T - d) = 0$.

Proof. Let $d_c = T - d$, then we need to show $(\mathcal{I} \circ f)(d_c) = 0$ for $d_c \in (0, \frac{T}{2})$. From Assumption 1, we have $f(t) > 0$ $\forall t \in (0, \frac{T}{2})$, and $f(t) < 0$ $\forall t \in (\frac{T}{2}, T)$, and hence we have $(\mathcal{I} \circ f)(0) < 0$ and $(\mathcal{I} \circ f)(\frac{T}{2}) > 0$. Then from the continuity of $(\mathcal{I} \circ f)(t)$, as guaranteed by A1.4, there exists a $d_c \in (0, \frac{T}{2})$ such that $(\mathcal{I} \circ f)(d_c) = 0$. For any other $d_c' \in (0, \frac{T}{2})$, and $d_c' \neq d_c$, it can be obtained that

$$\begin{aligned} &(\mathcal{I} \circ f)(d_c') - (\mathcal{I} \circ f)(d_c) \\ &= \int_{d_c'-\frac{T}{2}}^{d_c'} f(s)ds - \int_{d_c-\frac{T}{2}}^{d_c} f(s)ds \\ &= \int_{d_c'-\frac{T}{2}}^{d_c-\frac{T}{2}} f(s)ds + \int_{d_c'}^{d_c} -f(s)ds \end{aligned} \tag{7}$$

Both terms in the above equation are negative when $d_c' < d_c$ and positive when $d_c' > d_c$. Therefore we conclude d is unique.

We define the constant d described in Lemma 3.1 as the integral phase shift. For sinusoidal functions, this integral phase shift is $\frac{3}{4}T$. An important property is described in the following theorem.

Theorem 3.2 If a function $f(t)$ satisfies the conditions specified in Assumption 1, so does the function $g(t)$ defined by $\mathcal{D}(d) \circ \mathcal{I} \circ f(t)$, where d is the integral phase shift.

Proof. For A1.1, it can be obtained that

$$\begin{aligned} g(t+T) &= \int_{t+T-\frac{T}{2}-d}^{t+T-d} f(s)ds = \int_{t-\frac{T}{2}-d}^{t-d} f(s+T)ds \\ &= \int_{t-\frac{T}{2}-d}^{t-d} f(s)ds = g(t) \end{aligned} \tag{8}$$

For A1.2, it follows that

$$\begin{aligned} g(t+\frac{T}{2}) &= \int_{t-d}^{t+\frac{T}{2}-d} f(s)ds = \int_{t-\frac{T}{2}-d}^{t-d} f(s+\frac{T}{2})ds \\ &= -\int_{t-\frac{T}{2}-d}^{t-d} f(s)ds = -g(t) \end{aligned} \tag{9}$$

To complete the proof, we need to establish the condition specified in A1.3. For $t \in (0, \frac{T}{2} - d_c)$, we have

$$\begin{aligned} g(t) &= \int_{t-\frac{T}{2}-d}^{t-d} f(s)ds = \int_{t+d_c-\frac{T}{2}}^{t+d_c} f(s)ds \\ &= \int_{t+d_c-\frac{T}{2}}^{0} f(s)ds + \int_{0}^{d_c} f(s)ds + \int_{d_c}^{t+d_c} f(s)ds \end{aligned} \tag{10}$$

Note that

$$\begin{aligned} \int_{0}^{d_c} f(s)ds &= -\int_{d_c-\frac{T}{2}}^{0} f(s)ds \\ &= -\int_{d_c-\frac{T}{2}}^{t+d_c-\frac{T}{2}} f(s)ds - \int_{t+d_c-\frac{T}{2}}^{0} f(s)ds \end{aligned} \tag{11}$$

Substituting (11) into (10), we have

$$g(t) = 2 \int_{d_c}^{d_c+t} f(s)ds \tag{12}$$

Similarly, for $t \in (\frac{T}{2} - d_c, \frac{T}{2})$, we have

$$g(t) = 2 \int_{t+d_c-\frac{T}{2}}^{d_c} f(s)ds \tag{13}$$

Since the condition in Assumption A1.3 holds for any positive real $\delta' \in (0, \delta)$, we set $0 < \delta' < \min(d_c, \frac{T}{2} - d_c, \delta)$. Now, we choose another positive $\delta_1 \in (0, \min(d_c - \delta', \frac{T}{2} - (d_c + \delta')))$.

For $t \in (0, \delta_1)$, we have $(d_c, d_c + t) \subset (\delta', \frac{T}{2} - \delta')$, and therefore, from (12), we have

$$g(t) \geq 2K_b\delta'^{l}t := K_{b,1}t^{l_1} \tag{14}$$

with $K_{b,1} = 2K_b\delta'^{l}$, and $l_1 = 1$.

For $t \in (\frac{T}{2} - \delta_1, \frac{T}{2})$, we have $(d_c + t - \frac{T}{2}, d_c) \subset (\delta', \frac{T}{2} - \delta')$, and therefore, from (13), we have

$$g(t) \geq 2K_b\delta'^{l}(\frac{T}{2} - t) := K_{b,1}(\frac{T}{2} - t)^{l_1} \tag{15}$$

Furthermore, we need to consider $g(t)$ for $t \in (\delta_1, \frac{T}{2} - \delta_1)$. Consider $t \in (\delta_1, \frac{T}{2} - \delta_1) \cap (0, \frac{T}{2} - d_c)$. We have, $f(t) > 0$ for $t \in (0, \frac{T}{2} - d_c)$, and hence from (12), it can be concluded that

$$g(t) > \int_{d_c}^{d_c+\delta_1} f(s)ds = g(\delta_1) \geq K_{b,1}\delta_1^{l_1} \tag{16}$$

Similarly, for $t \in (\delta_1, \frac{T}{2} - \delta_1) \cap (\frac{T}{2} - d_c, \frac{T}{2})$, we have $f(t) > 0$. From (13), it can be concluded that

$$g(t) > \int_{(\frac{T}{2}-\delta_1)+d_c-\frac{T}{2}}^{d_c} f(s)ds = g(\frac{T}{2} - \delta_1) \geq K_{b,1}\delta_1^{l_1} \tag{17}$$

Therefore we have shown that A1.3 holds for $g(t)$ with δ_1, $K_{b,1}$ and l_1.

Finally, the delay operation and the half-period integration make no alteration to the properties specified in A1.4. This completes the proof.

Remark 2. When the basic wave pattern has a half-period symmetry, as shown in (Ding, 2006*a*), the integral phase shift is $\frac{3}{4}T$, the same as the integral phase shift of sinusoidal functions. The result shown in Theorem 3.2 deals with basic wave patterns which do not have half-period symmetry and are not even odd functions. For these general wave patterns, the integral phase shifts are dependent on the basic wave patterns, and the invariant property shown in Theorem 3.2 depends on the integral phase shifts. The half-period integration changes the phase, while for the half-period integration with the phase shift adjustment, the phase is kept the same. Therefore the half-period integration with phase shift adjustment plays the same role as the half-period integration with delay of $\frac{3}{4}T$ in (Ding, 2006*a*) for disturbance estimation.

With the key result described in Theorem 3.2, we can proceed with disturbance estimation and control design for asymptotic rejection of disturbances with asymmetric wave forms. Since the conditions specified in Assumption 1 are invariant under the half period integration with integral phase shift $\mathcal{D}(d) \circ \mathcal{I}$, we can introduce this operation repeatedly. Note that the integral phase shift is different at each iteration in general. With the half-period integration with phase shift adjustment, we define

$$w_{b,i}(t) := \mathcal{D}(d_{i-1}) \circ \mathcal{I} \circ w_{b,i-1}(t) \text{ for } i = 1, \ldots, m \tag{18}$$

where d_{i-1} is the integral phase shift for $w_{b,i-1}$ and $w_{b,0} := w_b$.

Consider a disturbance passing through a linear dynamic system described by the following differential equation

$$\frac{d^m y}{dt^m} + \beta_1 \frac{d^{m-1} y}{dt^{m-1}} + \ldots + \beta_m y = w(t) \tag{19}$$

where β_i, for $i = 1, \ldots, m$, are constants such that $s^m + \beta_1 s^{m-1} + \ldots + \beta_m$ is Hurwitz. The system (19) is stable with w as the input and y as the output, and there exists a periodic solution after the transient stage to the periodic input w. To simplify the notation, we use y to denote the periodic steady state output of (19). If the disturbance $w(t)$ satisfies Assumption 1, then the phase and gain can be calculated as shown in the following lemma.

Lemma 3.3 If y is the steady state output in (19) with the input $w(t)$ that satisfies Assumption 1, then the phase and gain can be calculated directly from $y(t)$ by

$$a = \frac{\mathcal{I} \circ |\bar{y}(t)|}{\mathcal{I} \circ |w_{b,m}(t)|} \tag{20}$$

$$\phi = \phi_1 - \phi_2 \tag{21}$$

where

$$\bar{y}(t) = \mathcal{D}(\bar{d}_m)\sum_{i=0}^{m}\beta_i\mathcal{I}^i(1-\mathcal{D}(\frac{T}{2}))^{m-i}y(t) \quad (22)$$

$$\phi_1(t) = \frac{1}{2}(\mathcal{I}\circ \mathrm{sign}(\bar{y}(t)) + \frac{T}{2})\mathrm{sign}(\bar{y}(t)) \quad (23)$$

$$\phi_2(t) = \frac{1}{2}(\mathcal{I}\circ \mathrm{sign}(w_b(t)) + \frac{T}{2})\mathrm{sign}(w_b(t)) \quad (24)$$

with $\beta_0 = 1$ and $\bar{d}_m = \mathrm{mod}(\sum_{i=0}^{m-1} d_i, T)$.

Proof. Observing that $\mathcal{I}\circ\frac{dy}{dt} = (1-\mathcal{D}(\frac{T}{2}))\circ y(t)$, applying the operation $\mathcal{D}(\bar{d}_m)\mathcal{I}^m$ to both sides of (19) gives

$$\begin{aligned}&\mathcal{D}(\bar{d}_m)\sum_{i=0}^{m}\beta_i\mathcal{I}^i(1-\mathcal{D}(\frac{T}{2}))^{m-i}y(t)\\ &= [\prod_{i=1}^{m}(\mathcal{D}(d_{i-1})\mathcal{I})]w(t)\end{aligned} \quad (25)$$

Therefore, we have, from the definition of $w_{b,i}(t)$,

$$\begin{aligned}\bar{y}(t) &= [\prod_{i=1}^{m}(\mathcal{D}(d_{i-1})\mathcal{I})]aw_b(t+\phi)\\ &= a[\prod_{i=j}^{m}(\mathcal{D}(d_{i-1})\mathcal{I})]w_{b,j-1}(t+\phi)\\ &= aw_{b,m}(t+\phi)\end{aligned} \quad (26)$$

where $1\le j\le m$. Then the result shown in (20) follows by integrating the absolute value of the above equation over half a period. For the primary period where the operant in $[0,T)$, a direct evaluation gives

$$\phi_1-\phi_2 = \begin{cases}\phi & \text{if } 0<t<\frac{T}{2}-\phi\\ \phi-T & \text{if } \frac{T}{2}-\phi<t<\frac{T}{2}\\ \phi & \text{if } \frac{T}{2}<t<T\end{cases} \quad (27)$$

Note that for the phase calculation, $\phi - T$ is equivalent to ϕ, and hence, the proof is completed.

4 Disturbance Estimation

Similar to observer design, we have the following filter:

$$\dot{p} = (A_c + kC)p + \phi(y) + bu - ky \quad (28)$$

where $p\in R^n$, $k\in R^n$ is chosen so that

$$\begin{aligned}K(s) &:= s^n - \sum_{i=1}^{n}k_is^{n-i}\\ &= B(s)(s^\rho + \lambda_1 s^{\rho-1}+\ldots+\lambda_\rho)/b_\rho\end{aligned} \quad (29)$$

with λ_i being positive real design parameters such that $(s^\rho + \lambda_1 s^{\rho-1}+\ldots+\lambda_\rho)$ is Hurwitz. An estimate of w is given by

$$\hat{w}(t) = \hat{a}w_b(\hat{\phi}_1) \quad (30)$$

where

$$\hat{a} = \frac{\mathcal{I}\circ|\bar{w}(t)|}{\mathcal{I}\circ|w_{b,\rho}(t)|} \quad (31)$$

$$\hat{\phi}_1(t) = \frac{1}{2}(\mathcal{I}\circ \mathrm{sign}(\bar{w}(t)) + \frac{T}{2})\mathrm{sign}(\bar{w}(t)) \quad (32)$$

with

$$\bar{w}(t) = \mathcal{Q}\circ(p_1 - y) \quad (33)$$

$$\mathcal{Q} = \mathcal{D}(\bar{d}_\rho)\sum_{i=0}^{\rho}\lambda_i\mathcal{I}^i(1-\mathcal{D}(\frac{T}{2}))^{\rho-i} \quad (34)$$

and p_1 being the first element of p, and $\bar{d}_\rho = \mathrm{mod}(\sum_{i=0}^{\rho-1} d_i, T)$

Theorem 4.1 If the disturbance in (1) satisfies the conditions specified in Assumptions 1, 2 and 3, then the estimate given in (30) converges to the actual disturbance in L_P, ie., $w-\hat{w}\in L_P$ for $P = 1, 2$ and ∞.

Proof. Since w and $\hat{w}$ are bounded signals, and $w-\hat{w}\in L_\infty$, we only need to show the case for L_1, as $w-\hat{w}\in L_1\cap L_\infty$ implies $w-\hat{w}\in L_2$. Consider a dummy filter

$$\dot{q} = (A_c + kC)q + bw \quad (35)$$

where $q\in R^n$ denotes the steady state only. Let $e = x-(p-q)$. It can be shown that

$$\|e(t)\|\le K_e e^{-\lambda_e t} \quad (36)$$

for some positive real constants K_e and λ_e. From the special structure of k chosen in (29), it can be shown that

$$\frac{d^\rho q_1}{dt^\rho} + \lambda_1\frac{d^{\rho-1}q_1}{dt^{\rho-1}} + \ldots + \lambda_\rho q_1 = w(t) \quad (37)$$

where q_1 denotes the first element of q. Based on Lemma 3.3, if we define

$$\bar{q}_1(t) = \mathcal{Q}\circ q_1(t) \quad (38)$$

$$\phi_1(t) = \frac{1}{2}(\mathcal{I}\circ \mathrm{sign}(\bar{q}_1(t)) + \frac{T}{2})\mathrm{sign}(\bar{q}_1(t)) \quad (39)$$

then we have $\phi_1 = t+\phi$ or $\phi_1 = t+\phi-T$, and

$$a = \frac{\mathcal{I}\circ|\bar{q}_1(t)|}{\mathcal{I}\circ|w_{b,\rho}(t)|} \quad (40)$$

$$w(t) = aw_b(\phi_1) \quad (41)$$

Let $\tilde{w} = w - \hat{w}$, and it can be expressed as

$$\tilde{w} = \tilde{a} w_b(\phi_1) + \hat{a}(w_b(\phi_1) - w_b(\hat{\phi}_1)) \quad (42)$$

where $\tilde{a} = a - \hat{a}$. The boundedness of $\hat{a}$ follows directly from (31) with the boundedness of $\bar{w}(t)$ which is in turn implied by the boundedness of $p_1 - y = q_1 - e_1$.

In order to show $\tilde{a} \in L_P$, the property of $e_w := \bar{q}_1 - \bar{w}$ needs to be investigated. From the definitions, it can be obtained that

$$e_w = \mathcal{Q} \circ q_1 - \mathcal{Q} \circ (p_1 - y) = \mathcal{Q} \circ e_1 \quad (43)$$

since $\mathcal{Q}$ is a linear combination of linear operators. From (36), and the above equation, it can be obtained that

$$e_w(t) \leq K_w e^{-\lambda_e t} \quad (44)$$

for a positive constant K_w. From (31) and (40) we have

$$|\tilde{a}| = \frac{|\mathcal{I} \circ (|\bar{q}_1(t)| - |\bar{w}(t)|)|}{\mathcal{I} \circ |w_{b,\rho}(t)|} \leq \frac{\mathcal{I} \circ |e_w|}{\mathcal{I} \circ |w_{b,\rho}(t)|} \quad (45)$$

Therefore from (44), it can be obtained that $\tilde{a} \in L_P$.

To establish $e_q := \text{sign}(\bar{q}(t)) - \text{sign}(\bar{w}(t)) \in L_P$, with the boundedness of e_q, we only need to show that

$$J_\infty := \sum_{i=0}^{\infty} J_i < \infty \quad (46)$$

where

$$J_i = \int_{(i-1)T}^{iT} |\text{sign}(\bar{q}_1(t)) - \text{sign}(\bar{q}_1(t) - e_w(t))| dt \quad (47)$$

Since $|e_w(t)|$ is bounded by an exponentially decaying function, there exists an $\bar{i}$ such that for $t > \bar{i}T$, $|e_w(t)| < K_b \delta^l$. Therefore, for $i > \bar{i}$, it can be shown that

$$J_i < 4\left(\frac{K_w}{K_{b,\rho}}\right)^{1/l_\rho} e^{-\lambda_e iT/l_\rho} \quad (48)$$

where $K_{b,\rho}$ and l_ρ are the respective coefficients for $w_{b,\rho}$ to satisfy A1.3, as guaranteed by Theorem 3.2. Hence J_∞ exists and $e_q \in L_P$ for $P = 1, 2$ and ∞.

With the result $e_q \in L_P$ for $P = 1, 2$ and ∞, it can be established $e_\phi := \phi_1 - \hat{\phi}_1 \in L_P$ from (23) and (32). From A1.4, we have

$$\begin{aligned} |w_b(\phi_1) - w_b(\hat{\phi}_1)| &= |w_b(\phi_1) - w_b(\phi_1 - e_\phi)| \\ &\leq K_d |e_\phi| \end{aligned} \quad (49)$$

where K_d is a constant, depending on the upper bound of the derivatives of w_b. Therefore, we have $(w_b(\phi_1) - w_b(\hat{\phi}_1)) \in L_P$, and this completes the proof.

Remark 3. The result shown in Theorem 4.1 for the disturbance estimation does not depend on the boundedness of the state variable. The only condition relating to the state variable is specified in (36) which does not depend on the boundedness of x. In the case of an unbounded x, p will be unbounded, and e will still satisfy (36) in the same way as the observer errors for unstable linear dynamic systems.

For the disturbances with straight lines in the wave form, the implementation of the estimation algorithm may be simplified. The simplified results are given for square waves and triangular waves in (Ding, 2006*a*) as those wave forms are symmetric. We show the result for sawtooth waves with alternating directions which has an asymmetric wave pattern.

Corollary 4.2 If the basic wave form for the disturbance in (1) is the alternating sawtooth wave form described by

$$w_b(t) = \begin{cases} 1 - \frac{2}{T}t & \text{for } 0 < t < \frac{T}{2} \\ \frac{2}{T}t - 2 & \text{for } \frac{T}{2} < t < T \\ 0 & \text{otherwise} \end{cases} \quad (50)$$

the disturbance $w(t) = a w_b(t + \phi)$ can be estimated by

$$\hat{w}(t) = \frac{\hat{a}}{T}\left(\frac{T}{2}\text{sign}(\bar{w}(t)) - \mathcal{I}\text{sign}(\bar{w}(t))\right) \quad (51)$$

with the property that $w - \hat{w} \in L_P$ for $P = 1, 2$ and ∞.

Proof. The alternating sawtooth wave form in (50) satisfies Assumption 1. From Lemma 3.3 and the definition of $\bar{q}_1$ in the proof of Theorem 4.1, we have

$$\text{sign}(\bar{q}_1(t)) = \begin{cases} 1 & \text{if } -\phi < t < \frac{T}{2} - \phi \\ -1 & \text{if } \frac{T}{2} - \phi < t < T - \phi \\ 0 & \text{otherwise} \end{cases} \quad (52)$$

and

$$\mathcal{I} \circ \text{sign}(\bar{q}_1(t)) = \begin{cases} 2(t + \phi) - \frac{T}{2} & \text{if } -\phi < t < \frac{T}{2} - \phi \\ \frac{3}{2}T - 2(t + \phi) & \text{if } \frac{T}{2} - \phi < t < T - \phi \\ 0 & \text{otherwise} \end{cases} \quad (53)$$

Hence we have

$$\left(\frac{1}{2}\text{sign}(\bar{q}_1(t)) - \frac{1}{T}\mathcal{I}\text{sign}(\bar{q}_1(t))\right) = w_b(t + \phi) \quad (54)$$

From the fact $(\mathrm{sign}(\bar{q}_1) - \mathrm{sign}(\bar{w})) \in L_P$ and $\tilde{a} \in L_P$, for $P = 1, 2$ and ∞, which is established in the proof of Theorem 4.1, it can be concluded in a similar way to the proof of Theorem 4.1 that $\tilde{w} \in L_P$. This completes the proof.

5 Disturbance Rejection with Stabilization

After the estimation of unknown disturbances, the control design follows in the same way as the control design for asymptotic rejection of disturbances with symmetric wave forms (Ding, 2006*a*). We include the key steps in control design for completeness of presentation without the proofs.

A state observer is designed as

$$\dot{\hat{x}} = (A_c + kC)\hat{x} + \psi(y) + b(u - \hat{w}) - ky \tag{55}$$

Control design can then be carried out using backstepping based on (55). Finally the control input is given by

$$u = \hat{w} + \frac{\alpha_\rho - \hat{x}_{\rho+1}}{b_\rho} \tag{56}$$

where

$$\begin{aligned}\alpha_i = {} & z_{i-1} - c_i z_i - \kappa_i(\frac{\partial \alpha_{i-1}}{\partial y})^2 z_i - k_i \hat{x}_1 + k_i y - \psi_i(y) \\ & + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}\dot{\hat{x}}_j + \frac{\partial \alpha_{i-1}}{\partial y}(\hat{x}_2 + \psi_1(y))\end{aligned} \tag{57}$$

for $\rho = 1, \ldots, \rho$, with $c_i,\ \kappa_i,\ i = 1, \ldots \rho$, being positive real design parameters, $z_1 := y = x_1$, $z_i = \hat{x}_i - \alpha_{i-1}$, $i = 2, \ldots \rho$, $z_0 = z_{\rho+1} = 0$ and $\frac{\partial \alpha_0}{\partial y} = 0$.

The proposed control ensures the asymptotic rejection of the disturbance and the boundedness of all the variables in the closed-loop system. The stability result is summarized in the following theorem.

Theorem 5.1 For a system (1) satisfying Assumptions 1 and 2, the control input u given in (56) with the estimated disturbance $\hat{w}$ ensures the asymptotic rejection of the unknown disturbance, ie., $\lim_{t\to\infty} y(t) = 0$, and the boundedness of the other variables in the system.

Remark 4. For linear systems, the term relating to the output is expressed by $\psi(y) = fy$, with $f \in R^n$. In this case, the control design is proposed as $u_l = k_l^T \hat{x} + \hat{w}$ where k_l is chosen so that $A_l = (A_c + fC + bk_l^T)$ is Hurwitz.

6 Example

Consider a nonlinear system in output feedback form

$$\begin{aligned}\dot{x}_1 &= x_2 - y^3 + (u - w) \\ \dot{x}_2 &= (u - w) \\ y &= x_1\end{aligned} \tag{58}$$

where $w = aw_b(t + \phi)$ is a periodic disturbance which satisfies Assumptions 1 and 2 with unknown a and ϕ. It is easy to see that the system (58) are in the format of (1) with $\phi(y) = [y^3\ 0]^T$ and $b = [1\ 1]^T$. The system is minimum phase, and therefore Assumption 2 is satisfied. The control input is designed as

$$u = \hat{w} - c_1 y - \kappa_1 y - y^3 - \hat{x}_2 \tag{59}$$

Simulation study has been carried out for the estimation and control design shown in this example. In the simulation study, the parameters were set as $k_1 = -2$, $k_2 = -1$, $c_1 = \kappa_1 = \lambda_1 = 1$, and the amplitude $a = 1$.

A non-smooth periodic disturbance with $w_b(t)$ with $T = 2$ as described in (50) was used in the simulation. The integral phase shift for this disturbance is $d_0 = \frac{T}{2}(1 + \frac{\sqrt{2}}{2})$. The control input and the system output are shown in Figure 1, in which the output converges to zero with the input to asymptotically cancel the disturbance. Figure 2 shows the disturbance and its estimate.

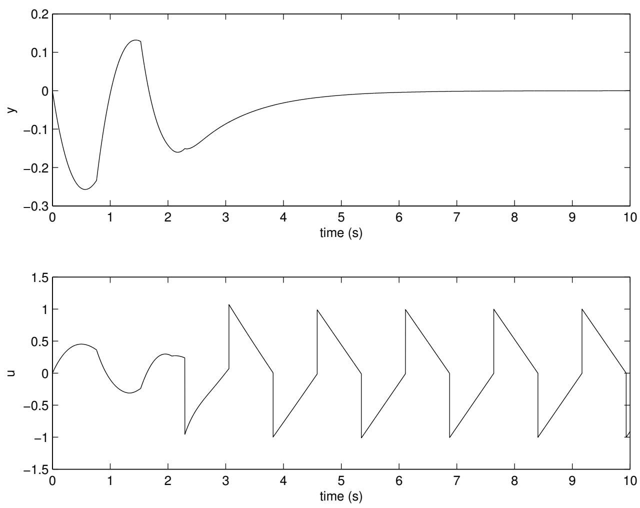

Fig. 1. The system input and output under alternating sawtooth wave

7 Conclusions

In this paper, we have introduced integral phase shift together with the half-period integration, and they have

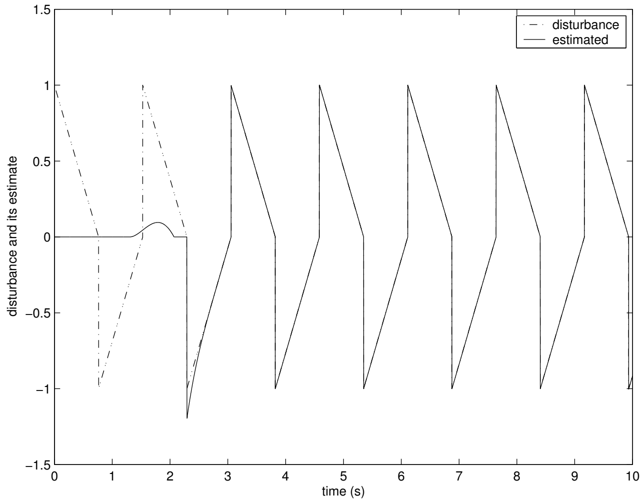

Fig. 2. The disturbance and its estimate for alternating sawtooth wave

been successfully used to establish the invariant properties of a large class of periodic disturbances whose basic wave forms may not be odd functions and nor symmetric in the half period. These invariant properties under the half-period integration with the delay of integral phase shift are instrumental in disturbance estimation together with a special observer designed to extract the contribution to the system state variables from the disturbances. The proposed disturbance estimation algorithm provides an estimate that converges in L_P. The proposed disturbance rejection algorithm with stability is based on the nonlinear systems in the output feedback form. In fact, the proposed estimation and rejection algorithms are novel for disturbance rejection in linear systems as well as in the nonlinear output feedback systems, and they can be used to asymptotically reject both non-smooth asymmetric disturbances and the disturbances which are generated from some nonlinear exosystems.

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