Tutorial 4

Question 19. Determine the z-transfer functions of the following plants (Hint: A ZOH should be added in each case):

1) $G_1(s) = \frac{1}{s(s+5)}$
2) $G_2(s) = \frac{s+2}{(s+1)(s+3)}$
3) $G_3(s) = \frac{10e^{-2Ts}}{s+5}$

Solution. 1)

$$\begin{aligned} G_1(z) &= \mathcal{Z}\{\frac{1-e^{-Ts}}{s}\frac{1}{s(s+5)}\} = (1-z^{-1})\mathcal{Z}\{\frac{1}{s^2(s+5)}\} \\ &= \frac{1}{25}(1-z^{-1})\mathcal{Z}\{-\frac{1}{s}+5\frac{1}{s^2}+\frac{1}{s+5}\} \\ &= \frac{1}{25}(1-z^{-1})\mathcal{Z}\{-\frac{1}{s}+5\frac{1}{s^2}+\frac{1}{s+5}\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{25}\frac{z-1}{z}[-\frac{z}{z-1}+5\frac{Tz}{(z-1)^2}+\frac{z}{z-e^{-5T}}\} \\
&= \frac{1}{25}\frac{(5T+e^{-5T}-1)z+(1-e^{-5T}-5Te^{-5T})}{(z-1)(z-e^{-5T})} \qquad (84)
\end{aligned}
$$

2)

$$
\begin{aligned}
G_2(z) &= \mathcal{Z}\{\frac{1-e^{-Ts}}{s}\frac{s+2}{(s+1)(s+3)}\} = (1-z^{-1})\mathcal{Z}\{\frac{s+2}{s(s+1)(s+3)}\} \\
&= (1-z^{-1})\mathcal{Z}\{\frac{2}{3}\frac{1}{s}-\frac{1}{2}\frac{1}{s+1}-\frac{1}{6}\frac{1}{s+3}\} \\
&= \frac{1}{6}\frac{z-1}{z}[4\frac{z}{z-1}-3\frac{z}{z-e^{-T}}-\frac{z}{z-e^{-3T}}] \\
&= \frac{1}{6}\frac{z-1}{z}[4\frac{z}{z-1}-3\frac{z}{z-e^{-T}}-\frac{z}{z-e^{-3T}}]
\end{aligned}
$$

$$= \frac{1}{6}\frac{(4-3e^{-T}-e^{-3T})z+(4e^{-4T}-3e^{-3T}-e^{-T})}{(z-e^{-T})(z-e^{-3T})} \tag{85}$$

3)

$$\begin{aligned}
G_3(z) &= \mathcal{Z}\{\frac{1-e^{-Ts}}{s}\frac{10e^{-2Ts}}{s+5}\} = 10z^{-2}(1-z^{-1})\mathcal{Z}\{\frac{1}{s(s+5)}\} \\
&= 10z^{-2}\frac{z-1}{z}\mathcal{Z}\{\frac{1}{5}\frac{1}{s}-\frac{1}{5}\frac{1}{s+5}\} \\
&= 2z^{-2}\frac{z-1}{z}[\frac{z}{z-1}-\frac{z}{z-e^{-5T}}] \\
&= 2z^{-2}\frac{1-e^{-5T}}{z-e^{-5T}} \qquad (86)
\end{aligned}$$

Question 20. A plant is described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{5}{s(s+5)} \tag{87}$$

and the systems input and output are sampled with a sampling interval $T = 0.1$ second.

1) Obtain the z transfer function between the input and the output.

2) Obtain the difference equation relating $y(k)$ and $u(k)$.

3) Determine the system output (first five steps) under a unit step using the difference equation obtained in 2).

Solution. 1) Using the solution of Question 19 1), we have

$$G_(z) = \mathcal{Z}\{\frac{1-e^{-Ts}}{s}\frac{5}{s(s+5)}\} = (1-z^{-1})\mathcal{Z}\{\frac{5}{s^2(s+5)}\}$$

$$= \frac{1}{5}\frac{(5T + e^{-5T} - 1)z + (1 - e^{-5T} - 5Te^{-5T})}{(z-1)(z-e^{-5T})}$$

$$= 0.2\frac{(1 + e^{-1} - 1)z + (1 - e^{-1} - e^{-1})}{(z-1)(z-e^{-1})} \quad (88)$$

and

$$\frac{Y(z)}{U(z)} = \frac{0.0736z + 0.0528}{z^2 - 1.3697z + 0.3697} \quad (89)$$

2) From the transfer function obtained in 1), we have

$$y(k+2) - 1.3697y(k+1) + 0.3697y(k) = 0.0736u(k+1) + 0.0528u(k) \quad (90)$$

3) From 2), we have

$$y(k) = 1.3697y(k-1) - 0.3697y(k-2) + 0.0736u(k-1) + 0.0528u(k-2) \quad (91)$$

Assuming $y(-2) = y(-1) = 0$, we have

$$\begin{aligned}
y(0) &= 1.3697y(-1) - 0.3697y(-2) + 0.0736u(-1) + 0.0528u(-2) = 0 \\
y(1) &= 1.3697y(0) - 0.3697y(-1) + 0.0736u(0) + 0.0528u(-1) \\
&= 1.3697 * 0 - 0.3697 * 0 + 0.0736 * 1 + 0.0528 * 0 = 0.0736 \\
y(2) &= 1.3697y(1) - 0.3697y(0) + 0.0736u(1) + 0.0528u(0) \\
&= 1.3697 * 0.0736 - 0.3697 * 0 + 0.0736 * 1 + 0.0528 * 1 = 0.2271 \\
y(3) &= 0.4104 \\
y(4) &= 0.6045 \\
y(4) &= 0.8207
\end{aligned}$$

Question 21. A first order system $\frac{Y(s)}{U(s)} = \frac{10}{s+5}$ is sampled at every T seconds. The control law for the system is designed as $u(kT) = -Ky(kT)$ where K is the controller gain.

1) Determine the range of the sampling interval T such that the closed-loop system is stable with $K = 10$.

2) Determine the range of controller gain K such that the closed-loop system is stable with $T = 0.1$ second.

Solution. 1) Similar to the solution 3) of Question 19, we have

$$\begin{aligned} \frac{Y(z)}{U(z)} &= \mathcal{Z}\{\frac{1-e^{-Ts}}{s}\frac{10}{s+5}\} = 10(1-z^{-1})\mathcal{Z}\{\frac{1}{s(s+5)}\} \\ &= 2\frac{1-e^{-5T}}{z-e^{-5T}} \end{aligned} \tag{92}$$

The closed-loop transfer function is given by

$$G(z) = \frac{K2\frac{1-e^{-5T}}{z-e^{-5T}}}{1+K2\frac{1-e^{-5T}}{z-e^{-5T}}}$$

$$= \frac{2K(1-e^{-5T})}{z-e^{-5T}+2K(1-e^{-5T})} \tag{93}$$

With $K = 10$, we have

$$z = e^{-5T} - 2K(1-e^{-5T}) = 21e^{-5T} - 20 \tag{94}$$

For $|z| < 1$, we have

$$-1 < 21e^{-5T} - 20 < 1 \tag{95}$$

and therefore we have $0 < T < 0.02$ second.

2) From 1), with $T = 0.1$, we have

$$z = e^{-5T} - 2K(1-e^{-5T}) = e^{-0.5} - 2K(1-e^{-0.5}) \tag{96}$$

For $|z| < 1$, we have

$$-1 < 2K(1 - e^{-0.5}) - e^{-0.5} < 1 \tag{97}$$

and therefore we have $-0.5 < K < 2.0415$.

Question 22. Consider a discrete-time system

$$\frac{Y(z)}{U(z)} = \frac{0.4z + 0.2}{z^2 - 1.4z + 0.4} \tag{98}$$

with feedback control $u(k) = -Ky(k)$.

1) Obtain the closed-loop transfer function and the characteristic equation of the closed-loop system, and then determine the stability of the system with $K = 1$ and $K = 10$ respectively.

2) Suggest a method to determine the range of the controller gain K such that the closed system is stable.

Solution. 1) The closed-loop transfer function is given by

$$\begin{aligned} G(z) &= \frac{K\frac{0.4z+0.2}{z^2-1.4z+0.4}}{1+K\frac{0.4z+0.2}{z^2-1.4z+0.4}} \\ &= \frac{K(0.4z+0.2)}{z^2+(0.4K-1.4)z+(0.2K+0.4)} \end{aligned} \tag{99}$$

The characteristic equation of the system is given by

$$D(z) = z^2 + (0.4K-1.4)z + (0.2K+0.4) \tag{100}$$

For $K = 1$, we have

$$D(z) = z^2 - z + 0.6 \tag{101}$$

and $D(z) = 0$ gives $z_{1,2} = 0.5000 \pm 0.5916j$. The system is stable as $|z_{1,2}| < 1$.