Tutorial 3

Question 13. Consider a dynamic system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (57)$$

$$y = [1 \ \ 0]x \quad (58)$$

Design a state feedback controller such that the system output tracks a constant r (with the closed-loop poles at -2,-2,-2).

Solution. Let $e = y - r = cx - r$, $z = \dot{x}$, and $v = \dot{u}$. We have

$$\begin{bmatrix} \dot{e} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} e \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v(t) \quad (59)$$

Let $v = -k_1 e - k_2 z_1 - k_3 z_2$. The closed-loop system matrix A_c is given by

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 + 3 & -k_3 + 2 \end{bmatrix} \tag{60}$$

and

$$|sI - A_c| = s^3 + (k_3 - 2)s^2 + (k_2 - 3)s + k_1 \tag{61}$$

Comparing it with the desired $(s+2)^3 = s^3 + 6s^2 + 12s + 8$, we have $k_1 = 8$, $k_2 = 15$, and $k_3 = 8$. Therefore the closed loop control for is given by

$$u(t) = -8 \int_0^t e(\tau) d\tau - 15x_2 - 8x_3 \tag{62}$$

Question 14. Consider a first order dynamic system

$$\dot{x} = x + u \tag{63}$$

The control input is designed as

$$u = -kx \tag{64}$$

such that the system is stale. Evaluate the performance index

$$J = \int_0^{\infty} x^2 dt \tag{65}$$

with $x(0) = 2$, and hence obtain an optimal value of k such that J is minimum.

Solution. The closed-loop system is given by

$$\dot{x} = -(k-1)x \tag{66}$$

and $x(t) = e^{-(k-1)t}x(0)$. Therefore

$$J(k) = \int_0^\infty e^{-2(k-1)t}x^2(0)dt = \frac{1}{2(k-1)}x^2(0) \tag{67}$$

We have $\lim_{k\to\infty} J(k) = 0$, and there does not exist an optimal k (because we did not consider control onput in the index).

Question 15. Repeat the optimal control design in Question 14, with the control performance index

$$J = \int_0^\infty [x^2 + ru^2]dt \tag{68}$$

where r is a constant.

Solution. Following the steps in the previous example, we have

$$J(k) = (1 + rk^2) \int_0^\infty e^{-2(k-1)t} x^2(0) dt = \frac{1 + rk^2}{2(k-1)} x^2(0) \tag{69}$$

Taking the partial derivative of J against k, we have

$$\frac{\partial J}{\partial k} = \frac{(k-1)2rk - (1 + rk^2)}{2(k-1)^2} = \frac{k^2 - 2k - 1/r}{2r(k-1)^2} \tag{70}$$

To seek an optimal k, we set $\frac{\partial J}{\partial k} = 0$ and we obtain

$$k^2 - 2k - 1/r = 0 \tag{71}$$

which gives $k = \frac{2+\sqrt{4+4/r}}{2} = 1 + \sqrt{1 + 1/r}$

Question 16. consider a dynamic system described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (72)$$

The initial value is given as $x(0) = [1,\ 1]^T$. With the feedback control in the form of

$$u = -kx_1 - kx_2 \quad (73)$$

obtain the relation between the performance index given by

$$J = \int_0^{\infty} x^T x dt \quad (74)$$

and the controller gain k.

Solution. The closed-loop system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix} x \tag{75}$$

Let

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \tag{76}$$

For $H^T P + PH = -Q$, We have

$$\begin{bmatrix} 0 & -k \\ 1 & -k \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{77}$$

which generates three equations

$$-p_2 k - p_2 k = -1 \tag{78}$$

$$-p_1 - p_2 k - p_3 k = 0 \tag{79}$$

$$p_2 - p_3 k + p_2 - p_3 k = -1 \tag{80}$$

Solving the above equations gives

$$P = \begin{bmatrix} 1 + \frac{1}{2k} & \frac{1}{2k} \\ \frac{1}{2k} & \frac{1}{2k}(1 + \frac{1}{k}) \end{bmatrix} \tag{81}$$

The performance index is given by

$$J = x^T(0)Px(0) = 1 + \frac{2}{k} + \frac{1}{2k^2} \tag{82}$$

Question 17. Re-design the optimal control gain k in Question 16 using the optimal control index

$$J = \int_0^\infty (x^T x + u^2) dt \tag{83}$$

Solution. Using Matlab lqr command, we have $k = [1 \;\; 1.7321]$ and
$$P = \begin{bmatrix} 1.7321 & 1 \\ 1 & 1.7321 \end{bmatrix}$$

Question 18. Determine the roots of the closed-loop control systems obtained in Questions 15, 16 and 17.

Solution. The closed-loop poles of Questions 15 and 16 depend on control gain k. For Question 17, the closed loop poles are at $-0.866 \pm 0.5j$.