Tutorial 2

Question 7. Design a full state feedback control $u = -Kx$ for the dynamic system described by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad (25)$$

such that the closed-loop system has the poles at $\{-2, -1 \pm 2j\}$.

Solution. With

$$d(s) = (s+2)(s+1+2j)(s+1-2j) = s^3 + 4s^2 + 9s + 10 \quad (26)$$

we have

$$K = [11 \quad 9 \quad 6] \quad (27)$$

Question 8. The dynamics of a rocket is described by

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{28}$$

$$y = [0 \;\; 1] x \tag{29}$$

The control input is designed as $u = -2x_1 - x_2$. Determine the roots of the characteristic equation.

Solution. The closed-loop system matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \tag{30}$$

and

$$d(s) = |sI - A| = \left| \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \right| = s^2 + 2s + 1 \tag{31}$$

and $s_1 = -1$ and $s_2 = -1$.

Question 9. For the dynamic system described in Question 8, how to change the control input such that the roots of the closed-loop systems are at $\{-2 \pm j\}$.

Solution. Let $u = -k_1 x_1 - k_2 x_2$, and the closed-loop system matrix

$$A = \begin{bmatrix} -k_1 & -k_2 \\ 1 & 0 \end{bmatrix} \tag{32}$$

and $d(s) = |sI - A| = s^2 + k_1 s + k_2$. Comparing this with $d(s) = (s + 2 - j)(s + 2 + j) = s^2 + 4s + 5$, we have $k_1 = 4$ and $k_2 = 5$.

Question 10. Consider a second order system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{33}$$

$$y = [0 \;\; 1] x \tag{34}$$

Design an observer such that the observer poles are at $\{-1 \pm j\}$.

Solution. Let the observer gain

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \tag{35}$$

and

$$A_o = A - LC = \begin{bmatrix} 1 & -l_1 \\ 3 & -l_2 + 1 \end{bmatrix} \tag{36}$$

With

$$d(s) = |sI - A_o| = s^2 + (l_2 - 2)s + (3l_1 - l_2 + 1) \tag{37}$$

and

$$(s + 1 - j)(s + 1 + j) = s^2 + 2s + 2 \tag{38}$$

we have $l_2 = 4$ and $l_1 = 5/3$. Therefore the observer is designed as

$$\dot{\hat{x}} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 5/3 \\ 4 \end{bmatrix} (y - [0 \;\; 1]\hat{x}) \quad (39)$$

Question 11. Design a full state observer for the dynamic system described by

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad (40)$$

$$y = [1 \;\; 0 \;\; 0]x \quad (41)$$

such that the observer has the poles at $\{-2, -1 \pm 2j\}$.

Solution. Let the observer gain

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \tag{42}$$

and

$$A_o = A - LC = \begin{bmatrix} -l_1 - 2 & 1 & 0 \\ -l_2 - 2 & 0 & 1 \\ -l_3 + 1 & 0 & 0 \end{bmatrix} \tag{43}$$

With

$$d(s) = |sI - A_o| = s^3 + (l_1 + 2)s^2 + (l_2 + 2)s + (l_3 - 1) \tag{44}$$

and

$$(s+2)(s+1-2j)(s+1+2j) = s^3 + 4s^2 + 9s + 10 \tag{45}$$

we have $l_1 = 2$, $l_2 = 7$ and $l_3 = 11$. Therefore the observer is designed as

$$\dot{\hat{x}} = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 2 \\ 7 \\ 11 \end{bmatrix} (y - [1 \;\; 0 \;\; 0]\hat{x}) \qquad (46)$$

Question 12. Consider a state space compensator for the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \qquad (47)$$

$$y = [0 \;\; 1]x \qquad (48)$$

with the closed loop poles at $\{-1 \pm j\}$, and the observer poles at $\{-2 \pm j\}$.

Solution. Let the control gain $K = [k_1 \;\; k_2]$. The closed characteristic polynomial is given by

$$\left| sI - \begin{bmatrix} -k_1 + 1 & -k_2 \\ 3 & 2 \end{bmatrix} \right| = s^2 + (k_1 - 3)s + (3k_2 - 2k_1 + 2) \tag{49}$$

Comparing with the desired

$$d(s) = (s + 1 + j)(s + 1 - j) = s^2 + 2s + 2 \tag{50}$$

we have $K = [5 \quad 10/3]$.

Let the observer gain

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \tag{51}$$

and

$$A_o = A - LC = \begin{bmatrix} 1 & -l_1 \\ 3 & -l_2 + 2 \end{bmatrix} \quad (52)$$

With

$$d(s) = |sI - A_o| = s^2 + (l_2 - 3)s + (3l_1 - l_2 + 2) \quad (53)$$

and

$$(s + 2 - j)(s + 2 + j) = s^2 + 4s + 5 \quad (54)$$

we have $l_2 = 5$ and $l_1 = 10/3$.

Finally, we have the compensator designed as

$$u = -5\hat{x}_1 - 10/3\hat{x}_2 \quad (55)$$

where $\hat{x}_1$ and $\hat{x}_2$ are observed states generated by

$$\dot{\hat{x}} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 10/3 \\ 5 \end{bmatrix} (y - [0 \;\; 1]\hat{x}) \qquad (56)$$