Tutorial 1

Question 1. An inverted pendulum can be described by the following set of differential equations

$$M\ddot{y} + ml\ddot{\theta} - u(t) = 0 \quad (1)$$

$$ml\ddot{y} + ml^2\ddot{\theta} - mlg\theta = 0 \quad (2)$$

where M is the mass of the cart, and m is the mass of the ball over the pendulum with $m << M$, y is the horizontal position of the cart and θ is the angle of the pendulum, u is the control input. Write the state space equation of the this system.

Solution. Let $(x_1, x_2, x_3, x_4) = (y, \dot{y}, \theta, \dot{\theta})$ the equations can be written as

$$M\dot{x}_2 + ml\dot{x}_4 - u(t) = 0 \quad (3)$$

$$\dot{x}_2 + l\dot{x}_4 - gx_3 = 0 \quad (4)$$

Solving $\dot{x}_2$ and $\dot{x}_4$ from the above equations with $M >> m$, we obtain

$$\dot{x}_2 = \frac{-mg}{M}x_3 + \frac{1}{M}u(t) \tag{5}$$

$$\dot{x}_4 = \frac{g}{l}x_3 - \frac{1}{Ml}u(t) \tag{6}$$

Noting $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$, we finally have

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -(mg/M) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & g/l & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/(ML) \end{bmatrix} x \tag{7}$$

Question 2. A system is described by the following differential equation

$$\dot{x} \quad = \quad \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{8}$$

Determine $\Phi(s)$ and $\Phi(t)$ of the system.

Solution.

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s+1 & 0 \\ -2 & s+3 \end{bmatrix} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+3 & 0 \\ 2 & s+1 \end{bmatrix} \quad (9)$$

$$\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ (e^{-t} - e^{-3t}) & e^{-3t} \end{bmatrix} \quad (10)$$

Question 3. Obtain the transfer function of the following state space system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (11)$$

$$y = [1 \quad 0]x \quad (12)$$

Solution.

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B \\ &= \frac{1}{(s+1)(s+3)}[1 \quad 0]\left[\begin{array}{cc} s+3 & 0 \\ 2 & s+1 \end{array}\right]\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = 0 \end{aligned} \qquad (13)$$

Question 4. Write the state space equations in the controller and observer canonical forms for the following systems described by the transfer functions

$$G_1(s) = \frac{s+1}{s^2+5s+5}, \quad G_2 = \frac{s+1}{4s^2+4s+1} \qquad (14)$$

Solution. For $G_1(s)$ in the controller canonical form:

$$A = \left[\begin{array}{cc} 0 & 1 \\ -5 & -5 \end{array}\right], B = \left[\begin{array}{c} 0 \\ 1 \end{array}\right], C = \left[\begin{array}{cc} 1 & 1 \end{array}\right] \qquad (15)$$

and the observer canonical form:

$$A = \begin{bmatrix} -5 & 1 \\ -5 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{16}$$

For $G_2(s)$ in the controller canonical form:

$$A = \begin{bmatrix} 0 & 1 \\ -1/4 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1/4 & 1/4 \end{bmatrix} \tag{17}$$

and the observer canonical form:

$$A = \begin{bmatrix} -1 & 1 \\ -1/4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{18}$$

Question 5. Determine the controllability and observability of the state space systems described in Question 3.

Solution.

$$P_c = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \tag{19}$$

$$P_o = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \tag{20}$$

The system is not controllable nor observable.

Question 6. Determine the controllability and observability of the state space systems

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{21}$$

$$y = [1 \quad 0]x \tag{22}$$

If the system is not controllable or observable, can you explain why?

Solution.

$$P_c = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \tag{23}$$

Since $|P_c| = 0$, P_c does not have full rank and therefore, the system is not controllable.

$$P_o = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \tag{24}$$

Since $|P_o| = 1$, P_o has full rank and the system is observable.