Control Systems II (EEEN30041)

Lecturer: Zhengtao Ding

Email: zhengtao.ding@manchester.ac.uk

Webpage: http://personalpages.manchester.ac.uk/staff/Zhengtao.Ding

Course Format and Assessment

The course will be delivered in 20 lectures with 4 tutorials and a computer-based assignment. Assessment will be based on an assignment (10%) and examination (90%). Course material will include online lecture notes and handouts.

Learning Outcomes

- Describe dynamic systems in continuous-time state space models, and dynamic models in discrete-time forms for industrial control applications.
- Design controllers in state space using state-feedback and output feedback methods including linear optimal control.
- Design and implement controllers in discrete-time using digital control including implementing PID in digital control.
- Appreciate important industrial control applications

References

Dorf and Bishop, *Modern Control Systems*

Stefani, Shahian, Savant and Hostetter, *Design of Feedback Control Systems*

Kailath, *Linear Systems*

Ogata, *Modern Control Engineering*

Kuo, *Automatic Control Systems*

Nise, *Control Systems Engineering*

Course Structure

Part 1. State space model and control design

Why state space control? Advantages and disadvantages of state space control design.

Part 2. Digital control

Why digital control?

1. State Space Variables

The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics.

Example 1. Consider a mass-spring-damper system

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t) \tag{1}$$

where M is the mass, b the friction coefficient, and k the spring constant.

Taking the position and velocity as the state variables, ie:

$$\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \end{aligned}$$

the dynamics can then be written as

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u\end{aligned}$$

Question: Why y and $\frac{dy}{dt}$ can be the state variables?

Example 2. RLC circuit

$$i_c = C\frac{dv_c}{dt} = -i_L + u(t) \tag{2}$$

$$L\frac{di_L}{dt} = -Ri_L + v_c \tag{3}$$

with the voltage across R as the output, $v_o = Ri_L$

By taking $x_1 = v_c$ and $x_2 = i_L$, we have

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t) \tag{4}$$

$$\frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2 \tag{5}$$

$$y = Rx_2 \tag{6}$$

Question: Is the set of state variables unique?

Alternative choice can be $x_1 = v_c$ and $x_2 = v_L$.

Question: Consider a different RLC circuit and obtain its dynamic model.

2. State Differential Equations

State Space Equation

A set of first order differential equations can be written as

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m
\end{aligned} \tag{7}$$

We can put the above in the matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \dots & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \quad (8)$$

The column vector consisting of the state variables is called state vector and it is written

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (9)$$

We may also write $x \in R^n$, as there are n state variables. Similarly we have the input vector. Using the state vector and input vector, we have the compact notation of the state differential equation

$$\dot{x} = Ax + Bu$$

where A is an $n \times n$ matrix or $A \in R^{n \times n}$, and B is an $n \times m$ matrix or $B \in R^{n \times m}$. We can also write the output in the compact form

$$y = Cx + Du$$

The dimensions of C and D depend on the dimension of the column vector y. Puting the above two equations together, we have the most common state space equation

$$\dot{x} = Ax + Bu \quad (10)$$

$$y = Cx + Du \quad (11)$$

Therefore a dynamic system is characterised by the four matrices $\{A, B, C, D\}$.

Example 3. Write down the state space equation for RLC system shown in Example 2.

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t) \quad (12)$$

$$y = [0 \; R]x \quad (13)$$

For this example, we have

$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \; B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}, \; C = [0 \; R], \; D = 0$$

Question: Determine the state space matrices A, B, C, D for Example 1.

Solution of state space equation

Consider the first order differential equation

$$\dot{x} = ax + bu \tag{14}$$

The solution is given by

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{15}$$

With the matrix exponential function defined by

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^kt^k}{k!} + \ldots \tag{16}$$

we have the solution of the state space equation given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{17}$$

Sometimes we denote $\Phi(t) = e^{At}$ and $\Phi(t)$ is referred to as the state transition matrix. Hence the above equation can be written as

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \tag{18}$$

It can be shown that $\Phi(t)$ equals the inverse Laplace transform of $(sI - A)^{-1}$.

3. Transfer Functions from State Equations

We can obtain the transfer function for a single-input-single-output (SISO) system from its state space equation. Taking the Laplace transform of the state space equation

$$\dot{x} = Ax + Bu \tag{19}$$

$$y = Cx \tag{20}$$

we have

$$sX(s) = AX(s) + BU(s) \tag{21}$$

$$Y(s) = CX(s) \tag{22}$$

From the first equation, we have

$$X(s)(sI - A) = BU(s) \tag{23}$$

and

$$X(s) = (sI - A)^{-1}BU(s) \tag{24}$$

Hence we have

$$Y(s) = C(sI - A)^{-1}BU(s) \tag{25}$$

Therefore the transfer function $G(s) = \frac{Y(s)}{U(s)}$ is given by

$$G(s) = C(sI - A)^{-1}B \tag{26}$$

Note that we can write $G(s) = C\Phi(s)B$, as $\Phi(s) = (sI - A)^{-1}$

Example 4. Determine the transfer function of the RLC system in Example 3.

Answer:

$$G(s) = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{27}$$

$\Phi(s) := (sI - A)^{-1}$ is a matrix with elements of Laplace transformed functions. Therefore we can use inverse Laplace transform to obtain $\Phi(t)$ which can be used to evaluate the time response.

Example 5. Obtain the time respnse of the RLC system shown in Example 2, assuming $R = 3$, $L = 1$, $C = 1/2$, $u(t) = 0$ and $x(0) = [1\ 1]^T$.

Solution: When $u(t) = 0$, we have

$$x(t) = \Phi(t)x(0) \tag{28}$$

In this case, we have

$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \tag{29}$$

and

$$\Phi(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix} \tag{30}$$

From the inverse Laplace transform, we obtain

$$\Phi(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \tag{31}$$

and subsequently

$$x(t) = \Phi(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} \tag{32}$$

Remark: $\Phi(t)$ can also be obtained by directly solving the differential equations. Indeed, let $z = Tx$ with

$$T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

It can be obtained that $\dot{z} = TAT^{-1}z := \bar{A}z$, with

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

With

$$e^{\bar{A}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix},$$

we have $\Phi(t) = T^{-1}e^{\bar{A}t}T$.

Stability of State Variable Systems

When we convert the state space model to the transfer function, we observe that the denominator of the transfer function is the determinant of $(sI - A)$, ie,

$$d(s) = |sI - A| \tag{33}$$

Therefore the stability of the state space model depends on the characteristic polynomial of the state matrix A. The system is stable if all the poles are in the left half plane, ie, the solutions of $d(s) = 0$ are all with negative real parts (equivalent to the eigenvalues of A are all with negative real parts).

4. State Space Realization and Canonical Forms

Given a transfer function, how to write its state space equations? For example, if the transfer function is given by

$$G(s) = \frac{6}{s^2 + 3s + 2} \tag{34}$$

we know that its state space realization is, based on the previous RLC example,

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \tag{35}$$

$$y = [0 \quad 3]x \tag{36}$$

Consider a general second transfer function

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \tag{37}$$

One realization is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (38)$$

$$y = [b_2 \quad b_1]x \quad (39)$$

The state space realization is not unique. For the same transfer function, we can have another realization as

$$\dot{x} = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \quad (40)$$

$$y = [1 \quad 0]x \quad (41)$$

Note that the state space variables in the two realizations are different. The first realization is referred to as the controller canonical form, and the second as the observer canonical form, as one is convenient for controller design and the other is convenient for observer design.

Controller Canonical Form

For a general transfer function given by (assuming the numerator and the denominator are coprime)

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} \tag{42}$$

the controller canonical form is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & \ldots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{43}$$

$$y = [b_n \quad b_{n-1} \quad \ldots \quad b_1]x \tag{44}$$

Observer Canonical Form

For the above transfer function, the observer canonical form is given by

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & 1 \\ -a_n & 0 & \dots & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u(t) \qquad (45)$$

$$y = [1 \quad 0 \quad \dots \quad 0]x \qquad (46)$$

Example 5. Put the state space equation shown for the RLC system into the controller and observer canonical forms.

5. Controllability and Observability

Controllability: A system is completely controllable if there exists a control input $u(t)$ that can transfer any initial state $x(0)$ to any other desired location x in a finite time.

We can check the controllability of the system

$$\dot{x} = Ax + Bu \tag{47}$$

$$y = Cx \tag{48}$$

by examining the algebraic condition

$$\text{rank}[B \;\; AB \;\; \ldots \;\; A^{n-1}B] = n \tag{49}$$

For a SISO system, we have B as a vector, and therefore if we define

$$P_c = [B \;\; AB \;\; \ldots \;\; A^{n-1}B] \tag{50}$$

then P_c is an $n \times n$ matrix. In this case, to check the controllability is equivalent to check if the determinant of P_c is nonzero.

Example 6. Check the controllability of the system

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ d & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad (51)$$

$$y = [0 \quad 1]x \quad (52)$$

where d is a constant.

Observability: A system is completely observable if and only if there exists a finite time T such that the initial state $x(0)$ can be determined from the observation history $y(t)$ given the control $u(t)$.

For a SISO system

$$\dot{x} = Ax + Bu \quad (53)$$

$$y = Cx \tag{54}$$

the system is completely observable is the determinant of the observability matrix

$$P_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{55}$$

is nonzero.

Example 7. Check the observability of the system

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \tag{56}$$

$$y = [1 \quad 1]x \tag{57}$$

Tutorial 1

Question 1. An inverted pendulum can be described by the following set of differential equations

$$M\ddot{y} + ml\ddot{\theta} - u(t) = 0 \quad (58)$$

$$ml\ddot{y} + ml^2\ddot{\theta} - mlg\theta = 0 \quad (59)$$

where M is the mass of the cart, and m is the mass of the ball over the pendulum with $m << M$, y is the horizontal position of the cart, l is the length of the pendulum and θ is the angle of the pendulum, u is the control input. Write the state space equation of the this system.

Question 2. A system is described by the following differential equation

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (60)$$

$$(61)$$

Determine $\Phi(s)$ and $\Phi(t)$ of the system.

Question 3. Obtain the transfer function of the following state space system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{62}$$

$$y = [1 \quad 0]x \tag{63}$$

Question 4. Write the state space equations in the controller and observer canonical forms for the following systems described by the transfer functions

$$G_1(s) = \frac{s+1}{s^2+5s+5}, \quad G_2 = \frac{s+1}{4s^2+4s+1} \tag{64}$$

Question 5. Determine the controllability and observability of the state space systems described in Question 3.

Question 6. Determine the controllability and observability of the state space systems

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{65}$$

$$y = [1 \quad 0]x \tag{66}$$

If the system is not controllable or observable, can you explain why?

6. Full State Feedback Design

In the first step in the state variable design, we assume all the state variables are available for feedback control. In this case, we can design the control input as

$$u = -Kx \tag{67}$$

where K is the gain matrix. The full state feedback design is to decide a suitable feedback gain matrix K.

For the closed-loop control system, we have

$$\dot{x} = Ax + B(-Kx) = (A - BK)x \tag{68}$$

The stability of the closed-loop system depends on the characteristic polynomial of $(A - BK)$

Pole Assignment. To assign the closed loop poles at given locations via full state feedback.

Pole assignment can be achieved if the system is completely controllable.

Example 8. Design a full state feedback control of the system described by

$$\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = u \tag{69}$$

such that the closed-loop poles are at $\{-1, -2, -3\}$ respectively.

Solution: The transfer function of the system is

$$G(s) = \frac{1}{s^3 + 5s^2 + 3s + 2} \tag{70}$$

If we realize the system in the controller canonical form, we have

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{71}$$

$$y = [1 \;\; 0 \;\; 0]x \tag{72}$$

If the control input is designed as

$$u = [k_1 \;\; k_2 \;\; k_3]x \tag{73}$$

The closed-loop system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (-[k_1 \;\; k_2 \;\; k_3]x) \tag{74}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1+2) & -(k_2+3) & -(k_3+5) \end{bmatrix} x := (A - BK)x \tag{75}$$

The closed-loop characteristic polynomial is

$$|sI - (A - BK)| = s^3 + (k_3+5)s^2 + (k_2+3)s + (k_1+2) \tag{76}$$

Comparing it with the desired characteristic polynomial

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6 \tag{77}$$

we have $k_3 = 1$, $k_2 = 8$, $k_1 = 4$, ie,

$$K = [4 \;\; 8 \;\; 1] \tag{78}$$

Therefore the control input is designed as

$$u = -[4 \;\; 8 \;\; 1]x = -4x_1 - 8x_2 - x_3 \tag{79}$$

Note that for this realization we have $x_1 = y$, $x_2 = \frac{dy}{dt}$ and $x_3 = \frac{d^2y}{dt^2}$. The control input is then expressed as

$$u = -4y - 8\frac{dy}{dt} - \frac{d^2y}{dt^2} \tag{80}$$

Question: Design a state feedback controller the RLC circuit described by

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \tag{81}$$

to place the closed-loop poles as $\{-2, -2\}$.

Answer: $u = -[\frac{1}{2}, -\frac{1}{2}]x = -(x_1 - x_2)/2$.

Question: How to implement the feedback control if only the output y is available?

From Example 8 we have seen the convenience of using the controller canonical form for the full state feedback design. In fact, for a given state space system, we can transform the system to the controller canonical form if the system is controllable. To avoid the state transform, we have Ackermann's formula for the full state feedback control design.

Ackermann's Formula. For a system $\{A, B\}$, the state feedback control gain K for the closed loop control $u = -Kx$ to achieve the desired closed-loop characteristic polynomial

$$d(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n \tag{82}$$

is given by

$$K = [0 \;\; 0 \;\; \ldots \;\; 1] P_c^{-1} d(A) \tag{83}$$

where

$$d(A) = A^n + \alpha_1 A^{n-1} + \ldots + \alpha_n I \tag{84}$$

and P_c is the controllability matrix.

Question: Applying Ackermann's Formula for state feedback control design for the RLC circuit shown in the previous question.

Hint: $d(s) = s^2 + 4s + 4$, and $d(A) = A^2 + 4A + 4I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$.

7. Observer Design

Full state feedback control needs the values of all the state variables. In industrial systems, it is common that not all the state are available, and in this case, an observer can be designed to provide an estimate of the unknown state variables.

Luenberger Observer. For a dynamic system

$$\dot{x} = Ax + Bu \tag{85}$$

$$y = Cx \tag{86}$$

a full-state observer is designed as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{87}$$

where $\hat{x}$ denotes the estimate of the state variable, and L is the observer gain.

Define the observer error as

$$e = x - \hat{x} \tag{88}$$

we have the error dynamics

$$\begin{aligned} \dot{e} &= (Ax + Bu) - (A\hat{x} + Bu + L(y - C\hat{x})) \\ &= (A - LC)e \end{aligned} \tag{89}$$

To ensure the error asymptotically converge to zero as $t \to \infty$, we need the characteristic equation $|sI - (A - LC)| = 0$ to have all the roots in the left half of the complex plan.

Example 9. Design a full state observer for the dynamic system

$$\dot{x} = \begin{bmatrix} -5 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{90}$$

$$y = [1 \;\; 0 \;\; 0]x \tag{91}$$

Solution: Note the system is in the observer canonical form. For $L = [l_1 \;\; l_2 \;\; l_3]^T$, we have

$$A - LC = \begin{bmatrix} -5 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} [1 \;\; 0 \;\; 0] = \begin{bmatrix} -(l_1+5) & 1 & 0 \\ -(l_2+3) & 0 & 1 \\ -(l_3+2) & 0 & 0 \end{bmatrix} \tag{92}$$

The characteristic polynomial is

$$|sI - (A - LC)| = s^3 + (l_1+5)s^2 + (l_2+3)s + (l_3+2) \tag{93}$$

If we place the poles for the observer at $\{-2, -2, -2\}$, the desired characteristic polynomial is

$$(s+2)^3 = s^3 + 6s^2 + 12s + 8 \tag{94}$$

By comparing the polynomials we have $l_1 = 1$, $l_2 = 9$, $l_3 = 6$, ie,

$$L = [1 \;\; 9 \;\; 6]^T \tag{95}$$

The full state observer is given by

$$\dot{\hat{x}} = \begin{bmatrix} -5 & 1 & 0 \\ -3 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} (y - [1 \;\; 0 \;\; 0]\hat{x}) \tag{96}$$

The observer canonical form makes the design of observer gain easier. For the system what is not in the observer canonical form, we can still evaluate the characteristic polynomial, and then by comparing the coefficients with the desired one to obtain the observer gain matrix. For design observer gains, we also have Ackermann's formula.

Ackermann's Formula. For a system $\{A, B, C\}$, the observer gain L to achieve the desired closed-loop characteristic polynomial

$$d(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n \qquad (97)$$

is given by

$$L = d(A)P_o^{-1}[0 \quad 0 \quad \ldots \quad 1]^T \qquad (98)$$

where

$$d(A) = A^n + \alpha_1 A^{n-1} + \ldots + \alpha_n I \qquad (99)$$

and P_o is the observability matrix.

8. Compensator Design

We aim at design dynamic feedback control from the system output. There are three steps in the compensator design.

- Full state feedback design
- Full state observer design
- Compensator design using the state estimate to replace the state variable in the full state control design

The final control design is given by

$$u = -K\hat{x}(t) \tag{100}$$

Question: How to ensure the stability of the closed-loop system?

The separation principle plays an important part.

Consider the system

$$\dot{x} = Ax + Bu \tag{101}$$

$$y = Cx \tag{102}$$

with the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{103}$$

and the control law

$$u = -K\hat{x}(t) \tag{104}$$

The closed-loop system with the observer can be written as

$$\dot{x} = Ax - BK\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} - BK\hat{x} + LC(x - \hat{x})$$

In terms of the state variable x and observer error $e = x - \hat{x}$, we have

$$\begin{aligned} \dot{x} &= (A - BK)x + BKe \\ \dot{e} &= (A - LC)e \end{aligned}$$

Treating $[x^T \; , \; e^T]^T$ as the augmented state variable, we have

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} \tag{105}$$

The characteristic polynomial of the augmented system is given by

$$\left| sI - \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right| = |sI - (A - BK)||sI - (A - LC)| \tag{106}$$

Therefore if $|sI - (A - BK)| = 0$ and $|sI - (A - LC)| = 0$ have all the roots in the left half of the complex plane, the augmented system is stable. This is the separation principle.

Example 9. Design a dynamic output feedback control (compensator) for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (107)$$

$$y = [1 \;\; 0]x \quad (108)$$

The poles for the closed loop controller are at $\{-1, -1\}$ and the poles for the observer error dynamics are at $\{-2, -2\}$.

Tutorial 2

Question 7. Design a full state feedback control $u = -Kx$ for the dynamic system described by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{109}$$

such that the closed-loop system has the poles at $\{-2, -1 \pm 2j\}$.

Question 8. The dynamics of a rocket is described by

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{110}$$

$$y = [0 \;\; 1]x \tag{111}$$

The control input is designed as $u = -2x_1 - x_2$. Determine the roots of the characteristic equation.

Question 9. For the dynamic system described in Question 8, how to change the control input such that the roots of the closed-loop systems are at $\{-2 \pm j\}$.

Question 10. Consider a second order system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{112}$$

$$y = [0 \;\; 1]x \tag{113}$$

Design an observer such that the observer poles are at $\{-1 \pm j\}$.

Question 11. Design a full state observer for the dynamic system described by

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{114}$$

$$y = [1 \;\; 0 \;\; 0]x \tag{115}$$

such that the observer has the poles at $\{-2, -1 \pm 2j\}$.

Question 12. Consider a state space compensator for the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad (116)$$

$$y = [0 \ \ 1]x \quad (117)$$

with the closed loop poles at $\{-1 \pm j\}$, and the observer poles at $\{-2 \pm j\}$.

9. Tracking and Internal Model Design

Consider

$$
\begin{aligned}
\dot{x} &= Ax + Bu && (118) \\
y &= Cx && (119)
\end{aligned}
$$

We need to design a control input such that the output asymptotically tracks a given reference r, which can be written as the output for the reference state space model

$$
\begin{aligned}
\dot{x}_r &= A_r x_r && (120) \\
r &= d_r x_r && (121)
\end{aligned}
$$

Consider the case of tracking a constant reference, we have the reference model as

$$
\dot{x}_r = 0 \qquad (122)
$$

$$r = x_r \tag{123}$$

Define the tracking error $e = y - r$. Taking the derivative of the tracking error gives

$$\dot{e} = \dot{y} = C\dot{x} \tag{124}$$

Let us use the notation $z = \dot{x}$ and $v = \dot{u}$. Take e and z as the state variable of an augmented state space system

$$\frac{d}{dt}\begin{bmatrix} e \\ z \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}\begin{bmatrix} e \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v \tag{125}$$

If the augmented system is controllable, we can design a state feedback controller in the form

$$v = -k_1 e - k_2 z \tag{126}$$

where k_2 is a matrix (vector) in general. The state feedback ensures the stability of the augmented system which implies the asymptotic tracking with e converging to zero. The control is input is given by integrating v as

$$u(t) = -k_1 \int_0^t e(\tau) d\tau - k_2 x(t) \tag{127}$$

The integration in the above equation reflects the dynamics of the tracking signal. It is clear to see in a block diagram that the controller acts as an internal model.

Example 10. Internal model design for a unit step input for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{128}$$

$$y = [1 \;\; 0] x \tag{129}$$

by placing the pools of the augmented system at $\{-1 \pm j, -10\}$.

The same control design can be extended to the case of tracking a polynomial of time.

Question: How to design an internal model based controller to track a sinusoidal signal?

10. Optimal Control

The performance of a control system can be represented by a performance index such as

$$J = \int_0^{t_f} g(x, u, t)dt \tag{130}$$

Optimal control is concerning with the control design to minimize a performance index. Consider a particular performance index

$$J = \int_0^{t_f} x^T Q x dt \tag{131}$$

where Q is an $n \times n$ positive definite matrix. To simplify the problem, we let t_f tend to infinity, that is, the index is given by

$$J = \int_0^{\infty} x^T Q x dt \tag{132}$$

How to design a full state feedback law to minimise the index?

Consider the control design $u = -Kx$ for $\dot{x} = Ax + Bu$. The closed-loop system is

$$\dot{x} = (A - BK)x := Hx \tag{133}$$

If we have a positive definite matrix P such that

$$\frac{d}{dt}(x^T Px) = -x^T Qx \tag{134}$$

then substituting the above into the performance index, we have

$$J = -\int_0^\infty \frac{d}{dt}(x^T Px)dt = -x^T Px \mid_0^\infty = x^T(0)Px(0) \tag{135}$$

where we assume that the closed-loop system is stable ($x(\infty) = 0$).

A direct evaluation gives

$$\begin{aligned}\frac{d}{dt}(x^T P x) &= \dot{x}^T P x + x^T P x \\ &= x^T H^T P x + x^T P H x \\ &= x^T (H^T P + P H) x \end{aligned} \tag{136}$$

From the differential equation $\frac{d}{dt}(x^T P x) = -x^T Q x$ we have

$$H^T P + P H = -Q \tag{137}$$

Therefore the optimal control design with the performance index $J = \int_0^\infty x^T Q x dt$ can be carried out in two steps:

- Solve the matrix equation

$$H^T P + P H = -Q \tag{138}$$

to obtain matrix P that depends on the control gain K.

- Minimize the index

$$J = x^T(0)Px(0) \tag{139}$$

to determine the control gain or other parameters in the system.

Example 11. Design the state feedback control $K = [k_1 \;\; k_2]$ by restricting $k_1 = 1$ with the optimal control index of $Q = I$ and $x(0) = [1 \;\; 1]^T$ for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{140}$$

$$\tag{141}$$

Example 12. Continue from Example 11 by restricting the control gain as $K = [k \;\; k]$ (ie., $k_1 = k_2$) and $x(0) = [1 \;\; 0]^T$.

Solution: Solving the matrix equation $H^T P + PH = -I$ gives

$$p_{11} = \frac{1 + 2k}{2k} \tag{142}$$

It is easy to see the control index is given by

$$J = x^T(0)Px(0) = p_{11} = 1 + \frac{1}{2k} \tag{143}$$

The minimum value of J is obtained when k tends to infinity.

11. Linear Quadratic Regulator

Consider the solution of Example 12. The bigger the controller gain, the smaller the performance index; there is no optimal solution of the control gain. This is due to the fact that we did not consider the control effort in the performance index.

In engineering systems, the bigger control input efforts often mean the bigger energy consumption. To consider the input in the performance index, we often define

$$J = \int_0^\infty [x^T Q x + u^T R u] dt \tag{144}$$

where R is a positive definite matrix. For SISO case, we have R as a constant scaler.

Consider the full state feedback control $u = -Kx$, the performance index can be written as

$$J = \int_0^\infty [x^T Q x + x^T K^T R K x] dt := \int_0^\infty x^T S x dt \tag{145}$$

where

$$S = Q + K^T R K \tag{146}$$

Similar to the case with no control in the performance index, we need to solve a matrix equation

$$H^T P + P H = -S \tag{147}$$

and the performance index is then given by

$$J = \int_0^\infty [x^T Q x + x^T K^T R K x] dt := \int_0^\infty x^T S x dt = x(0)^T P x(0) \tag{148}$$

Example 13. Repeat Example 12 with the new performance index

$$J = \int_0^\infty [x^T x + ru^2] dt \tag{149}$$

General Solution for Linear Quadratic Regulator: Consider a dynamic system described by

$$\dot{x} = Ax + Bu \tag{150}$$

The optimal control input for the performance index

$$J = \int_0^\infty [x^T Q x + u^T R u] dt \tag{151}$$

is given by

$$u = -R^{-1} B^T P x \tag{152}$$

where P satisfies

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{153}$$

This equation is often referred to as the matrix algeraic Riccati equation, and it can be easily solved using Matlab.

Tutorial 3

Question 13. Consider a dynamic system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{154}$$

$$y = [1 \quad 0]x \tag{155}$$

Design a state feedback controller such that the system output tracks a constant r.

Question 14. Consider a first order dynamic system

$$\dot{x} = x + u \tag{156}$$

The control input is designed as

$$u = -kx \tag{157}$$

such that the system is stale. Evaluate the performance index

$$J = \int_0^\infty x^2 dt \tag{158}$$

with $x(0) = 2$, and hence obtain an optimal value of k such that J is minimum.

Question 15. Repeat the optimal control design in Question 14, with the control performance index

$$J = \int_0^\infty [x^2 + ru^2] dt \tag{159}$$

where r is a constant.

Question 16. consider a dynamic system described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{160}$$

The initial value is given as $x(0) = [1,\ 1]^T$. With the feedback control in the form of

$$u = -kx_1 - kx_2 \tag{161}$$

obtain the relation between the performance index given by

$$J = \int_0^\infty x^T x dt \tag{162}$$

and the controller gain k.

Question 17. Re-design the optimal control gain k in Question 16 using the optimal control index

$$J = \int_0^\infty (x^T x + u^2) dt \tag{163}$$

Question 18. Determine the roots of the closed-loop control systems obtained in Questions 15, 16 and 17.

12. Digital Control and Sampled Data System

Question: Why digital control?

Due to the application of digital computers in industrial control systems. [There are also some inherently discrete-time systems.]

Sampled Data. Sampled data (or a discrete signal) are data obtained for system variables only at discrete time intervals.

We assume the sampling at the same fixed period, T, which is called the sampling period. The sampled data for a continuous time variable $x(t)$ are denoted by $x(kT)$ with k taking the values of integers.

How to decide the sampling period?

It depends on the dynamics of the system, the required accuracy and hardware constraints.

Sampler. An ideal sampler is a switch that closes for every T seconds for a instant. For an continuous time signal $r(t)$, sampled at kT, the output from the sampler $r^*(t)$ is an impulse signal,

$$r^*(t) = r(kT)\delta(t - kT)$$

where δ is the impulse function.

Zero-order-hold. After the sampling, it is assumed that the value is kept at same until next sampling, ie, the value of $x(t)$ is assumed to be at the constant of $x(kT)$ for $kT \leq t < (k+1)T$.

The impulse response of the zero-order-hold is given by

$$g_0(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise.} \end{cases} \tag{164}$$

and therefore its transfer function is given by

$$G_0(s) = \frac{1}{s} - \frac{1}{s}e^{-Ts} = \frac{1 - e^{-Ts}}{s} \tag{165}$$

Quantization error. An error due to a computer's finite word size.

13. The z-Transform

The output from an ideal sampler is a sequence of impulses with values $r(kT)$, and we write

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t - kT) \tag{166}$$

Taking the Laplace transform of the above equation, we have

$$\mathcal{L}\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT)e^{-kTs} = \sum_{k=0}^{\infty} r(kT)(e^{Ts})^{-k} \tag{167}$$

We define the z-transform as

$$R(z) = \mathcal{Z}\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT)z^{-k} \tag{168}$$

with $z = e^{Ts}$. Similar to the notation for the Laplace transformed functions, a capital letter R denotes the z-transformed functions of $r(t)$.

Example 14. Determine the z-transform for the unit step function $u(t)$.

$$U(z) = \sum_{k=0}^{\infty} u(kT) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \tag{169}$$

Example 15. Determine the z-transform of e^{-at}.

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} e^{-akT} z^{-k} = \sum_{k=0}^{\infty} (e^{aT} z)^{-k} \\ &= \frac{1}{1 - (e^{aT} z)^{-1}} = \frac{z}{z - e^{-aT}} \end{aligned}$$

The z-transform table

$x(t)$	$X(s)$	$X(z)$
$\delta(t)$	1	1
$\delta(t-kT)$	e^{-kTs}	z^{-k}
$u(t)$, unit step	$1/s$	$\frac{z}{z-1}$
t	$1/s^2$	$\frac{Tz}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$

Inverse z transform

- Partial fraction method (similar to inverse Laplace transform)
- Long division

Transfer function of an open loop system

We need to multiply the transfer function of the zero-order-hold to the system transfer function $G_p(s)$.

$$G(z) = \mathcal{Z}\{G_0(s)G_p(s)\} \tag{170}$$

Example 16. Determine the z-transfer function of $G_p(s) = \frac{1}{s(s+1)}$.

$$G(s) = \frac{1 - e^{-Ts}}{s}\frac{1}{s(s+1)} = (1 - e^{-Ts})(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}) \tag{171}$$

It follows that

$$\begin{aligned} G(z) &= (1 - z^{-1})\mathcal{Z}(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}) \\ &= (1 - z^{-1})(\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z - e^{-T}}) \\ &= \frac{(ze^{-T} - z + Tz) + (1 - e^{-T} - Te^{-T})}{(z-1)(z - e^{-T})} \end{aligned}$$

14. Discrete-time Systems

Difference equation

Consider

$$y(k+n) + a_1 y(k+n-1) + \ldots + a_n y(k) = b_1 u(k+n) + \ldots + b_n u(k) \quad (172)$$

The solution of this difference equation can be obtained by iteration using

$$y(k+n) = -a_1 y(k+n-1) - \ldots - a_n y(k) + b_1 u(k+n-1) + \ldots + b_n u(k) \quad (173)$$

if necessary initial values are unknown.

Transfer function

Note that z^{-1} and z can be used as a shift operator. It can be shown that

$$\mathcal{Z}\{y(k-1)\} = z^{-1}\mathcal{Z}\{y(k)\} = z^{-1}y(z)$$

$$\mathcal{Z}\{y(k+1)\} \quad = \quad z\mathcal{Z}\{y(k)\} = zy(z) \tag{174}$$

In fact, from the definition, we have $z^{-1} = e^{-Ts}$, a delay operator in Laplace domain.

Taking z-transform of the difference equation

$$z^n Y(z) + a_1 z^{n-1} Y(z) + \ldots + a_n Y(z) = b_1 z^{n-1} U(z) + \ldots + b_n U(z) \tag{175}$$

and the transfer function is given by

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_1 z^{n-1} + \ldots + b_n}{z^n + a_1 z^{n-1} + \ldots + a_n} \tag{176}$$

and finally we have

$$G(z) = \frac{b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} \tag{177}$$

System response using difference equation

For a given transfer function, we can obtain the difference and then obtain the system responses using the difference equation.

Example 17. For the system considered in Example 16 with $T = 1$, determine the first four terms of the output subject to the impulse input, assuming $y(-2) = y(-1) = 0$.

For $T = 1$, we have the transfer function

$$G(z) = \frac{Y(z)}{U(z)} = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678} \tag{178}$$

The difference equation is obtained as

$$y(k+2) - 1.3678y(k+1) + 0.3678y(k) = 0.3678u(k+1) + 0.2644u(k) \quad (179)$$

or

$$y(k+2) = 1.3678y(k+1) - 0.3678y(k) + 0.3678u(k+1) + 0.2644u(k) \quad (180)$$

Note the $\{u(k)\} = \{1,\ 0, \ldots, 0, \ldots\}$. We have

$$y(0) = 1.3678y(-1) - 0.3678y(-2) + 0.3678u(-1) + 0.2644u(-2) = 0 \quad (181)$$

$$y(1) = 1.3678y(0) - 0.3678y(-1) + 0.3678u(0) + 0.2644u(-1) = 0.3678 \quad (182)$$

and $y(2) = 0.7675$ and $y(3) = 0.9145$.

The method shown in the above example can also be used for evaluation of inverse z transform.

Indeed, from partial fraction expansion, we have, taking $U(z) = 1$,

$$Y(z) = G(z)U(z) = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678} = \frac{1}{z-1} + 0,6322\frac{1}{z - 0.3678}$$

and

$$y(k) = z^{-1}(\mathcal{Z}^{-1}\{\frac{z}{z-1} + 0,6322\frac{z}{z-0.3678}\}) = (1)^{k-1} - 0.6322(0.3678)^{k-1}$$

15. Closed-loop Transfer Functions and Stability

Closed-loop transfer function

Consider a closed-loop system

$$
\begin{aligned}
Y(z) &= G(z)U(z) && (183)\\
U(z) &= D(z)E(z) && (184)\\
E(z) &= R(z) - Y(z) && (185)
\end{aligned}
$$

where $D(z)$ denotes the transfer function of a digital controller, and $E(z)$ and $R(z)$ denote the feedback error and the reference signal.

The closed-loop transfer function is obtained as

$$\frac{Y(z)}{R(z)} = \frac{G(z)D(z)}{1 + G(z)D(z)} \qquad (186)$$

The above result is the same for the closed-loop transfer function of the continuous-time system. In general, the block diagram manipulation follows in the same way as the block diagram manipulation of the continuous-time systems.

Stability analysis

Conside the mapping between s-plane and z-plane. Let $s = \sigma + j\omega$ and we have

$$z = e^{Ts} = e^{\sigma + j\omega} \tag{187}$$

It can be seen that for $\sigma < 0$ we have $|z| < 1$, that is, the left-half of the s plane corresponds to the area within the unit circle in z-plane. Therefore we have the following statement.

A sampled (discrete-time) system is stable if all the poles of the closed-loop transfer function lie within the unit circle of the z-plane.

Example 18. The open-loop transfer function considered in Example 17 is under a closed-loop control with the controller $D(z) = K$, a constant. Evaluate the stability for the closed-loop system when $K = 1$ and $K = 10$.

The closed-loop transfer function is given by

$$\frac{KG(z)}{1 + KG(z)} = \frac{K\frac{0.3678z+0.2644}{z^2-1.3678z+0.3678}}{1 + K\frac{0.3678z+0.2644}{z^2-1.3678z+0.3678}} \tag{188}$$

and therefore the characteristic equation is

$$d(z) = z^2 + (0.3678K - 1.3678)z + (0.2644K + 0.3678) \tag{189}$$

For $K = 1$, we have

$$d(z) = z^2 - z + 0.6832 = 0 \tag{190}$$

and the poles, $z_{1,2} = 0.50 \pm j0.6182$, are within the unit circle and the system is stable. For $K = 10$, we have

$$d(z) \quad = \quad z^2 + 2.310z + 3.012 = 0 \tag{191}$$

and the poles, $z_{1,2} = -1.155 \pm j1.295$, are outside the unit circle and the system is unstable.

Example 19. Range of T for stability

Consider a unit feedback sampled-data system with the plant transfer function $G_p = \frac{10}{s+1}$. Determine the range of the sampling interval to ensure the closed-loop stability.

Let

$$G(s) = \frac{1 - e^{-Ts}}{s} \frac{10}{s+1} = (1 - e^{-Ts})10(\frac{1}{s} - \frac{1}{s+1})$$

Taking z-transform, we have

$$G(z) = (1 - z^{-1})10(\frac{1}{z-1} - \frac{z}{z - e^{-T}} = 10\frac{1 - e^{-T}}{z - e^{-T}}$$

The closed-loop transfer function is obtained as

$$G_c(z) = \frac{G(z)}{1 + G(z)} = \frac{10\frac{1-e^{-T}}{z-e^{-T}}}{1 + 10\frac{1-e^{-T}}{z-e^{-T}}} = \frac{10(1 - e^{-T})}{z - e^{-T} + 10(1 - e^{-T})}$$

The characteristic equation is given by $z - e^{-T} + 10(1 - e^{-T}) = 0$ or $z = 10 - 11e^{-T}$. For the stability, we need $|z| < 1$, ie,

$$-1 < 10 - 11e^{-T} < 1$$

which gives $0 < T < -\ln\frac{9}{11}$ or $0 < T < 0.2007$.

Tutorial 4

Question 19. Determine the z-transfer functions of the following plants (Hint: A ZOH should be added in each case):

1) $G_1(s) = \frac{1}{s(s+5)}$
2) $G_2(s) = \frac{s+2}{(s+1)(s+3)}$
3) $G_3(s) = \frac{10e^{-2Ts}}{s+5}$

Question 20. A plant is described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{5}{s(s+5)} \tag{192}$$

and the systems input and output are sampled with a sampling interval $T = 0.1$ second.

1) Obtain the z transfer function between the input and the output.

2) Obtain the difference equation relating $y(k)$ and $u(k)$.

3) Determine the system output (first five steps) under a unit step using the difference equation obtained in 2) .

Question 21. A first order system $\frac{Y(s)}{U(s)} = \frac{10}{s+5}$ is sampled at every T seconds. The control law for the system is designed as $u(kT) = -Ky(kT)$ where K is the controller gain.

1) Determine the range of the sampling interval T such that the closed-loop system is stable with $K = 10$.

2) Determine the range of controller gain K such that the closed-loop system is stable with $T = 0.1$ second.

Question 22. Consider a discrete-time system

$$\frac{Y(z)}{U(z)} = \frac{0.4z + 0.2}{z^2 - 1.4z + 0.4} \tag{193}$$

with feedback control $u(k) = -Ky(k)$.

1) Obtain the closed-loop transfer function and the characteristic equation of the closed-loop system, and then determine the stability of the system with $K = 1$ and $K = 10$ respectively.

2) Suggest a method to determine the range of the controller gain K such that the closed system is stable.

16. Discrete-time State Space Systems

State Difference Equation

Similar to the state differential equations, we have state difference equation

$$x(k+1) = Ax(k) + Bu(k) \quad (194)$$

$$y(k) = Cx(k) + Du(k) \quad (195)$$

where $x(k)$ is the state vector at the step k.

System Response

The response can be evaluated repeatedly with the given initial values:

$$x(1) = Ax(0) + Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$

$$\vdots$$

$$x(k) = A^k x(0) + A^{k-1}Bu(0) + \ldots + Bu(k-1)$$

Controllability and Observability

The controllability and observability can be checked in the same ways as for the continuous-time systems. The system is controllable if the controllability matrix

$$P_c = [B \quad AB \quad \ldots \quad A^{n-1}B] \tag{196}$$

has full rank, and the system is observable if the observability matrix

$$P_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{197}$$

has full rank.

Transfer functions from state difference equations

Taking z-transform of $x(k+1) = Ax(k) + Bu(k)$, we have

$$\begin{aligned} (zI - A)X(z) &= BU(z) \\ X(z) &= (zI - A)^{-1}BU(z) \end{aligned} \tag{198}$$

and

$$Y(z) = CX(z) = C(zI - A)^{-1}BU(z) \tag{199}$$

Therefore the transfer function is given by

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B \tag{200}$$

Note that the transfer function is in the same form as for the continuous-time case, with the only difference that s is replaced by z. We would expect the same kind of state space realizations for discrete-time transfer functions, and this indeed is the case.

State space realization

The canonical forms corresponding to the transfer functions in s domain work in the same way for z domain.

Example 20. Obtain the state difference equation for the transfer function

$$G(z) = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678} \tag{201}$$

The realization in the controller canonical form is given by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.3678 & 1.3678 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{202}$$

$$y \quad = \quad [0.2644 \quad\quad 0.3678]x \tag{203}$$

Stability

The characteristic equation for the transfer function in z domain is given by

$$d(z) = |zI - A| = 0 \tag{204}$$

The system is stable if all the roots of the characteristic equation are within the unit circle.

Full State Feedback Control Design

Design the full state feedback control as

$$u(k) = -Kx(k) \tag{205}$$

which gives the closed-loop system

$$x(k+1) = (A - BK)x(k) \tag{206}$$

The full-state feedback control can be used to place the poles of the closed-loop system at the desired positions, if the system is controllable.

Example 21. Design the full state feedback control law for the system in Example 20 to place the poles at $\{-0.5, -0.5\}$.

17. Implementation of Digital Controllers

There are two methods for design a digital controller.

- Emulation method, ie, design the controller in continuous-time and then convert to discrete-time.
- Direct digital design, including discrete-time pole placement via full state feedback shown in the previous section.

Digital implementation of PID controllers

In s domain, we have the transfer function for a PID controller as

$$\frac{U(s)}{X(s)} = G_c(s) = k_1 + \frac{k_2}{s} + k_3 s \tag{207}$$

We need approximations for differentiation and integration in discrete-time.

Backward difference rule for differentiation

$$u(KT) = \frac{dx}{dt}|_{t=kT} = \frac{1}{T}(x(kT) - x((k-1)T) \tag{208}$$

The z-transform for this equation is given by

$$U(z) = \frac{1 - z^{-1}}{T} X(z) = \frac{z-1}{Tz} X(z) \tag{209}$$

Forward-rectangular integration

$$u(kT) = u((k-1)T) + Tx(kT) \tag{210}$$

and the z-transform gives

$$\frac{U(z)}{X(z)} = \frac{Tz}{z-1} \tag{211}$$

The z domain transfer function for the PID controller is given by

$$\frac{U(z)}{X(z)} = k_1 + \frac{k_2 T z}{z-1} + k_3 \frac{z-1}{Tz} \tag{212}$$

The difference equation for this transfer function is

$$\begin{aligned} u(kT) \quad &= \quad u((k-1)T) + (k_1 + k_2 T + \frac{k_3}{T})x(kT) \\ &\quad -(k_1 + \frac{2k_3}{T})x((k-1)T) + \frac{k_3}{T}x((k-2)T) \end{aligned} \tag{213}$$

This is called the velocity form of PID implementation, as the control input at the current step is calculated based on the control input in the previous step, and the contribution in the current step only contributes to the change of the control input.

Question: A PI controller is designed as $\frac{U(s)}{X(s)} = 10(1 + \frac{1}{s})$. Find the difference equation for controller implementation.

Hint: $\frac{U(z)}{X(z)} = \frac{10 - 9z^{-1}}{1 - z^{-1}}$.