Math 67201/47201

Solutions (Jan. 2017)

A1.

(i) Proof. Let $B_1 = A_1$, $B_2 = A_2 \setminus A_1, \cdots,$

$B_k = A_k \setminus \bigcup_{m=1}^{k-1} A_m, \cdots$ 2

Then $B_1, B_2, \cdots$ are disjoint and

$$A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$ 1

Hence,

$$P(A) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$$

$$\leq \sum_{n=1}^{\infty} P(A_n)$$ 2

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(ii) Proof.

$$P(A) \leq \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n = \frac{1}{8} \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^{n-1}$$ 2 1

$$= \frac{1}{8} \times \frac{1}{1-\frac{1}{8}} = \frac{1}{7}$$ 2

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A2.:

(i). Proof.

$$E[|X|^\alpha] = \int_\Omega |X|^\alpha dp = \int_{\{|X|\geq\varepsilon\}} |X|^\alpha dp + \int_{\{|X|<\varepsilon\}} |X|^\alpha dp \quad 2$$

$$\geq \int_{\{|X|\geq\varepsilon\}} |X|^\alpha dp \gneq \varepsilon^\alpha \int_{\{|X|\geq\varepsilon\}} 1 \, dp \quad 2$$

$$= \varepsilon^\alpha P(|X|\geq\varepsilon). \quad 1$$

This yields that

$$P(|X|\geq\varepsilon) \leq \frac{E[|X|^\alpha]}{\varepsilon^\alpha}$$

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(ii)

Proof. By the Chebyshev's inequality,

$$P(|X_n| \geq \frac{1}{n^\beta}) \leq \frac{E[|X_n|]}{\frac{1}{n^\beta}} \leq \frac{n^\beta}{n} \quad 3$$

$$= \frac{1}{n^{1-\beta}}$$

Hence.

$$\lim_{n\to\infty} P(|X_n| \geq \frac{1}{n^\beta}) \leq \lim_{n\to\infty} \frac{1}{n^{1-\beta}} = 0. \quad 2$$

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A.3

(i) We say that $\{Z_n, n \geq 0\}$ is a martingale w.r.t. $\{\mathcal{F}_n, n \geq 0\}$ if

(a) Z_n is $\mathcal{F}_n$-measurable, for $n \geq 0$, 1

(b) $E[|Z_n|] < +\infty$ 1

(c) $E[Z_{n+1} | \mathcal{F}_n] = Z_n$. 3

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(ii)

Proof of (a):

$$E[Z_{n+1} Z_n] = E[E[Z_{n+1} Z_n | \mathcal{F}_n]]$$ 1

$$= E[Z_n E[Z_{n+1} | \mathcal{F}_n]] = E[Z_n \cdot Z_n]$$ 1

$$= E[Z_n^2]$$ 1

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Proof of (b):

$$E[Z_{n+2} Z_n] = E[E[Z_{n+2} Z_n | \mathcal{F}_{n+1}]]$$ 1

$$= E[Z_n E[Z_{n+2} | \mathcal{F}_{n+1}]] = E[Z_n Z_{n+1}]$$ 1

2

A4

(i) Proof.

(a) Since Z_n is a function of $X_1, X_2, \cdots, X_n$, Z_n is $\mathcal{F}_n$-measurable.

(b) $E[|Z_n|] \leq \sum_{k=1}^{n} E[|X_k - \mu|] < +\infty$

(c) $E[Z_{n+1} | \mathcal{F}_n] = E[Z_n + X_{n+1} - \mu | \mathcal{F}_n]$

$= E[Z_n | \mathcal{F}_n] + E[X_{n+1} - \mu | \mathcal{F}_n]$

$= Z_n + E[X_{n+1} - \mu] = Z_n + 0 = Z_n$ 2

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(ii). Proof.

(a) Since M_n is a function of $X_1, X_2, \cdots, X_n$, M_n is also $\mathcal{F}_n$-measurable

(b) $E[|M_n|] \leq n E[\sum_{k=1}^{n} (X_k - \mu)^2] + E[\sum_{k=1}^{n} (X_k - \mu)^2]$

$= (n+1) \sum_{k=1}^{n} Var(X_k) < +\infty.$

(c) $E[M_{n+1} | \mathcal{F}_n] = E[(\sum_{k=1}^{n} (X_k - \mu) + (X_{n+1} - \mu))^2 | \mathcal{F}_n]$

$- E[\sum_{k=1}^{n} (X_k - \mu)^2 + (X_{k+1} - \mu)^2 | \mathcal{F}_n]$

$$= E\left[\left(\sum_{k=1}^{n}(X_k-\mu)\right)^2 + 2(X_{n+1}-\mu)\sum_{k=1}^{n}(X_k-\mu) + (X_{n+1}-\mu)^2 \,\middle|\, \mathcal{F}_n\right]$$ ⑥

$$- \sum_{k=1}^{n}(X_k-\mu)^2 - E\left[(X_{n+1}-\mu)^2 \,\middle|\, \mathcal{F}_n\right]$$ 2

$$= M_n + 2\sum_{k=1}^{n}(X_k-\mu)\, E\left[X_{n+1}-\mu \,\middle|\, \mathcal{F}_n\right]$$ 1

$$= M_n + 2\sum_{k=1}^{n}(X_k-\mu)\, E[(X_{k+1}-\mu)]$$

1

$$= M_n$$

$\underline{\underline{5}}$

B5. For $n \geq 1$,

(i) $\{T \leq n\} = \bigcup_{k=1}^{n} \{S_k = -a \text{ or } S_k = b\}$

$= \bigcup_{k=1}^{n} [\{S_k = -a\} \cup \{S_k = b\}] \in \mathcal{F}_n$ 3

$\{T = 0\} = \phi \in \mathcal{F}_n$. 1

Hence T is a stopping time. 4

(ii) Proof.

(a) Z_n is a function of $X_1, X_2, \cdots, X_n$. Hence Z_n is $\mathcal{F}_n$-measurable. 1

(b) $E[|Z_n|] \leq \left(\frac{q}{p}\right)^n + \left(\frac{q}{p}\right)^{-n} < +\infty$

(c) $E[Z_{n+1} | \mathcal{F}_n] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} | \mathcal{F}_n\right]$

$= E\left[\left(\frac{q}{p}\right)^{S_n} \cdot \left(\frac{q}{p}\right)^{X_{n+1}} | \mathcal{F}_n\right]$ 2

$= \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}} | \mathcal{F}_n\right]$ 2

$= Z_n \quad E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] = Z_n \left[\frac{q}{p} \times p + \left(\frac{q}{p}\right)^{-1} \cdot q\right]$ 1

$= Z_n$

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(iii).

Proof.

(a) Note that

$$S_{T\wedge n} = S_T I_{\{T\le n\}} + S_n I_{\{T>n\}}$$
$$= \sum_{k=1}^{n} S_k I_{\{T=k\}} + S_n I_{\{T>n\}}$$

Since each term on the right is $\mathcal{F}_n$- measurable, 1

$S_{T\wedge n}$ is $\mathcal{F}_n$- measurable.

As a function of $S_{T\wedge n}$, $Z_{T\wedge n}$ is also $\mathcal{F}_n$- measurable.

(b) Since $|S_{T\wedge n}| \le a+b$, we have

$$E[|Z_{T\wedge n}|] \le \left(\frac{q}{p}\right)^{a+b} + \left(\frac{q}{p}\right)^{-(a+b)} < +\infty$$

(c) Note that

$$Z_{T\wedge(n+1)} = Z_{T\wedge n} + I_{\{T\ge n+1\}} (Z_{n+1} - Z_n)$$

and $I_{\{T\ge n+1\}} = I_{\{T\le n\}^c}$ is $\mathcal{F}_n$- measurable. 2

We have.

$$E[Z_{T\wedge(n+1)} \mid \mathcal{F}_n] = E[Z_{T\wedge n} \mid \mathcal{F}_n]$$
$$+ E[I_{\{T\ge n+1\}} (Z_{n+1} - Z_n) \mid \mathcal{F}_n]$$ 1
$$= Z_{T\wedge n} + I_{\{T\ge n+1\}} \left(E[Z_{n+1} \mid \mathcal{F}_n] - Z_n\right) = Z_{T\wedge n}.$$

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(iv)

As $|Z_{T\wedge n}| \leq \left(\frac{q}{p}\right)^{a+b} + \left(\frac{q}{p}\right)^{-(a+b)}$,

We can apply Doob's Optional Stopping Theorem to get

$$E[Z_T] = E[Z_0] = \left(\frac{q}{p}\right)^{S_0} = \left(\frac{q}{p}\right)^0 = 1$$

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B6.

(i) We say that a portfolio ϕ is self-financing if

$$\phi(t)\cdot S(t) = \phi(t+1)\cdot S(t), \qquad t=0,\dots,T-1. \quad 2$$

This means that the investor adjusts his portfolio from $\phi(t)$ to $\phi(t+1)$ without bringing in or consuming any wealth. 2

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(ii)
$$\widetilde{V}_0(\phi) + \sum_{k=1}^{t} \phi(k)\cdot(\widetilde{S}(k) - \widetilde{S}(k-1))$$

$$= \phi(0)\cdot\widetilde{S}(0) + \sum_{k=1}^{t} \phi(k)\cdot\widetilde{S}(k) - \sum_{k=1}^{t} \phi(k)\widetilde{S}(k-1) \quad 1$$

$$= \phi(0)\cdot\widetilde{S}(0) + \phi(t)\cdot\widetilde{S}(t) + \sum_{k=1}^{t-1} \phi(k)\cdot\widetilde{S}(k) - \sum_{\ell=0}^{t-1} \phi(\ell+1)\cdot\widetilde{S}(\ell) \quad 2$$

$$= \phi(0)\cdot\widetilde{S}(0) - \phi(1)\cdot\widetilde{S}(0) + \sum_{k=1}^{t-1} (\phi(k)\cdot\widetilde{S}(k) - \phi(k+1)\cdot\widetilde{S}(k)) + \phi(t)\widetilde{S}(t) \quad 2$$

$$= \phi(t)\widetilde{S}(t) = \widetilde{V}_t(\phi). \quad 1$$

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~~(iii)~~

~~A probability measure Q is called ~~an~~ equivalent martingale probability measure if the discounted price process $\tilde{S}(t)$ is a martingale under Q and Q is equivalent to P.~~

(iii). We need to find the condition under which there exists a martingale probability measure Q.

Let $Q(\{\omega_1\}) = q$, $Q(\{\omega_2\}) = 1-q$.

In order that Q is a martingale probability measure we have.

$$E^Q[\tilde{S}_1(1)] = E^Q[\tilde{S}_1(0)] = \frac{S_1(0)}{S_0(0)} = \frac{6}{4} \qquad 3$$

But. $$E^Q[\tilde{S}_1(1)] = E^Q\left[\frac{S_1(1)}{S_0(1)}\right] = \frac{1}{4(1+r)} E^Q[S_1(1)]$$

$$= \frac{1}{4(1+r)} [5 \times Q(\{\omega_1\}) + 10 \times Q(\{\omega_2\})]$$

$$= \frac{1}{4(1+r)} [5q + 10(1-q)] \qquad 2$$

Hence, we have

$$\frac{1}{4(1+r)} [5q + 10(1-q)] = \frac{6}{4}. \qquad 1$$

Solve the above equation to get

$$q = \frac{10 - 6(1+r)}{5}$$ 1

$0 < q < 1$ gives

$$0 \leq r < \frac{2}{3}.$$ 2

The equivalent martingale measure is

$$Q(\{\omega_1\}) = \frac{10-6(1+r)}{5}$$

$$Q(\{\omega_2\}) = \frac{6(1+r)-5}{5}$$ 1

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B.7

(i)

$$E[\exp(aX)] = \sum_{k=0}^{\infty} \exp(ak)\, p(X=k)$$

$$= \sum_{k=0}^{\infty} (e^{a})^{k} \frac{e^{-\lambda}\lambda^{k}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{a})^{k}}{k!} \quad 3$$

$$= e^{-\lambda} \exp(\lambda e^{a})$$

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(ii).

(a) As a function of N_t, V_t is $\mathcal{F}_t$-measurable

(b). Since $N_t \sim \mathrm{Poiss}(\lambda t)$,

$$E[|V_t|] < +\infty.$$

(c) Let $s<t$, We have

$$E[V_t \mid \mathcal{F}_s] = E[\exp(-\theta N_t + \lambda t(1-e^{-\theta})) \mid \mathcal{F}_s]$$

$$= \exp(\lambda t(1-e^{-\theta}))\, E[\exp(-\theta N_t) \mid \mathcal{F}_s]$$

$$= \exp(\lambda t(1-e^{-\theta}))\, E[\exp(-\theta N_s) \cdot \exp(-\theta(N_t-N_s)) \mid \mathcal{F}_s]$$

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$$= \exp(\lambda t(1-e^{-\theta}))\, \exp(-\theta N_s)\, E[\exp(-\theta(N_t-N_s))]$$

$= \exp(\lambda t(1-e^{-\theta})) \exp(-\theta N_s) e^{-\lambda(t-s)} \exp(\lambda(t-s)e^{-\theta})$ 2

$= \exp(-\theta N_s + \lambda s(1-e^{-\theta})) = Y_s$

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(iii)

(a) Since Z_t is a functional of $B_u, u \le t$,

Z_t is $\mathcal{F}_t$- measurable

(b) As $B_t \sim N(0,t)$, we have 1

$E[|Z_t|] \le \underbrace{E[|B_t|^3] + 3\int_0^t E[|B_u|]\,du}_{<+\infty}$

(c) for $s<t$

$E[Z_t | \mathcal{F}_s] = E[B_t^3 - 3\int_0^t B_u du \,|\, \mathcal{F}_s]$

$= E[(B_t - B_s + B_s)^3 - 3\int_0^s B_u du - 3\int_s^t B_u du \,|\, \mathcal{F}_s]$ 3

$= E[(B_t - B_s)^3 + 3(B_t - B_s)^2 B_s + 3(B_t - B_s)B_s^2 \,|\, \mathcal{F}_s]$

$+ E[B_s^3 | \mathcal{F}_s]$

$- 3\int_0^s B_u du - E[3\int_s^t B_u du \,|\, \mathcal{F}_s]$ 1

$= E[(B_t - B_s)^3] + 3B_s E[(B_t - B_s)^2] + 3B_s^2 E[(B_t - B_s)]$

$+ B_s^3 - 3\int_0^s B_u du - 3(t-s)B_s - 3E[\int_s^t (B_u - B_s)du \,|\, \mathcal{F}_s]$ 2

$= Z_s - 3E[\int_s^t (B_u - B_s)du] = Z_s$ 1

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