Math37001

Solutions (Jan. 2017)

A1.

(i) Proof. Let $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $\cdots$,

$B_k = A_k \setminus \bigcup_{m=1}^{k-1} A_m$, $\cdots$, 2

Then $B_1, B_2, \cdots$ are disjoint and 1

$$A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

Hence,

$$P(A) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$$

2

$$\leq \sum_{n=1}^{\infty} P(A_n).$$

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(ii) Proof.

$$P(A) \leq \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \left(\tfrac{1}{8}\right)^n = \tfrac{1}{8} \sum_{n=1}^{\infty} \left(\tfrac{1}{8}\right)^{n-1}$$ 1

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$$= \frac{1}{8} \times \frac{1}{1-\frac{1}{8}} = \frac{1}{7}$$

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A2.

(i) We say that a random variable $Y = E[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$ if

(a) Y is $\mathcal{G}$-measurable, 2

(b) $\int_A Y\,dP = \int_A X\,dP$ for every $A \in \mathcal{G}$. 3

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(ii) Set $Y = P(B|A)I_A + P(B|A^c)I_{A^c}$.

We prove $Y = E[I_B|\mathcal{G}]$.

(a) Since I_A, I_{A^c} are $\mathcal{G}$-measurable, we see that as a function of I_A, I_{A^c}, Y is $\mathcal{G}$-measurable. 1

(b) On the other hand, we have

$$\int_A Y\,dP = \int_A P(B|A)I_A\,dP + \int_A P(B|A^c)I_{A^c}\,dP$$

$$= P(B|A)\int_A I_A\,dP = P(B|A)P(A) = P(B\cap A)$$

$$= \int_A I_B\,dP,$$ 1

$$\int_{A^c} Y\,dP = \int_{A^c} P(B|A)I_A\,dP + \int_{A^c} P(B|A^c)I_{A^c}\,dP$$

$$= \int_{A^c} P(B|A^c) I_{A^c} \, dP = P(B|A^c) \int_{A^c} I_{A^c} \, dP$$

$$= P(B|A^c)\, P(A^c) = P(A^c \cap B) = \int_{A^c} I_B \, dP,$$

$$\int_\Omega Y \, dP = E[Y] = P(B|A)\, E[I_A] + P(B|A^c)\, E[I_{A^c}]$$

$$= P(B|A)\, P(A) + P(B|A^c)\, P(A^c)$$

$$= P(A \cap B) + P(A^c \cap B) = P(B) = E[I_B]$$

$$= \int_\Omega I_B \, dP$$

$$\int_\phi Y \, dP = 0 = \int_\phi I_B \, dP$$

By the definition of conditional expectation $E[I_B | \mathcal{G}]$,

we conclude that

$$E[I_B | \mathcal{G}] = Y.$$

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A.3

(i) We say that $\{Z_n, n \geq 0\}$ is a martingale w.r.t. $\{\mathcal{F}_n, n \geq 0\}$ if

(a) Z_n is $\mathcal{F}_n$- measurable, for $n \geq 0$, 1

(b) $E[|Z_n|] < +\infty$ 1

(c) $E[Z_{n+1} | \mathcal{F}_n] = Z_n$. 3

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(ii)

<u>Proof of (a):</u>

$$E[Z_{n+1} Z_n] = E[E[Z_{n+1} Z_n | \mathcal{F}_n]]$$ 1

$$= E[Z_n E[Z_{n+1} | \mathcal{F}_n]] = E[Z_n \cdot Z_n]$$ 1

$$= E[Z_n^2]$$ 1

3

<u>Proof of (b):</u>

$$E[Z_{n+2} Z_n] = E[E[Z_{n+2} Z_n | \mathcal{F}_{n+1}]]$$ 1

$$= E[Z_n E[Z_{n+2} | \mathcal{F}_{n+1}]] = E[Z_n Z_{n+1}]$$ 1

2

A4

(i) Proof.

(a) Since Z_n is a function of $X_1, X_2, \cdots, X_n$, Z_n is $\mathcal{F}_n$-measurable. 1

(b) $E[|Z_n|] \leq \sum_{k=1}^{n} E[|X_k - \mu|] < +\infty$

(c) $E[Z_{n+1} | \mathcal{F}_n] = E[Z_n + X_{n+1} - \mu | \mathcal{F}_n]$ 1

$= E[Z_n | \mathcal{F}_n] + E[X_{n+1} - \mu | \mathcal{F}_n]$ 1

$= Z_n + E[X_{n+1} - \mu] = Z_n + 0 = Z_n$ 2

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(ii). Proof.

(a) Since M_n is a function of $X_1, X_2, \cdots, X_n$, M_n is also $\mathcal{F}_n$-measurable

(b) $E[|M_n|] \leq n E[\sum_{k=1}^{n} (X_k - \mu)^2] + E[\sum_{k=1}^{n} (X_k - \mu)^2]$ 1

$= (n+1) \sum_{k=1}^{n} Var(X_k) < +\infty$.

(c) $E[M_{n+1} | \mathcal{F}_n] = E[(\sum_{k=1}^{n} (X_k - \mu) + (X_{n+1} - \mu))^2 | \mathcal{F}_n]$

$- E[\sum_{k=1}^{n} (X_k - \mu)^2 + (X_{k+1} - \mu)^2 | \mathcal{F}_n]$ 1

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$$= E\left[\left(\sum_{k=1}^{n}(X_k-\mu)\right)^2 + 2(X_{n+1}-\mu)\sum_{k=1}^{n}(X_k-\mu) + (X_{n+1}-\mu)^2 \,\middle|\, \mathcal{F}_n\right]$$

$$- \sum_{k=1}^{n}(X_k-\mu)^2 - E\left[(X_{n+1}-\mu)^2 \,\middle|\, \mathcal{F}_n\right]$$ 2

$$= M_n + 2\sum_{k=1}^{n}(X_k-\mu)\, E\left[X_{n+1}-\mu \,\middle|\, \mathcal{F}_n\right]$$ 1

$$= M_n + 2\sum_{k=1}^{n}(X_k-\mu)\, E\left[(X_{k+1}-\mu)\right]$$

$$= M_n$$ 1

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B5. For $n \geq 1$,

(i) $\{T \leq n\} = \bigcup_{k=1}^{n} \{S_k = -a \text{ or } S_k = b\}$ 2

$= \bigcup_{k=1}^{n} [\{S_k = -a\} \cup \{S_k = b\}] \in \mathcal{F}_n$ 2

$\{T = 0\} = \phi \in \mathcal{F}_n$. 1

Hence T is a stopping time.

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(ii) Proof.

(a) Z_n is a function of $X_1, X_2, \cdots, X_n$. Hence Z_n is $\mathcal{F}_n$-measurable. 1

(b) $E[|Z_n|] \leq \left(\frac{q}{p}\right)^n + \left(\frac{q}{p}\right)^{-n} < +\infty$ 1

(c) $E[Z_{n+1} | \mathcal{F}_n] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} | \mathcal{F}_n\right]$

$= E\left[\left(\frac{q}{p}\right)^{S_n} \cdot \left(\frac{q}{p}\right)^{X_{n+1}} | \mathcal{F}_n\right]$ 2

$= \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}} | \mathcal{F}_n\right]$ 2

$= Z_n \; E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] = Z_n \left[\frac{q}{p} \times p + \left(\frac{q}{p}\right)^{-1} \cdot q\right]$ 3

$= Z_n$ 1

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(iii) Since $|S_{T\wedge n}| \le a+b$, we have

$$|Z_{T\wedge n}| \le \left(\frac{q}{p}\right)^{a+b} + \left(\frac{p}{q}\right)^{a+b}, \quad \forall n \ge 0.$$ 5

Hence we can use Doob's optional stopping theorem to deduce

$$E[Z_T] = E[Z_0] = 1$$

(iv) Note that

$$\langle 1 \rangle \quad P(S_T = -a) + P(S_T = b) = 1$$ 2

On the other hand, by (iii)

$$E[Z_T] = E\left[\left(\frac{q}{p}\right)^{S_T}\right]$$

$$\langle 2 \rangle. \quad = \left(\frac{q}{p}\right)^{-a} P(S_T = -a) + \left(\frac{q}{p}\right)^{b} P(S_T = b) = 1$$ 4

Solve $\langle 1 \rangle$ and $\langle 2 \rangle$ to obtain

$$P(S_T = -a) = \frac{1 - \left(\frac{q}{p}\right)^{b}}{\left(\frac{q}{p}\right)^{-a} - \left(\frac{q}{p}\right)^{b}}$$ 4

$$P(S_T = b) = \frac{1 - \left(\frac{q}{p}\right)^{-a}}{\left(\frac{q}{p}\right)^{b} - \left(\frac{q}{p}\right)^{-a}}.$$

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B6.

(i) We say that a portfolio ϕ is self-financing if

$$\phi(t)\cdot S(t) = \phi(t+1)\cdot S(t), \qquad t=0,\cdots,T-1. \qquad 2$$

This means that the investor adjusts his portfolio from $\phi(t)$ to $\phi(t+1)$ without bringing in or consuming any wealth. 3

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(ii)
$$\tilde{V}_0(\phi) + \sum_{k=1}^{t} \phi(k)\cdot(\tilde{S}(k) - \tilde{S}(k-1))$$

$$= \phi(0)\cdot\tilde{S}(0) + \sum_{k=1}^{t} \phi(k)\cdot\tilde{S}(k) - \sum_{k=1}^{t} \phi(k)\tilde{S}(k-1) \qquad 3$$

$$= \phi(0)\cdot\tilde{S}(0) + \phi(t)\cdot\tilde{S}(t) + \sum_{k=1}^{t-1} \phi(k)\cdot\tilde{S}(k) - \sum_{\ell=0}^{t-1} \phi(\ell+1)\cdot\tilde{S}(\ell) \qquad 4$$

$$= \phi(0)\cdot\tilde{S}(0) - \phi(1)\cdot\tilde{S}(0) + \sum_{k=1}^{t-1} (\phi(k)\cdot\tilde{S}(k) - \phi(k+1)\cdot\tilde{S}(k)) + \phi(t)\tilde{S}(t) \qquad 2$$

$$= \phi(t)\tilde{S}(t) = \tilde{V}_t(\phi). \qquad 1$$

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(iii)

A probability measure Q is called ~~an~~ equivalent martingale probability measure if the discounted price process $\tilde{S}(t)$ is a martingale under Q. and Q is equivalent to P.

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(iv). We need to find the condition under which there exists a martingale probability measure Q.

Let $Q(\{\omega_1\}) = q$, $Q(\{\omega_2\}) = 1-q$.

In order that Q is a martingale probability measure we have.

$$E^Q[\tilde{S}_1(1)] = E^Q[\tilde{S}_1(0)] = \frac{S_1(0)}{S_0(0)} = \frac{6}{4}$$

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But.

$$E^Q[\tilde{S}_1(1)] = E^Q\left[\frac{S_1(1)}{S_0(1)}\right] = \frac{1}{4(1+r)} E^Q[S_1(1)]$$

$$= \frac{1}{4(1+r)}\left[5 \times Q(\{\omega_1\}) + 10 \times Q(\{\omega_2\})\right]$$

$$= \frac{1}{4(1+r)}\left[5q + 10(1-q)\right]$$

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Hence, we have

$$\frac{1}{4(1+r)}\left[5q + 10(1-q)\right] = \frac{6}{4}.$$

1

Solve the above equation to get

$$q = \frac{10 - 6(1+r)}{5}$$

2

$0 < q < 1$ gives

$$0 \leq r < \frac{2}{3}.$$

The equivalent martingale measure is

$$Q(\{\omega_1\}) = \frac{10-6(1+r)}{5}$$

$$Q(\{\omega_2\}) = \frac{6(1+r)-5}{5}$$

1

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