Solutions to Math37001/67001 exam (January, 2014)

A.1:

(i) (b) is correct. 5

(ii). Since $A^c = \bigcup_{n=1}^{\infty} A_n^c$, we have 2

$$P(A^c) = P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \le \sum_{n=1}^{\infty} P(A_n^c) = 0. \quad 2$$

Hence $P(A) = 1 - P(A^c) = 1$. 1

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A.2

(i) We say that $\{Z_n, n \ge 0\}$ is a martingale w.r.t. $\{\mathcal{F}_n, n \ge 0\}$ if

(a) Z_n is $\mathcal{F}_n$-measurable 1

(b) $E[|Z_n|] < +\infty$, 1

(c) $E[Z_{n+1} | \mathcal{F}_n] = Z_n$. 2

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(ii)

(a) $E[Z_j Z_i] = E[\, E[Z_j Z_i \mid \mathcal{F}_i]\,]$ 1

$= E[Z_i\, E[Z_j \mid \mathcal{F}_i]\,] = E[Z_i \cdot Z_i] = E[Z_i^2]$

1

(2)

(b) $E[(Z_{n+1}-Z_n)^2 \mid \mathcal{F}_n] = E[Z_{n+1}^2 - 2Z_nZ_{n+1} + Z_n^2 \mid \mathcal{F}_n]$ 1

$= E[Z_{n+1}^2 \mid \mathcal{F}_n] - 2E[Z_nZ_{n+1} \mid \mathcal{F}_n] + Z_n^2$ 1

$= E[Z_{n+1}^2 \mid \mathcal{F}_n] - 2Z_n E[Z_{n+1} \mid \mathcal{F}_n] + Z_n^2$ 1

$= E[Z_{n+1}^2 \mid \mathcal{F}_n] - 2Z_n^2 + Z_n^2 = E[Z_{n+1}^2 \mid \mathcal{F}_n] - Z_n^2.$ 1

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<u>A.3</u>:

(i) (a) Z_n is $\mathcal{F}_n$-measurable because Z_n is a function of $X_1, X_2, \dots, X_n$

(b) $|Z_n| \le |S_n| + |\mu| n$ 1

$\le |X_1| + \dots + |X_n| + n|\mu|$

$\Rightarrow E[|Z_n|] \le \sum_{i=1}^{n} E[|X_i|] + n|\mu| < +\infty$

(c) $E[Z_{n+1} \mid \mathcal{F}_n] = E[S_{n+1} - (n+1)\mu \mid \mathcal{F}_n]$

$= E[S_n + X_{n+1} - n\mu - \mu \mid \mathcal{F}_n]$ 2

$= E[Z_n + X_{n+1} - \mu \mid \mathcal{F}_n]$

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$$= E[Z_n \mid \mathcal{F}_n] + E[X_{n+1} - \mu \mid \mathcal{F}_n]$$ 1

$$= Z_n + E[X_{n+1}] - \mu = Z_n + \mu - \mu = Z_n$$

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(ii)

(a) As a function of $X_1, X_2, \dots, X_n$, Y_n is $\mathcal{F}_n$-measurable. 1

(b).

$$E[Y_n] = E[e^{X_1}] \cdots E[e^{X_n}]\, e^{-n\mu - \frac{1}{2}\sigma^2 n} < +\infty.$$ 1

(c) $$E[Y_{n+1} \mid \mathcal{F}_n] = E[e^{S_{n+1} - (n+1)\mu - \frac{1}{2}\sigma^2 (n+1)} \mid \mathcal{F}_n]$$

$$= E[e^{S_n + X_{n+1} - n\mu - \mu - \frac{1}{2}\sigma^2 n - \frac{1}{2}\sigma^2} \mid \mathcal{F}_n]$$

$$= E[e^{S_n - n\mu - \frac{1}{2}n\sigma^2} e^{-\mu - \frac{1}{2}\sigma^2} e^{X_{n+1}} \mid \mathcal{F}_n]$$ 2

$$= Y_n\, E[e^{X_{n+1}}]\, e^{-\mu - \frac{1}{2}\sigma^2}$$ 1

$$= Y_n\, e^{\mu + \frac{1}{2}\sigma^2} \cdot e^{-\mu - \frac{1}{2}\sigma^2} = Y_n.$$ 1

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A.4

(i) We say that ϕ is self-financing if

$$S(t)\cdot\phi(t) = \phi(t+1)\cdot S(t).$$ 3

This means that the investor adjusts his strategy at time t from $\phi(t)$ to $\phi(t+1)$ without changing the total wealth. 2

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(ii) The value process is given by

$$V_\phi(t) = \sum_{i=0}^{d} \phi_i(t)\, S_i(t) = \phi(t)\cdot S(t).$$ 2

We say that ϕ is an ~~a~~ arbitrage opportunity if $V_\phi(0)=0$, $V_\phi(T)\geq 0$ and $P(V_\phi(T)>0)>0$.

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B.5 (5)

(i) As a function of $X_1, X_2, \dots, X_n$, Z_n is $\mathcal{F}_n$-measurable

(a) 1

(b) $E[|Z_n|] \leq \left(\frac{1-p}{p}\right)^n + \left(\frac{1-p}{p}\right)^{-n} < +\infty$ 1

(c) $E[Z_{n+1} \mid \mathcal{F}_n] = E\left[\left(\frac{1-p}{p}\right)^{S_{n+1}} \mid \mathcal{F}_n\right]$

$= E\left[\left(\frac{1-p}{p}\right)^{S_n} \left(\frac{1-p}{p}\right)^{X_{n+1}} \mid \mathcal{F}_n\right]$ 2

$= Z_n \, E\left[\left(\frac{1-p}{p}\right)^{X_{n+1}}\right]$ 1

$= Z_n \times \left\{ \frac{1-p}{p} p(X_{n+1}=1) + \left(\frac{p}{1-p}\right) p(X_{n+1}=-1) \right\}$ 1

$= Z_n \times \left\{ \frac{1-p}{p} \times p + \frac{p}{1-p} \times 1-p \right\} = Z_n$ 1

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(ii)

As $|Z_{T \wedge n}| \leq \max_{a \leq l \leq b} \left(\frac{1-p}{p}\right)^l = K$

for all $n \geq 0$, 4

By Doob's optional Stopping Theorem, we have

$E[Z_T] = E[Z_0] = 1$ 3

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(iii)

Note that

$$E[Z_T] = E\left[\left(\frac{1-p}{p}\right)^{S_T}\right]$$

$$= \cancel{E}\left(\frac{1-p}{p}\right)^a p(T_a < T_b) + \left(\frac{1-p}{p}\right)^b p(T_b < T_a) \quad 4$$

$$= 1 \qquad (1)$$

We also have

$$p(T_a < T_b) + p(T_b < T_a) = 1 \quad (2) \qquad 2$$

Solve (1) and (2) to get

$$p(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}, \quad 4$$

where $\varphi(x) = \left(\frac{1-p}{p}\right)^x$.

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(iv). If $\frac{1}{2} < p < 1$, then $\frac{1-p}{p} < 1$.

Thus

$$\lim_{b \to +\infty} \varphi(b) = \lim_{b \to \infty} \left(\frac{1-p}{p}\right)^b = 0 \qquad 1$$

and hence,

$$P(T_a < +\infty)$$

$$= \lim_{b \to +\infty} P(T_a < T_b) = \lim_{b \to +\infty} \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}$$

$$= \frac{\varphi(0)}{\varphi(a)} = \frac{1}{\left(\frac{1-p}{p}\right)^a} = \left(\frac{1-p}{p}\right)^{-a}.$$ 2

On the other hand,

$$P(T_b < T_a) = 1 - \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}$$

$$= \frac{\varphi(0) - \varphi(a)}{\varphi(b) - \varphi(a)}$$

$$= \cancel{\varphi(a)} \frac{\frac{\varphi(0)}{\varphi(a)} - 1}{\frac{\varphi(b)}{\varphi(a)} - 1}$$ 2

As $\lim_{a \to -\infty} \varphi(a) = \lim_{a \to -\infty} \left(\frac{1-p}{p}\right)^a = +\infty$, we obtain

$$P(T_b < +\infty) = \lim_{a \to -\infty} P(T_b < T_a)$$

$$= \lim_{a \to -\infty} \frac{\frac{\varphi(0)}{\varphi(a)} - 1}{\frac{\varphi(b)}{\varphi(a)} - 1} = 1$$ 1

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B.6

(i) Suppose a martingale probability measure P^* exists. Then the discounted value process $\tilde{V}_\phi(t) = \frac{1}{S_0(t)} V_\phi(t)$ is a martingale 1

under P^*.

If $V_\phi(0) = 0$, $V_\phi(T) \geq 0$, then $\tilde{V}_\phi(0) = 0$, 2

$\tilde{V}_\phi(T) \geq 0$. By the property of martingale,

$$E^{P^*}[\tilde{V}_\phi(T)] = E^{P^*}[\tilde{V}_\phi(0)] = 0.$$ 3

This implies that $\tilde{V}_\phi(T) = 0$ almost surely.

Hence $V_\phi(T) = 0$ a.s. The market is free of 2

arbitrage.

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(ii) In order that $\tilde{S}_1(t) = \frac{1}{S_0(t)} S_1(t)$ is a martingale, we must have

$$E^{P}[\tilde{S}_1(t+1) \mid \mathcal{F}_t] = \tilde{S}_1(t).$$ 3

i.e,

$$E\left[\frac{S_1(t+1)}{S_0(t+1)} \,\middle|\, \mathcal{F}_t \right] = \frac{S_1(t)}{S_0(t)}$$

$$E\left[\frac{Z(1)\cdots Z(t)Z(t+1)}{(1+r)^{t+1}} \,\middle|\, \mathcal{F}_t \right] = \frac{Z(1)\cdots Z(t)}{(1+r)^t} \qquad 2$$

Divide both sides by $\tilde{S}_1(t)$,

$$E[\, Z(t+1) \mid \mathcal{F}_t] = 1+r \qquad 3$$

By the independence, we ~~have~~ deduce that

$$E[Z(t+1)] = 1+r. \qquad 1$$

Note that $E[Z(t+1)] = up + \ell(1-p)$. 2

Hence, we have

$$up + \ell(1-p) = 1+r \qquad 1$$

Solve this equation to get.

$$p = \frac{1+r-\ell}{u-\ell}. \qquad 1$$

~~In order to p~~

$0 < p < 1$ implies that

$$\ell < 1+r < u. \qquad 1$$

(iii).

$$S_1(T) = Z(1) \cdots Z(T).$$

The probability law of $S_1(T)$ is given by

$$P(S_1(T) = u^k \ell^{T-k}) = \binom{T}{k} p^k (1-p)^{T-k},$$

4

$k = 0, 1, \cdots, T, \qquad p = \frac{1+r-\ell}{u-\ell}.$

The price of the option is

$$\Pi = E\left[\frac{X}{S_0(T)}\right] = E\left[\frac{1}{(1+r)^T} \exp(S_1(T))\right]$$

2

$$= \frac{1}{(1+r)^T} \sum_{k=0}^{T} \exp(u^k \ell^{T-k}) \binom{T}{k} p^k (1-p)^{T-k}.$$

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