Heme

Solutions to Math 7001 ( January, 2013 ) ①

A1.

~~If~~

~~In~~ (i) Monotone Convergence Theorem

Let $\{X_n, n\geq 1\}$ be a sequence of random variables such that $0\leq X_1 \leq X_2 \leq X_3 \leq \cdots$ and

b).

$$X_n \longrightarrow X \quad \text{as} \quad n\to\infty.$$

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Then

$$E[X_n]=\int_\Omega X_n(\omega)\,dP \longrightarrow \int_\Omega X(\omega)\,dP \quad \text{as } n\to+\infty$$

A2.

(ii)

a) If $\omega\notin B_n$, then $X_n(\omega)=0$ and hence $X_n(\omega)\leq X_{n+1}(\omega)$.

If $\omega\in B_n$, then $X_n(\omega)=1$. But $\omega\in B_n\subset B_{n+1}$, we also have $X_{n+1}(\omega)=1$. Thus $X_n(\omega)\leq X_{n+1}(\omega)$.

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In all the cases, we have $X_n(\omega)\leq X_{n+1}(\omega)$.

If $\omega\notin B$, then $\omega\notin B_n$ for all $n\geq 1$, In this case $X_n(\omega)=X(\omega)=0$.

If $\omega\in B$, then there exists $N\geq 1$ such that $\omega\in B_N$

(3)

(ii) ~~Let~~ Proof.

(a) As a function of $I_A$ and $I_{A^c}$,

$P(B|A) 1_A + P(B|A^c) 1_{A^c}$ is $\mathcal{G}$-measurable.

(b). Now.

$$\int_A \left( P(B|A) 1_A + P(B|A^c) 1_{A^c} \right) dP$$

$$= P(B|A) \int_A 1_A \, dP = P(B|A) P(B)$$

$$= P(B \cap A) = \int_A 1_B \, dP = \int_A X \, dP$$

2

$$\int_{A^c} \left( P(B|A) 1_A + P(B|A^c) I_{A^c} \right) dP$$

$$= P(B|A^c) \int_{A^c} I_{A^c} \, dP = P(B|A^c) P(A^c)$$

$$= P(B \cap A^c) = \int_{A^c} I_B \, dP = \int_{A^c} X \, dP$$

2

$$\int_\Omega \left( P(B|A) I_A + P(B|A^c) I_{A^c} \right) dP$$

$$= P(B|A) P(A) + P(B|A^c) P(A^c)$$

$$= P(B\cap A) + P(B\cap A^c) = P(B) = \int_\Omega X\,dP$$

1

$$\int_\phi \left( P(B|A) I_A + P(B|A^c) I_{A^c} \right) dP = \int_\phi X\,dP = 0$$

From the definition, we conclude

$$E[X|\mathcal{G}] = P(B|A) I_A + P(B|A^c) I_{A^c}$$

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A3.

(i)

(1) Since $S_n$ is a function of $X_1, \cdots X_n$, $S_n$ is $\mathcal{F}_n$- measurable.

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(2) $|S_n| \le n$, $E[|S_n|] < +\infty$.

(3) $E[S_{n+1} | \mathcal{F}_n] = E[S_n + X_{n+1} | \mathcal{F}_n]$

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$= E[S_n | \mathcal{F}_n] + E[X_{n+1} | \mathcal{F}_n]$

$= S_n + E[X_{n+1}] = S_n + \frac{1}{2} - \frac{1}{2} = S_n$.

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(ii) ⑤

(1) As a function of $X_1, X_2, \cdots, X_n$, $Z_n$ is $\mathcal{F}_n$-measurable

1

(2) $|Z_n| \leq n^2+n, \Rightarrow E[|Z_n|] < +\infty$

(3) $E[Z_{n+1} | \mathcal{F}_n] = E[S_{n+1}^2 - (n+1) | \mathcal{F}_n]$

$= E[(S_n + X_{n+1})^2 - (n+1) | \mathcal{F}_n]$

2

$= E[S_n^2 + 2X_{n+1}S_n + X_{n+1}^2 - n - 1 | \mathcal{F}_n]$

$= S_n^2 - n - 1 + 2E[X_{n+1}S_n | \mathcal{F}_n] + E[X_{n+1}^2 | \mathcal{F}_n]$

1

$= Z_n - 1 + 2S_n E[X_{n+1}] + E[X_{n+1}^2]$ 1

$= Z_n - 1 + 2S_n \times 0 + (\frac{1}{2} \times 1 + \frac{1}{2} \times 1)$

1

$= Z_n.$

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(6)

A4

(i) $\phi$ is said to be self-financing if 3

$$\phi(t)\cdot S(t) = \phi(t+1)\cdot S(t)$$

This means that the investor adjusts his portfolio from $\phi(t)$ to $\phi(t+1)$ without bringing in or consuming any wealth. 2

(ii) A market is said to be complete if every bounded claim is attainable. 2 A market is complete if there exists a unique equivalent probability measure. 3

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B5

(i) $\{Z_n, n \geq 0\}$ is said to be a martingale if

(1) $Z_n$ is $\mathcal{F}_n$- measurable, 1

(2) $E[|Z_n|] < +\infty$, 1

(3) $E[Z_{n+1} | \mathcal{F}_n] = Z_n$. 3

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(iii) Proof.

(1) As a function of $X_1, \cdots X_n$, $Z_n$ is $\mathcal{F}_n$- measurable.

(2) $|Z_n| \leq \phi(\theta)^{-n} e^{\theta n} \Rightarrow E[|Z_n|] < +\infty$ 1

(3) $E[Z_{n+1} | \mathcal{F}_n] = E[\phi(\theta)^{-n-1} e^{\theta S_{n+1}} | \mathcal{F}_n]$

$= E[e^{\theta S_n} e^{\theta X_{n+1}} | \mathcal{F}_n] \phi(\theta)^{-n-1}$ 2

$= \phi(\theta)^{-n-1} e^{\theta S_n} E[e^{\theta X_{n+1}} | \mathcal{F}_n]$ 1

$= Z_n \phi(\theta)^{-1} E[e^{\theta X_{n+1}}]$

$= Z_n \phi(\theta)^{-1} \phi(\theta) = Z_n$ 1

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(iii) For any $n \geq 1$, (8)

$$\{T \leq n\} = \bigcup_{k=1}^{n} \{S_k = -a \text{ or } S_k = b\} \quad 3$$

As $\{S_k = -a \text{ or } S_k = b\} \in \mathcal{F}_n$, for $k \leq n$, 1

we have $\{T \leq n\} \in \mathcal{F}_n$. 1

It follows from the definition that $T$ is a stopping time

$\underline{5}$

(iv) We know from (ii), $Z_n = \phi(\theta)^{-n} e^{\theta S_n}$, $n \geq 1$, is a martingale. On the other hand,

$$|Z_{n \wedge T}| = \phi(\theta)^{-(n \wedge T)} e^{\theta S_{n \wedge T}}$$

$$\leq 1 \times \left(e^{\theta a} + e^{\theta b}\right) \quad \text{for all } n \geq 1. \quad 4$$

By Doob's optional stopping theorem, we conclude that

$$E[Z_T] = E[\phi(\theta)^{-T} e^{\theta S_T}] = E[Z_0] = 1.$$

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$\underline{5}$

(v) As $\phi(\theta_0) = 1$, $E[\phi(\theta_0)^{-T} e^{\theta_0 S_T}] = 1$

reduces to

(*) $E[e^{\theta_0 S_T}] = 1$ 4

But $E[e^{\theta_0 S_T}] = e^{\theta_0 b} P(S_T = b) + e^{-\theta_0 a} P(S_T = -a)$

Thus we have

(**) $e^{\theta_0 b} P(S_T = b) + e^{-\theta_0 a} P(S_T = -a) = 1$ 3

Note that

(***) $P(S_T = b) + P(S_T = -a) = 1$ 2

Solve (**), (***) to get

$$P(S_T = -a) = \frac{e^{\theta_0 b} - 1}{e^{\theta_0 b} - e^{-\theta_0 a}}$$ 1

$$P(S_T = b) = \frac{1 - e^{-\theta_0 a}}{e^{\theta_0 b} - e^{-\theta_0 a}}$$

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(10)

B2

(i) The value process of a portfolio $\phi$ is 2

$$V_\phi(t) = \phi(t) \cdot S(t) = \phi_1(t) S_1(t) + \phi_2(t) S_2(t).$$

We say that $\phi$ is an arbitrage opportunity if

$$V_\phi(0) = 0, \quad V_\phi(T) \geq 0 \quad \text{and} \quad P(V_\phi(T) > 0) > 0.$$ 3

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(ii) Assume that an equivalent martingale probability $Q$ exists. For any portfolio $\phi$ with $V_\phi(0) = 0$, $V_\phi(T) \geq 0$, we also have 1

$$\widetilde{V}_\phi(0) = \frac{1}{S_0(0)} V_\phi(0) = 0, \quad \widetilde{V}_\phi(T) = \frac{1}{S_0(T)} V_\phi(T) \geq 0.$$

Since $\widetilde{V}_\phi$ is a martingale under $Q$, we must have

$$E_Q[\widetilde{V}_\phi(T)] = E_Q[\widetilde{V}_\phi(0)] = 0.$$ 3

This implies $\widetilde{V}_\phi(T) = \frac{1}{S_0(T)} V_\phi(T) = 0$ a.s.

Therefore, it is not possible to have arbitrage opportunities. 1

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(iii). We need to determine the condition on $r$ so that an equivalent martingale probability $Q$ exists. Suppose $q = Q(\{\omega_1\})$, $q_1 = Q(\{\omega_2\}) = 1-q$. 1

Under the equivalent probability measure $Q$, the discounted price process $\widetilde{S}_1(t) = \frac{S_1(t)}{S_0(t)}$ is a martingale. In particular, we have

$$(1) \quad E_Q[\widetilde{S}_1(T)] = E_Q[\widetilde{S}_1(0)] = E_Q\left[\frac{S_1(0)}{S_0(0)}\right] = \frac{15}{10}$$

3

On the other hand,

$$E_Q[\widetilde{S}_1(T)] = E_Q\left[\frac{S_1(T)}{S_0(T)}\right]$$

$$= \frac{1}{10(1+r)} E_Q[S_1(T)]$$

$$= \frac{1}{10(1+r)} \{ S_1(T,\omega_1)\, q + S_1(T,\omega_2)(1-q) \}$$

$$= \frac{1}{10(1+r)} \{ 12q + 30(1-q) \}$$

2

(1) Becomes

$$\frac{1}{10(1+r)}\left\{12q + 30(1-q)\right\} = \frac{15}{10} \qquad 1$$

Solve the above equation to get

$$q = \frac{15(1-r)}{18} \qquad 1$$

In order that Q is an equivalent probability we must have

$$0 < q = \frac{15(1-r)}{18} < 1 \qquad 2$$

This gives

$$0 \leq r < 1.$$

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(iv) An European put option gives one the right to sell ~~at~~ on a specified date the expiry date, at a specified price K, the strike price. 3

The payoff is

$$X = (K - S_1(T))^+$$

$$= \begin{cases} K - S_1(T) & \text{if } S_1(T) \leq K, \\ 0 & \text{if } S_1(T) > K. \end{cases} \qquad 2$$

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(V) In this case, the equivalent martingale probability is given by

$$Q(\{\omega_1\}) = q = \frac{15(1-r)}{18} = \frac{5(1-r)}{6} = \frac{5(1-\frac{3}{5})}{6}$$

$$= \frac{1}{3}$$

2

$$Q(\{\omega_2\}) = q_1 = 1-q = \frac{2}{3}.$$

The price is given by

$$\Pi = S_0(0)\, E_Q\left[\frac{X}{S_0(T)}\right]$$

$$= 10 \times \frac{1}{10(1+r)} E_Q\left[(K - S_1(T))^+\right]$$

3

$$= \frac{1}{1+\frac{3}{5}} \left\{ (18-12) \times \frac{1}{3} + 0 \times \frac{2}{3} \right\}$$

$$= \frac{5}{4}.$$

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