Solutions to Math 37001 (Jan. 2012)

A.1

(i) $P(A) \leq \sum_{n=1}^{\infty} P(B_n) = \sum_{n=1}^{\infty} \frac{1}{8^n}$

$$= \frac{1}{8}\sum_{n=1}^{\infty} \frac{1}{8^{n-1}} = \frac{1}{8} \times \frac{1}{1-\frac{1}{8}} = \frac{1}{8} \times \frac{8}{7} = \frac{1}{7}$$

(ii) Set $Y_n = \sum_{j=1}^{n} 1_{A_j}$. Then $Y_n \uparrow Y$

By Monotone convergence theorem,

$$E[Y] = \lim_{n\to\infty} E[Y_n] = \lim_{n\to\infty} \sum_{j=1}^{n} E[1_{A_j}]$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} P(A_j) = \sum_{j=1}^{\infty} P(A_j)$$

A.2

(i) We say that a random variable Y is the conditional expectation of X given $\mathcal{G}$ if

(1) Y is $\mathcal{G}$ - measurable

(2) $\int_A Y\,dP = \int_A X\,dP$ for any $A \in \mathcal{G}$.

(ii) Put $$Y = a^2 1_A + \frac{b^2 P_2 + c^2 P_3}{P_2 + P_3} 1_{A^c}.$$

As a function of 1_A and 1_{A^c}, Y is $\mathcal{G}$-measurable.

On the other hand,

$$\int_A Y \, dP = a^2 \int_A 1_A \, dP = a^2 \int_A dP$$

$$= \int_A X^2 \, dP$$

and

$$\int_{A^c} Y \, dP = \frac{b^2 P_2 + c^2 P_3}{P_2 + P_3} \int_{A^c} 1_{A^c} \, dP$$

$$= \frac{b^2 P_2 + c^2 P_3}{P_2 + P_3} P(A^c) = \frac{b^2 P_2 + c^2 P_3}{P_2 + P_3} (P_2 + P_3)$$

$$= b^2 P_2 + c^2 P_3 = \int_{\{X=b\}} X^2 \, dP + \int_{\{X=c\}} X^2 \, dP$$

$$= \int_{A^c} X^2 \, dP$$

$$\int_\Omega Y \, dP = a^2 P_1 + b^2 P_2 + c^2 P_3 = \int_\Omega X^2 \, dP$$

$$\int_\phi Y \, dP = \int_\phi X^2 \, dP = 0$$

Therefore, $Y = E[X^2 | \mathcal{G}]$ by definition.

A.3

(i) $\{Z_n, n \geq 0\}$ is a martingale w.r.t $\{\mathcal{F}_n, n \geq 0\}$

if

(1) Z_n is $\mathcal{F}_n$- measurable,

(2) $E[|Z_n|] < +\infty$,

(3) $E[Z_{n+1} | \mathcal{F}_n] = Z_n$.

(ii)

(a) $E[Z_{n+2} | \mathcal{F}_n] = E[\, E[Z_{n+2} | \mathcal{F}_{n+1}] \,|\, \mathcal{F}_n]$

$= E[Z_{n+1} | \mathcal{F}_n] = Z_n$.

(b) $E[(Z_{j+1} - Z_j) X]$

$= E[\, E[(Z_{j+1} - Z_j) X \,|\, \mathcal{F}_j]]$

$= E[X \, E[Z_{j+1} - Z_j \,|\, \mathcal{F}_j]]$

$= E[X \times 0] = 0$

A.4

(i) The value process is

$$V_\phi(t) = \phi(t) \cdot S(t) = \phi_0(t) S_0(t) + \phi_1(t) S_1(t) + \cdots + \phi_d(t) S_d(t)$$ 2

We say that ϕ is an arbitrage opportunity if

$$V_\phi(0) = 0, \quad V_\phi(T) \geq 0 \text{ and } P(V_\phi(T) > 0) > 0.$$ 2

4

P^* is said to be an equivalent martingale probability if the discounted price is a martingale under P^*. 1

(ii) Suppose that an equivalent martingale probability measure P^* exists. and suppose that there exists an arbitrage opportunity ϕ. Then discounted value process $\tilde{V}_\phi(t) = \frac{1}{S_0(t)} V_\phi(t)$ satisfies

$$\tilde{V}_\phi(0) = 0, \quad \tilde{V}_\phi(T) \geq 0, \quad P^*(\tilde{V}_\phi(T) > 0) > 0.$$ 2

But $\tilde{V}_\phi$ is a martingale under P^*. Therefore,

$$E^{P^*}[\tilde{V}_\phi(T)] = E^{P^*}[\tilde{V}_\phi(0)] = 0.$$ 2

This implies $\tilde{V}_\phi(T) = 0$ P^*-a.s., which is a contradiction.

16

B.5

(i) S_n is $\mathcal{F}_n$-measurable because S_n is a function of $X_1, X_2, \cdots, X_n$.

$E[|S_n|] \le K+n < +\infty$ 1

$E[S_{n+1} | \mathcal{F}_n] = E[S_n + X_{n+1} | \mathcal{F}_n]$ 1

$= S_n + E[X_{n+1} | \mathcal{F}_n] = S_n + E[X_{n+1}]$ 1

$= S_n + (\frac{1}{2} - \frac{1}{2}) = S_n$ 2

So $\{S_n, n \ge 0\}$ is a martingale.

5

(ii) (1) Y_n is $\mathcal{F}_n$-measurable because Y_n is a function of $X_1, X_2, \cdots, X_n$. 1

(2) $E[|Y_n|] \le (K+n)^2 + n < +\infty$

(3) $E[Y_{n+1} | \mathcal{F}_n] = E[S_{n+1}^2 - (n+1) | \mathcal{F}_n]$

$= E[(S_n + X_{n+1})^2 - (n+1) | \mathcal{F}_n]$ 1

$$= E[\, S_n^2 + 2 S_n X_{n+1} + X_{n+1}^2 - (n+1) \mid \mathcal{F}_n]$$

$$= E[\, Y_n + 2 S_n X_{n+1} + X_{n+1}^2 - 1 \mid \mathcal{F}_n]$$ 1

$$= Y_n + 2 S_n E[X_{n+1} \mid \mathcal{F}_n] + E[X_{n+1}^2] - 1$$ 1

$$= Y_n + 2 S_n E[X_{n+1}] + \left(1 \times \tfrac{1}{2} + 1 \times \tfrac{1}{2} \right) - 1$$ 1

$$= Y_n + 0 = Y_n .$$ 5

$\{ Y_n \}$ is a martingale.

(iii). Because $|S_{T \wedge n}| \leq N$, ~~and~~ by Doob's optional stopping theorem,

$$E[S_T] = E[S_0] = k .$$ 2

But $E[S_T] = 0 \times p(S_T = 0) + N p(S_T = N) \cancel{=k}$.

Thus $N p(S_T = N) = k$, $p(S_T = N) = \frac{k}{N}$.

$$p(S_T = 0) = 1 - p(S_T = N) = \frac{N-k}{N} .$$ 2

$\{ Y_{T\wedge n}, \}_{n\geq 0}$ is a martingale.

We have

$$E[Y_{T\wedge n}] = E[Y_0] = E[S_0^2] = k^2$$

Namely,

2

$$E[S_{T\wedge n}^2] = E[T\wedge n] + k^2$$

By Monotone, dominated convergence theorem,

$$E[S_T^2] = \lim_{n\to\infty} E[S_{T\wedge n}^2]$$

$$= \lim_{n\to\infty} \{ E[T\wedge n] + k^2 \}$$

1

$$= E[T] + k^2$$

So $E[T] = E[S_T^2] - k^2 = N^2 p(S_T = N) - k^2$

$$= N^2 \times \frac{k}{N} - k^2 = Nk - k^2 = k(N-k).$$

1

8

(iv)

(1) Z_n is $\mathcal{F}_n$ - measurable because Z_n is a function of $X_1, X_2, \cdots, X_n$.

(2) $E[|Z_n|] \le \dfrac{e^{8\lambda(k+n)}}{\cosh(\lambda)^n} < +\infty$

(3) $E[Z_{n+1} | \mathcal{F}_n] = E\left[\dfrac{e^{\lambda S_{n+1}}}{(\cosh(\lambda))^{n+1}} \middle| \mathcal{F}_n \right]$

$$= E\left[\frac{e^{\lambda S_n}}{(\cosh(\lambda))^n} \frac{e^{\lambda X_{n+1}}}{\cosh(\lambda)} \middle| \mathcal{F}_n \right]$$

$$= Z_n \frac{1}{\cosh(\lambda)} E[e^{\lambda X_{n+1}} | \mathcal{F}_n]$$

$$= Z_n \frac{1}{\cosh(\lambda)} E[e^{\lambda X_{n+1}}]$$

$$= Z_n \frac{1}{\cosh(\lambda)} \left\{ e^{\lambda} \tfrac{1}{2} + e^{-\lambda} \tfrac{1}{2} \right\} = Z_n$$

$\{Z_n\}$ is a martingale.

5

(V). As $\left|Z_{n\wedge T_N}\right| \leq \dfrac{e^{\lambda N}}{\cosh(\lambda)^{n\wedge T_N}} \leq e^{\lambda N}$,

2

by Doob's optional stopping theorem, we have

$$E[Z_{T_N}] = E[Z_0] = E[e^{\lambda S_0}] = e^{\lambda K}$$

1

But

$$E[Z_{T_N}] = E\left[\frac{e^{\lambda S_{T_N}}}{\cosh(\lambda)^{T_N}}\right] \cancel{= e^{\lambda N}}$$

$$= e^{\lambda N} E\left[\frac{1}{\cosh(\lambda)^{T_N}}\right].$$

2

So $E[\cosh(\lambda)^{-T_N}] = e^{\lambda K} \cdot e^{-\lambda N} = e^{\lambda(K-N)}$.

2

7

10

B.6

(i) ϕ is said to be self-financing if

$$\phi(t)\cdot S(t) = \phi(t+1)\cdot S(t)$$ 3

meaning that the invester does not bring in or consume any wealth. 2

$\underline{\underline{5}}$

(ii) A market is complete if every contingent claim X is attainable, i.e., there exists a 2 self-financing portfolio $\phi(t) = (\phi_0(t), \phi_1(t))$ such that $V_\phi(T) = X$. An arbitrage-free market is complete if there exists a unique equivalent 3 martingale probability measure P^*.

$\underline{\underline{5}}$

(iii) The discounted price is

$$\tilde{S}_1(t) = \frac{1}{S_0(t)} S_1(t).$$

$\{\tilde{S}_1(t), t \geq 0\}$ is a martingale under P^*

if and only if

$$E^{P^*}[\tilde{S}_1(t+1) \mid \mathcal{F}_t] = \tilde{S}_1(t)$$ 2

i.e

(1) $E\left[\widetilde{S}_1(t+1) / \widetilde{S}_1(t) \mid \mathcal{F}_t \right] = 1$ 2

Note that

$$\widetilde{S}_1(t+1) = \frac{x\, Z(1) \cdots Z(t+1)}{(1+r)^{t+1}}$$ 2

$$\widetilde{S}_1(t) = \frac{x\, Z(1) \cdots Z(t)}{(1+r)^{t}}$$

$$\frac{\widetilde{S}_1(t+1)}{\widetilde{S}_1(t)} = \frac{Z(t+1)}{1+r}$$ 2

(1) becomes

$$E\left[Z(t+1) \mid \mathcal{F}_t \right] = 1+r$$

By the independence,

$$E\left[Z(t+1) \mid \mathcal{F}_t \right] = E\left[Z(t+1) \right]$$

$$= up + \ell(1-p).$$ 2

Thus $up + \ell(1-p) = 1+r$, which gives

$$p = \frac{1+r-\ell}{u-\ell}.$$ 1

In order that $0 < p < 1$ we must have

$$\ell < 1+r < u.$$ 1

12

(iv) The price is given by

$$\Pi = S_0(0)\, E\left[\frac{X}{S_0(T)} \right] = \frac{1}{(1+r)^T} E[X] \qquad 2$$

$$= \frac{1}{(1+r)^T} E\left[\left(x\, Z(1) \cdots Z(T) \right)^2 \right] \qquad 1$$

$$= \frac{x^2}{(1+r)^T} E\left[\left(Z(1) \cdots Z(T) \right)^2 \right]$$

$$= \frac{x^2}{(1+r)^T} \sum_{j=0}^{T} \left(u^j \ell^{T-j} \right)^2 \binom{T}{j} p^j (1-p)^{T-j}, \qquad 5$$

where $\quad p = \dfrac{1+r-\ell}{u-\ell}$.

8