Solutions to M37001 (2010).

A.1:

(i) (b) is right.

(ii) We say that $E[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$ if

1. $E[X|\mathcal{G}]$ is $\mathcal{G}$ - measurable (determined),

~~② $E[X|\mathcal{G}]$ is integrable, i.e.,~~

$\cancel{E[\,|E[X|\mathcal{G}]|\,] < \infty}$,

2. for any event $A \in \mathcal{G}$,

$$\int_A E[X|\mathcal{G}]\, dP = \int_A X(\omega)\, dP$$

Apply the property 2 to get

$$E[E[X|\mathcal{G}]] = \int_\Omega E[X|\mathcal{G}]\, dP$$

$$= \int_\Omega X\, dP = E[X] \qquad \text{as} \quad \Omega \in \mathcal{G}.$$

A.2

(i) We say that $\{Z_n, n \geq 0\}$ is a martingale w.r.t. $\{\mathcal{F}_n, n \geq 0\}$ if

① Z_n is $\mathcal{F}_n$- determined (measurable),

② Z_n is integrable, i.e. $E[|Z_n|] < +\infty$,

③ $E[Z_{n+1} | \mathcal{F}_n] = Z_n$.

(ii) (a) $E[Z_{n+3} | \mathcal{F}_n] = E[E[Z_{n+3} | \mathcal{F}_{n+2}] | \mathcal{F}_n]$

$= E[Z_{n+2} | \mathcal{F}_n] = E[E[Z_{n+2} | \mathcal{F}_{n+1}] | \mathcal{F}_n]$

$= E[Z_{n+1} | \mathcal{F}_n] = Z_n$.

(b)

① Z_n is $\mathcal{G}_n$- determined by the assumption.

② Z_n is integrable since $Z_n, n \geq 0$, is a martingale w.r.t. $\mathcal{F}_n$. $n \geq 0$.

③ $E[Z_{n+1} | \mathcal{G}_n] = E[E[Z_{n+1} | \mathcal{F}_n] | \mathcal{G}_n]$

$= E[Z_n | \mathcal{G}_n] = Z_n$

Hence $\{Z_n, n \geq 0\}$ is a martingale with respect to $\{\mathcal{G}_n, n \geq 0\}$.

A.3:

(i)

1. Since Z_n is a function of $X_1, X_2, \cdots, X_n$, Z_n is $\mathcal{F}_n$- measurable

2. As

$$|Z_n| \leq n + S_n^2 \leq n + (|X_1| + |X_2| + \cdots + |X_n|)^2$$
$$\leq n + n^2,$$

Z_n is integrable.

3.
$$E[Z_{n+1} | \mathcal{F}_n] = E[S_{n+1}^2 - (n+1) | \mathcal{F}_n]$$
$$= E[(S_n + X_{n+1})^2 - (n+1) | \mathcal{F}_n]$$
$$= E[S_n^2 + 2 S_n X_{n+1} + X_{n+1}^2 - (n+1) | \mathcal{F}_n]$$
$$= E[Z_n | \mathcal{F}_n] + 2E[S_n X_{n+1} | \mathcal{F}_n] + E[X_{n+1}^2 - 1 | \mathcal{F}_n]$$
$$= Z_n + 2 S_n E[X_{n+1} | \mathcal{F}_n] + E[X_{n+1}^2] - 1$$
$$= Z_n + 2 S_n E[X_{n+1}] + (1 \times \tfrac{1}{2} + 1 \times \tfrac{1}{2}) - 1$$
$$= Z_n + 0 = Z_n.$$

(ii)

① Because Z_n is a function of $X_1, X_2, \cdots, X_n$,

Z_n is $\mathcal{F}_n$- determined.

② As $|Z_n| \leq \left(\frac{q}{p}\right)^n + \left(\frac{q}{p}\right)^{-n}$,

Z_n is integrable.

③ $E[Z_{n+1} | \mathcal{F}_n] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} | \mathcal{F}_n\right]$

$= E\left[\left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p}\right)^{X_{n+1}} | \mathcal{F}_n\right] = \left(\frac{q}{p}\right)^{S_n} E\left[\left(\frac{q}{p}\right)^{X_{n+1}} | \mathcal{F}_n\right]$

$= Z_n \; E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right]$

$= Z_n \left\{ \frac{q}{p} P(X_{n+1}=1) + \left(\frac{q}{p}\right)^{-1} P(X_{n+1}=-1) \right\}$

$= Z_n \left\{ \frac{q}{p} \times p + \frac{p}{q} \times q \right] = Z_n.$

A.4 :

(i) ϕ is self- financeing if

$\phi(t) \cdot S(t) = \phi(t+1) \cdot S(t)$, $t \geq 1$.

This means the investor does not bring in or consume any wealth.

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(ii). Suppose $\widehat{V_\phi}(t) = \widehat{V_\phi}(0) + \widehat{G_\phi}(t), \quad t \geq 1.$

Let $t=1$ to get

$$\begin{aligned}
\widehat{V_\phi}(1) &= \widehat{V_\phi}(0) + \widehat{G_\phi}(1) \\
&= \phi(0)\cdot\widehat{S}(0) + \phi(1)\cdot(\widehat{S}(1) - \widehat{S}(0)) \\
&= \phi(1)\cdot\widehat{S}(1) + \phi(0)\cdot\widehat{S}(0) - \phi(1)\cdot\widehat{S}(0) \\
&= \widehat{V_\phi}(1) + \Phi(0)\cdot\widehat{S}(0) - \phi(1)\cdot\widehat{S}(0).
\end{aligned}$$

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This gives $\quad \phi(0)\cdot\widehat{S}(0) = \phi(1)\cdot\widehat{S}(0).$

Taking $t=2$ we have

$$\begin{aligned}
\widehat{V_\phi}(2) &= \widehat{V_\phi}(0) + \widehat{G_\phi}(2) \\
&= \phi(0)\cdot\widehat{S}(0) + \phi(1)\cdot(\widehat{S}(1) - \widehat{S}(0)) \\
&\quad + \phi(2)\cdot(\widehat{S}(2) - \widehat{S}(1)) \\
&= \phi(0)\cdot\widehat{S}(0) - \phi(1)\cdot\widehat{S}(0) + \phi(1)\cdot\widehat{S}(1) + \widehat{V_\phi}(2) - \Phi(2)\cdot\widehat{S}(1) \\
&= 0 + \widehat{V_\phi}(2) + \phi(1)\cdot\widehat{S}(1) - \phi(2)\cdot\widehat{S}(1).
\end{aligned}$$

We deduce that $\quad \phi(1)\cdot\widehat{S}(1) = \phi(2)\cdot\widehat{S}(1).$

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Continuing the above procedure, we obtain

$$\phi(t)\cdot\widehat{S}(t) = \phi(t+1)\cdot\widehat{S}(t)$$

for all $t \geq 1$.

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B.5:

(i) For any $n \geq 1$, we have

$$\{T \leq n\} = \bigcup_{k=1}^{n} \{S_k = -a \text{ or } S_k = b\} \in \mathcal{F}_n.$$

This proves that T is a stopping time.

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(ii)

① As a function of $X_1, X_2, \dots, X_n$, Y_n is $\mathcal{F}_n$-determined.

② $|Y_n| \leq |S_n| + n|p-q| \leq n + n|p-q|$ implies that Y_n is integrable.

③ $E[Y_{n+1} | \mathcal{F}_n] = E[S_{n+1} - (n+1)(p-q) | \mathcal{F}_n]$

$$= E[\cancel{S_{n+1}}\,p\; S_n - n\overset{(p-q)}{\cancel{+}} X_{n+1} - (p-q) | \mathcal{F}_n] \quad 1$$

$$= S_n - n(p-q) + E[X_{n+1} - (p-q) | \mathcal{F}_n] \quad 1$$

$$= Y_n + E[X_{n+1}] - (p-q)$$

$$= Y_n + (p-q) - (p-q) = Y_n. \quad 2$$

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(iii) By the definition of T, we have

$$|Z_{T\wedge n}| = \left|\left(\frac{q}{p}\right)^{S_{T\wedge n}}\right| \leq \left(\frac{q}{p}\right)^{a+b} + \left(\frac{p}{q}\right)^{a+b}.$$

Thus Doob's Optional Theorem can be applied to get

$$E[Z_T] = E[Z_0] = E\left[\left(\frac{q}{p}\right)^0\right] = 1.$$ 4

For the martingale $Y_n, n \geq 0$, we have

$$|Y_{n+1} - Y_n| = |\cancel{\otimes}\, S_{n+1} - S_n - (n+1)(p-q) + n(p-q)|$$

$$= |X_{n+1} - (p-q)| \leq |X_{n+1}| + |p-q| \leq 1 + |p-q|.$$

Hence the Doob's Optional Theorem implies

$$E[Y_T] = E[Y_0] = 0$$ 4

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(iv) It is known from (iii) that $E[Z_T] = 1$.

On the other hand,

$$E[Z_T] = E\left[\left(\frac{q}{p}\right)^{S_T}\right]$$

$$= \left(\frac{q}{p}\right)^{-a} P(S_T = -a) + \left(\frac{q}{p}\right)^b P(S_T = b)$$

$$= \left(\frac{p}{q}\right)^a (1 - P(S_T = b)) + \left(\frac{q}{p}\right)^b P(S_T = b).$$ 3

Hence, it follows that

$$\left(\frac{p}{q}\right)^a (1 - P(S_T = b)) + \left(\frac{q}{p}\right)^b P(S_T = b) = 1$$ 1

(8)

Solve the above equation to get

$$P(S_T=b) = \frac{1-\left(\frac{p}{q}\right)^a}{\left(\frac{q}{p}\right)^b - \left(\frac{p}{q}\right)^a}$$ 1

$$P(S_T=-a) = 1 - P(S_T=b)$$

$$= 1 - \frac{1-\left(\frac{p}{q}\right)^a}{\left(\frac{q}{p}\right)^b - \left(\frac{p}{q}\right)^a} = \frac{1-\left(\frac{q}{p}\right)^b}{\left(\frac{p}{q}\right)^a - \left(\frac{q}{p}\right)^b}$$ 2

$E[Y_T]=0$ is equivalent to

$$E[S_T - T(p-q)] = 0.$$

Hence. $E[T] = \frac{1}{p-q} E[S_T]$ 2

$$= \frac{1}{p-q}\left\{ (-a)P(S_T=-a) + b\,P(S_T=b) \right\}$$

$$= \frac{1}{p-q}\left\{ (-a) \times \frac{1-\left(\frac{q}{p}\right)^b}{\left(\frac{p}{q}\right)^a - \left(\frac{q}{p}\right)^b} + b\,\frac{1-\left(\frac{p}{q}\right)^a}{\left(\frac{q}{p}\right)^b - \left(\frac{p}{q}\right)^a} \right\}$$

$$= \frac{a\left(1-\left(\frac{q}{p}\right)^b\right) + b\left(1-\left(\frac{p}{q}\right)^a\right)}{\left(\left(\frac{q}{p}\right)^b - \left(\frac{p}{q}\right)^a\right)(p-q)}$$

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B6.

(i)

① As a function of $N(t)$, $U(t)$ is $\mathcal{F}_t$- measurable.

② $U(t)$ is integrable as

$$E[|U(t)|] \leq E[N(t)] + \lambda t = \lambda t + \lambda t = 2\lambda t$$ 1

③ For $s<t$,

$$E[U(t) | \mathcal{F}_s] = E[U(t) - U(s) + U(s) | \mathcal{F}_s]$$ 1

$$= E[U(t) - U(s) | \mathcal{F}_s] + E[U(s) | \mathcal{F}_s]$$ 1

$$= E[U(t) - U(s)] + U(s)$$ 1

$$= E[N(t)] - \lambda t - E[N(s)] + \lambda s + U(s)$$

$$= \lambda t - \lambda t - \lambda s + \lambda s + U(s) = U(s).$$ 2

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(ii)

① Since $V(t)$ is a function of $N(t)$, $V(t)$ is $\mathcal{F}_t$-determined.

② $$E[|V(t)|] \leq E[U^2(t)] + \lambda t$$ 2

$$= E[(N(t) - \lambda t)^2] + \lambda t = \lambda t + \lambda t = 2\lambda t < +\infty.$$

③ For $s<t$,

$$E[V(t) | \mathcal{F}_s] = E[U^2(t) - \lambda t \, | \mathcal{F}_s]$$

$$= E\left[(U(t) - U(s) + U(s))^2 - \lambda t \mid \mathcal{F}_s \right] \quad 1$$

$$= E\left[(U(t) - U(s))^2 + 2(U(t) - U(s))U(s) + U^2(s) - \lambda t \mid \mathcal{F}_s \right] \quad 2$$

$$= E\left[(U(t) - U(s))^2 \mid \mathcal{F}_s \right] + 2E\left[(U(t) - V(s))U(s) \mid \mathcal{F}_s \right]$$

$$+ E\left[U^2(s) - \lambda s \mid \mathcal{F}_s \right] - \lambda(t-s) \quad 2$$

$$= E\left[(U(t) - U(s))^2 \right] + 2U(s)\, E\left[U(t) - U(s) \mid \mathcal{F}_s \right]$$

$$+ U^2(s) - \lambda s - \lambda(t-s) \quad 2$$

$$= E\left[(N(t) - N(s) - \lambda(t-s))^2 \right] + 2U(s)\, E\left[U(t) - U(s) \right]$$

$$+ V(s) - \lambda(t-s) \quad 1$$

$$= \lambda(t-s) + 2U(s)\left[E[N(t)] - \lambda t - E[N(s)] + \lambda s \right]$$

$$+ V(s) - \lambda(t-s) \quad 2$$

$$= V(s).$$

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(iii)

① Again as a function of $N(t)$, $W(t)$ is $\mathcal{F}_t$- measurable.

② As $N(t)$ has the Poisson distribution with parameter λt, $E[W(t)] < +\infty$.

3 For $s<t$,

$$E[W(t)\mid\mathcal{F}_s] = E[\exp[-\theta N(t) + \lambda t(1-e^{-\theta})] \mid \mathcal{F}_s]$$

$$= E[\cancel{\exp[-\theta N(t)}\ W(t)W^{-1}(s)W(s)\mid\mathcal{F}_s] \qquad 2$$

$$= W(s)\,E[W(t)W^{-1}(s)\mid\mathcal{F}_s] \qquad 1$$

$$= W(s)\,E[\exp[-\theta(N(t)-N(s)) + \lambda(t-s)(1-e^{-\theta})]\mid\mathcal{F}_s]$$

$$= W(s)\,E[\exp[-\theta(N(t)-N(s))]\mid\mathcal{F}_s]\,\exp(\lambda(t-s)(1-e^{-\theta})) \quad 2$$

$$= W(s)\,E[\exp[-\theta(N(t)-N(s))]]\,\exp(\lambda(t-s)(1-e^{-\theta})) \quad 1$$

$$= W(s)\exp(\lambda(t-s)(1-e^{-\theta}))$$
$$\times \sum_{k=0}^{\infty} e^{-\theta k}\,p(N(t)-N(s)=k) \qquad 2$$

$$= W(s)\exp(\lambda(t-s)(1-e^{-\theta}))$$
$$\times \sum_{k=0}^{\infty} e^{-\theta k}\, e^{-\lambda(t-s)}\frac{(\lambda(t-s))^k}{k!} \qquad 1$$

$$= W(s)\exp(\lambda(t-s)(1-e^{-\theta}))$$
$$\times e^{-\lambda(t-s)}\sum_{k=0}^{\infty}\frac{(e^{-\theta}\lambda(t-s))^k}{k!} \qquad 2$$

$$= W(s)\exp(\lambda(t-s)(1-e^{-\theta}))\,e^{-\lambda(t-s)}\,e^{\lambda(t-s)e^{-\theta}}$$
$$= W(s)\exp(\lambda(t-s)(1-e^{-\theta}))\exp(\lambda(t-s)(e^{-\theta}-1)) \qquad 12$$

$= W(s)$

B7.

(i) We say that $\phi = (\phi_0(t), \phi_1(t))$ is an arbitrage opportunity if $V_\phi(0) = 0$, ~~and~~

$V_\phi(T) \geq 0$ and $P(V_\phi(T) > 0) > 0$.

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(ii) The market is free of arbitrage if and only if there exists a probability measure Q under which the discounted price $\tilde{S}_1$ is a martingale.

1

Suppose $Q(\{w_1\}) = p^*$, $Q(\{w_2\}) = q^* = 1 - p^*$.

In order that $\tilde{S}_1$ is a martingale under Q we must have

$$E^{Q^*}[\tilde{S}_1] = E^{Q^*}[\tilde{S}_0] = E^{Q^*}\left[\frac{S_1(0)}{S_0(0)}\right]$$

$$= \frac{d}{5}$$

2

But $E^{Q^*}[\tilde{S}_1] = E^{Q^*}\left[\frac{S_1(1)}{S_0(1)}\right]$

$$= \frac{S_1(1,\omega_1)}{5(1+r)} Q^*(\{\omega_1\}) + \frac{S_1(1,\omega_2)}{5(1+r)} Q^*(\{\omega_2\})$$

$$= \frac{d_1}{5(1+r)} p^* + \frac{d_2}{5(1+r)} (1-p^*).$$ 2

Thus we have the following equation for p^*:

$$\frac{d_1}{5(1+r)} p^* + \frac{d_2}{5(1+r)} (1-p^*) = \frac{d}{5}$$ 1

Solve the above equation to get

$$p^* = \frac{d_2 - d(1+r)}{d_2 - d_1}$$ 1

In order that Q is a well defined martingale probability, we must have

$$0 < p^* = \frac{d_2 - d(1+r)}{d_2 - d_1} < 1$$ 1

This is equivalent to

$$\begin{cases} d_2 - d(1+r) > 0 \\ d_2 - d(1+r) < d_2 - d_1 \end{cases}$$

Combining these two we get

$$d_1 < d(1+r) < d_2.$$ 2

The condition is:

$$d_1 < d(1+r) < d_2.$$

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The martingale probability measure is given by

$$Q(\{w_1\}) = p^* = \frac{d_2 - d(1+r)}{d_2 - d_1}$$

$$Q(\{w_2\}) = 1 - p^* = \frac{d(1+r) - d_1}{d_2 - d_1}$$

(iii) An European call option with strike price K gives one the right to buy on the expiry date at a specified price K. The payoff (claim) is

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$$X = (S_1(1) - K)^+ = \begin{cases} S_1(1) - K & \text{if } S_1(1) \geq K, \\ 0 & \text{if } S_1(1) < K. \end{cases}$$

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iv)
(a) Under the assumption, the martingale probability is

$$Q(\{w_1\}) = p^* = \frac{8-5}{8-3} = \frac{3}{5}.$$

$$Q(\{w_2\}) = 1 - p^* = \frac{2}{5}.$$

1

The price of the call option is

$$\Pi_X(0) = S_0(0) \, E^Q\left[\frac{X}{S_0(1)}\right]$$

$$= 5 \, E^Q\left[\frac{X}{5(1+r)}\right] = \frac{1}{1+r} E^Q[X]$$

$$= \frac{1}{1+r} \{ X(w_1) \, Q(\{w_1\}) + X(w_2) \, Q(\{w_2\}) \}$$

2

$$= \frac{1}{1+r}\left\{ (d_1-K)^+ \frac{3}{5} + \frac{2}{5}(d_2-K)^+ \right\}$$

$$= \frac{1}{1+r}\left\{ (3-7)^+ \times \frac{3}{5} + \frac{2}{5}(8-7)^+ \right]$$ 2

$$= \frac{1}{1+r} \times \frac{2}{5} = \frac{2}{5(1+r)}$$ 5

(b) Let $\phi = (\phi_0, \phi_1)$ be the replicating strategy for the claim of the X. Then

$$V_\phi(1) = \phi_0 S_0(1)^{(\omega)} + \phi_1 S_1(1)^{(\omega)} = X(\omega).$$ 2

Let $w = w_1$, and $w = w_2$ respectively to get

$$\begin{cases} \phi_0 5(1+r) + \phi_1 d_1 = X(\omega_1) = (d_1-K)^+ = 0 \\ \phi_0 5(1+r) + \phi_1 d_2 = X(\omega_2) = (d_2-K) \end{cases}$$

i.e

$$\begin{cases} \phi_0 5(1+r) + \phi_1 \times 3 = 0 \\ \phi_0 5(1+r) + \phi_1 \times 8 = 8-7 = 1 \end{cases}$$ 2

Solve the above equation to obtain

$$\phi_1 = \frac{1}{5}, \quad \phi_0 = -\frac{3}{25(1+r)}.$$ 1

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