Martingale Theory for Finance

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1 Introduction

2 Probability spaces and σ-fields

3 Integration with respect to a probability measure.

4 Conditional expectation.

5 Discrete time random models in finance

The theory of martingales is widely used in the modern theory of finance. In this part, we will gradually introduce the basic concepts by considering the discrete time models in finance. We will work on a fixed probability space $(\Omega, \mathcal{F}, P)$, where Ω is finite , i.e., $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$, $P(\omega_i) > 0$ and $\sum_{i=1}^{N} P(\omega_i) = 1$. The probability space is equipped with a sequence of increasing σ-fields $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{T-1} \subset \mathcal{F}_T$ with $\mathcal{F}_0 = \{\Omega, \emptyset\}$. $\mathcal{F}_t$ represents the collection of events (information) up to time t. In the model, the initial time (current) is $t = 0$ and the terminal date is T. The time runs from $0, 1, 2, ..$ to T.

Definition 5.1 *A financial market contains $d + 1$ traded financial assets, whose prices at time t are denoted by $S(t) = (S_0(t), S_1(t), ..., S_d(t))$, where $S_1(t), ..., S_d(t)$ are random variables representing the prices of risky assets, for example, stocks, houses, and $S_0(t)$ is the price of a riskless asset (e.g. bank account). We assume that $S(t)$ is $\mathcal{F}_t$-measurable (determined) meaning that $S(t)$ is determined by what happened up to time t on the market.*

Definition 5.2 *A portfolio at time t is a division $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ of the investor's capital between different assets. It means that $\phi_0(t)$ units are put in the bank account, $\phi_1(t)$ units for the asset 1 (e.g stock),..., and $\phi_d(t)$ units for the asset d.*

We assume that $\phi(t)$ is $\mathcal{F}_{t-1}$-measurable (determined) meaning that $\phi(t)$ is determined by what happened up to time $t - 1$. This is reasonable because the investor selects his time t portfolio after observing what happened before time t (up to time $t-1$). For example, the investor needs to decide how to invest his capital in the market tomorrow based on the information available up to today. A portfolio is also called a trading strategy.

Definition 5.3 *The value of a portfolio $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ at time t is defined (given) by*

$$V_\phi(t) = \phi_0(t)S_0(t) + \phi_1(t)S_1(t) + ... + \phi_{d-1}(t)S_{d-1}(t) + \phi_d(t)S_d(t)$$

$$= \sum_{i=0}^{d} \phi_i(t)S_i(t) = \phi(t) \cdot S(t)$$

It is the sum of the number of units $\phi_i(t)$ invested in asset number i multiplied by the price of the asset nr. i. For example, if one holds 10 units of stock and 5 units of bound (bank account), then portfolio is $\phi = (\phi_0, \phi_1) = (5, 10)$. If the prices are $S_0 = 1, S_1 = 150$ then the value of the portfolio is $V_\phi = 5 \times 1 + 10 \times 150 = 1505$. Next we introduce the gain process of a portfolio.

Now $\phi(t) \cdot S(t-1) = \sum_{i=0}^{d} \phi_i(t)S_i(t-1)$ reflects the market value of the portfolio just after it has been established at time $t-1$, while $\phi(t) \cdot S(t) = \sum_{i=0}^{d} \phi_i(t)S_i(t)$ is the value just after time t prices are observed, but before changes are made in the portfolio. (remember that $\phi(t)$ is the portfolio one holds from time $t-1$ until time t). Hence

$$\phi(t) \cdot (S(t) - S(t-1)) = \phi(t) \cdot \Delta S(t)$$

is the change in the market value due to changes in asset prices which occur between time $t-1$ and t.

Definition 5.4 *The gain (cumulative) process G_ϕ of a portfolio $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ is given by*

$$G_\phi(t) = \sum_{\tau=1}^{t} \phi(\tau) \cdot (S(\tau) - S(\tau-1))$$

Definition 5.5 *Recall $S_0(t)$ is the price of the riskless asset.*

$$\tilde{S}(t) = \frac{1}{S_0(t)}S(t) = \frac{1}{S_0(t)}(S_0(t), S_1(t), ..., S_d(t))$$

$$= (1, \frac{1}{S_0(t)}S_0(t), \frac{1}{S_0(t)}S_1(t), ..., \frac{1}{S_0(t)}S_d(t))$$

is called the discounted price process.

Definition 5.6 *The discounted value process of a portfolio $\phi(t)$ is defined (given) by*

$$\tilde{V}_\phi(t) = \phi_0(t)\tilde{S}_0(t) + \phi_1(t)\tilde{S}_1(t) + ... + \phi_{d-1}(t)\tilde{S}_{d-1}(t) + \phi_d(t)\tilde{S}_d(t)$$

$$= \sum_{i=0}^{d} \phi_i(t)\tilde{S}_i(t) = \phi(t) \cdot \tilde{S}(t)$$

The discounted gain process is

$$\tilde{G}_\phi(t) = \sum_{\tau=1}^{t} \phi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1))$$

Definition 5.7 *A portfolio* $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ *is called self-financing if by*

$$\phi(t) \cdot S(t) = \phi(t + 1) \cdot S(t) \quad (t = 1, 2, ..., T - 1)$$

Interpretation: With the new prices $S(t)$ in hand at the time t, the investor has fortune (value) $V_\phi(t) = \phi(t) \cdot S(t)$. Now at time t the investor adjusts his portfolio (strategy) from $\phi(t)$ to $\phi(t + 1)$ according to the market situation. (re-allocate his money in different assets) The self-financing says that although the investor adjusts his strategy, the total wealth (value) $\phi(t + 1) \cdot S(t)$ remains the same as $\phi(t) \cdot S(t)$. This means that the investor does not bring in or consume any wealth. It is therefore called self-financing.

Remark. A trading strategy $\phi(t)$ is self-financing with respect to $S(t)$ if and only if it is self-financing with respect to the discounted price $\tilde{S}(t)$ (i.e., $\phi(t) \cdot \tilde{S}(t) = \phi(t + 1) \cdot \tilde{S}(t)$

Proof. ϕ is self-financing if and only if

$$\phi(t) \cdot S(t) = \phi(t + 1) \cdot S(t) \quad (t = 1, 2, ..., T - 1)$$

which holds if and only if

$$\frac{1}{S_0(t)}\phi(t) \cdot S(t) = \frac{1}{S_0(t)}\phi(t + 1) \cdot S(t) \quad (t = 1, 2, ..., T - 1)$$

This is just

$$\phi(t) \cdot \tilde{S}(t) = \phi(t + 1) \cdot \tilde{S}(t)$$

Namely it is self-financing with respect to the discounted price $\tilde{S}(t)$.

Proposition 5.8 *A trading strategy* ϕ *is self-financing if and only if*

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \tilde{G}(t)$$

meaning that the discounted value at time t *is the sum of the initial value and the net gain.*

Proof. Recall

$$\tilde{G}_\phi(t) = \sum_{\tau=1}^{t} \phi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1))$$

Suppose that ϕ is self-financing. It is also self-financing w.r.t. the discounted price process. Using the self-financing property, we have

$$G_\phi(t) = \sum_{\tau=1}^{t} \phi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1))$$

$$= \sum_{\tau=1}^{t} \phi(\tau) \cdot \tilde{S}(\tau) - \sum_{\tau=1}^{t} \phi(\tau) \cdot \tilde{S}(\tau - 1)$$

$$= \sum_{\tau=1}^{t-1} \phi(\tau) \cdot \tilde{S}(\tau) + \phi(t) \cdot \tilde{S}(t) - \sum_{\tau=0}^{t-1} \phi(\tau + 1) \cdot \tilde{S}(\tau)$$

$$= \sum_{\tau=1}^{t-1} \phi(\tau) \cdot \tilde{S}(\tau) - \sum_{\tau=1}^{t-1} \phi(\tau + 1) \cdot \tilde{S}(\tau) - \phi(1) \cdot \tilde{S}(0) + \phi(t) \cdot \tilde{S}(t)$$

$$= -\phi(0) \cdot \tilde{S}(0) + \tilde{V}_\phi(t)$$

i.e.,

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \tilde{G}(t)$$

Conversely, suppose

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \tilde{G}(t)$$

for $t \geq 1$. Let $t = 1$ to get

$$\tilde{V}_\phi(1) = \tilde{V}_\phi(0) + \tilde{G}(1)$$

$$= \phi(0) \cdot \tilde{S}(0) + \phi(1) \cdot (\tilde{S}(1) - \tilde{S}(0))$$

$$= \tilde{V}_\phi(1) + \phi(0) \cdot \tilde{S}(0) - \phi(1) \cdot \tilde{S}(0)$$

which implies that

$$\phi(0) \cdot \tilde{S}(0) = \phi(1) \cdot \tilde{S}(0)$$

Let $t = 2$ in () to obtain

$$\tilde{V}_\phi(2) = \tilde{V}_\phi(0) + \tilde{G}(2)$$

$$= \phi(0) \cdot \tilde{S}(0) + \phi(1) \cdot (\tilde{S}(1) - \tilde{S}(0)) + \phi(2) \cdot (\tilde{S}(2) - \tilde{S}(1))$$

This yields

$$\phi(1) \cdot \tilde{S}(1) = \phi(2) \cdot \tilde{S}(1)$$

Continuing the above procedure, we obtain

$$\phi(t) \cdot \tilde{S}(t) = \phi(t + 1) \cdot \tilde{S}(t)$$

for all $t \geq 1$. Hence ϕ is self-financing.

The next proposition says that if one has decided what to do about the risky assets (i.e.,$\phi_1(t), ..., \phi_d(t)$ are given), it is always possible to choose $\phi_0(t)$ in such a way that $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ is self-financing.

Proposition 5.9 *If $(\phi_1(t), ..., \phi_d(t))$ is $\mathcal{F}_{t-1}$ measurable and V_0 is $\mathcal{F}_0$ measurable, there is a unique process $\phi_0(t)$ such that $\phi_0(t)$ is $\mathcal{F}_{t-1}$ measurable and $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ is a self-financing strategy with initial value $V_\phi(0) = V_0$.*

Proof. We will determine $\phi_0(t)$ according to the condition for self-financing. If $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ is self-financing, then by Proposition 5.8,

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \tilde{G}_\phi(t)$$

$$= \frac{1}{S_0(0)} V_0 + \sum_{\tau=1}^{t} (\phi_1(\tau)\Delta\tilde{S}_1(\tau) + ... + \phi_d(\tau)\Delta\tilde{S}_d(\tau)),$$

where $\Delta\tilde{S}_i(\tau) = \tilde{S}_i(\tau) - \tilde{S}_i(\tau - 1)$. On the other hand,

$$\tilde{V}_\phi(t) = \sum_{k=0}^{d} \phi_k(t)\tilde{S}_k(t)$$

$$= \phi_0(t) + \sum_{k=1}^{d} \phi_k(t)\tilde{S}_k(t)$$

Comparing the above two equations, we obtain that

$$\phi_0(t) = \frac{1}{S_0(0)} V_0 + \sum_{\tau=1}^{t} (\phi_1(\tau)\Delta\tilde{S}_1(\tau) + ... + \phi_d(\tau)\Delta\tilde{S}_d(\tau))$$

$$-\sum_{k=1}^{d} \phi_k(t)\tilde{S}_k(t)$$

So $\phi_0(t)$ is uniquely determined and self-financing.

The No-Arbitrage Conditions

Definition 5.10 *A self-financing portfolio $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ is called an arbitrage opportunity or arbitrage strategy if $V_\phi(0) = 0$ and the terminal wealth of ϕ satisfies*
(1) $V_\phi(T) \geq 0$,
(2) $P(V_\phi(T) > 0) > 0$.

Remark. Using the arbitrage strategy ϕ, one starts with nothing $V_\phi(0) = 0$, but end up with something $P(V_\phi(T) > 0) > 0$. Arbitrage is " making something out of nothing".

Definition 5.11 *We say that a market is arbitrage free if there are no arbitrage opportunities in the class of self-financing trading strategies.*

The question is under which conditions the market is free of arbitrage. We will show that no-arbitrage is related to fair games(martingales)

Definition 5.12 *A probability measure P^* on $(\Omega, \mathcal{F})$ with $P^*(\omega) > 0$ is called a martingale probability measure if the discounted prices $\tilde{S}(t), t = 0, ..., T$ is P^*-martingale, i.e.,*

$$E^*[\tilde{S}(t+1)|\mathcal{F}_t] = \tilde{S}(t)$$

or

$$E^*[\tilde{S}_i(t+1)|\mathcal{F}_t] = \tilde{S}_i(t)$$

Denote by $\mathcal{P}$ the collection of all equivalent martingale probability measures.

Proposition 5.13 *Let P^* be an equivalent martingale probability measure and ϕ is any self-financing strategy. The the discounted value process $\tilde{V}_\phi(t)$ is a P^*-martingale, i.e.,*

$$E^*[\tilde{V}_\phi(t+1)|\mathcal{F}_t] = \tilde{V}_\phi(t),$$

where E^* denotes the expectation computed under the new probability measure P^*.

Proof. Since $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ is self-financing, by Proposition 5.8,

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \tilde{G}_\phi(t)$$

Hence,

$$\begin{aligned} \tilde{V}_\phi(t+1) - \tilde{V}_\phi(t) &= \tilde{G}_\phi(t+1) - \tilde{G}_\phi(t) \\ &= \phi(t+1) \cdot (\tilde{S}(t+1) - \tilde{S}(t)) \end{aligned}$$

Recall that $\phi(t+1)$ is determined by $\mathcal{F}_t$ and $\tilde{S}_t$ is P^* -martingale. We have

$$E^*[\tilde{V}_\phi(t+1)|\mathcal{F}_t]$$

$$\begin{aligned} &== E^*[\tilde{V}_\phi(t) + \phi(t+1) \cdot (\tilde{S}(t+1) - \tilde{S}(t))|\mathcal{F}_t] \\ &= \tilde{V}_\phi(t) + \phi(t+1) \cdot E^*[(\tilde{S}(t+1) - \tilde{S}(t))|\mathcal{F}_t] = \tilde{V}_\phi(t) \end{aligned}$$

completing the proof.

Theorem 5.14 *A market is arbitrage-free if and only if there exists an equivalent martingale probability measure P^* (i.e., $\tilde{S}(t)$ is a martingale under P^*).*

Proof. We only prove the sufficiency. Assume that an equivalent martingale measure P^* exists. For any portfolio ϕ with $V_\phi(0) = 0$ and $V_\phi(T) \geq 0$, we have

$$\tilde{V}_\phi(0) = \frac{1}{S_0(t)} V_\phi(0) = 0, \quad \tilde{V}_\phi(T) = \frac{1}{S_0(t)} V_\phi(T) \geq 0$$

On the other hand, $\tilde{V}_\phi(t), t \geq 0$ is a martingale with respect to P^*. So we have

$$E^*[\tilde{V}_\phi(T)] = E^*[\tilde{V}_\phi(0)] = 0$$

This yields that $\tilde{V}_\phi(T) = 0, a.s.$ and hence, $V_\phi(T) = 0$. If one starts with nothing, one ends up still with nothing. So the market is free of arbitrage.

Example 5.15 *Consider a single time period model, where we have the current time $t = 0$) and the terminal time $t = T$. Suppose that the sample space $\Omega = \{\omega_1, \omega_2\}$ consists of two outcomes. The market has two assets: a riskless bank account with price S_0 and a stock with price S_1. Suppose the price at $t = 0$ is given by*

$$S(0) = (S_0(0), S_1(0)) = (1, 150)$$

The prices at time $t = T$ are determined by the following rules. $S_0(T) = 1+r$, where r is the interest rate.

$$S_1(T) = \begin{cases} 180 & \text{if } w = w_1 \\ 90 & \text{if } w = w_2. \end{cases}$$

Determine for which values of r the market is free of arbitrage.

Solution. Recall that a market is arbitrage-free if and only if there exists an equivalent martingale probability measure P^*. Suppose $P^*(\{\omega_1\}) = p^*, P^*(\{\omega_2\}) = 1-p^*$.. If P^* is an equivalent martingale probability measure, then the discounted price process $\tilde{S}(t)$ is a martingale. In particular,

$$E^*[\tilde{S}_1(T)] = E^*[\tilde{S}_1(0)]$$

That is

$$E^*[\frac{1}{S_0(T)}S_1(T)] = E^*[\frac{1}{S_0(0)}S_1(0)] = 150 \tag{5.1}$$

On the other hand,

$$E^*[\frac{1}{S_0(T)}S_1(T)] = \frac{1}{1+r}E^*[S_1(T)]$$

$$= \frac{1}{1+r}[S_1(T)(\omega_1)P^*(\{\omega_1\}) + S_1(T)(\omega_2)P^*(\{\omega_2\})]$$

$$= \frac{1}{1+r}[180p^* + 90(1 - p^*)]$$

(6.1) becomes the following equation:

$$\frac{1}{1+r}[180p^* + 90(1 - p^*)] = 150$$

Rearranging the terms,

$$(1 + r)150 = 90p^* + 90$$

We obtain,

$$p^* = \frac{5(1+r)-3}{3} = \frac{5r+2}{3}$$

To make sure P^* exists, we must have $0 < p^* = \frac{5r+2}{3} < 1$, which is equivalent to $0 < r < \frac{1}{5}$.

Completeness of the market.

Definition 5.16 *A contingent claim (payoff) X with maturity date T is an arbitrary non-negative $\mathcal{F}_T$ measurable random variable representing a cash-flow.*

Definition 5.17 *A contingent claim X is said to be attainable if there exists a self-financing portfolio $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ such that $V_\phi(T) = X$.*

In this case, we say that the portfolio $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ is a replicating strategy of the claim X.

Remark. If X is attainable, then one can find a trading strategy ϕ such that the terminal wealth $V_\phi(T)$ is equal to X.

Definition 5.18 *A market is complete if every contingent claim X is attainable, i.e., there exists a replicating self-financing portfolio $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ such that $V_\phi(T) = X$.*

The question is: When is a market complete?

Theorem 5.19 *An arbitrage-free market is complete if and only if there exists a unique equivalent martingale probability measure P^*.*

Proof. We only prove the necessity. Suppose that the market is complete. We will show that there is only one equivalent martingale probability measure. If P_1, P_2 be two martingale probability measures, we will show $P_1 = P_2$. First we prove that

$$E_1[\frac{1}{S_0(T)}X] = E_2[\frac{1}{S_0(T)}X]$$

for any $\mathcal{F}_T$ measurable random non-negative random variable X, where E_1, E_2 denote the expectation w.r.t. P_1 and respectively P_2. Given $X \geq 0$. Since the market is complete, there is a self-financing strategy ϕ replicating X: $X = V_\phi(T)$. The terminal wealth of the discounted value is $\tilde{V}_\phi(T) = \frac{1}{S_0(T)}V_\phi(T) = \frac{1}{S_0(T)}X$. On the other hand, by Proposition 5.13, $\{\tilde{V}_\phi(t), t \geq 0\}$ is a martingale both under P_1 and P_2. It follows that

$$E_1[\frac{1}{S_0(T)}X] = E_1[\tilde{V}_\phi(T)] = \tilde{V}_\phi(0)$$

$$E_2[\frac{1}{S_0(T)}X] = E_2[\tilde{V}_\phi(T)] = \tilde{V}_\phi(0)$$

This yields

$$E_1[\frac{1}{S_0(T)}X] = E_2[\frac{1}{S_0(T)}X] \tag{5.2}$$

Now for any $A \in \mathcal{F}_T$, choose $X = S_0(T)I_A$ and use (6.2) to get

$$P_1(A) = E_1[I_A] = E_1[\frac{1}{S_0(T)}X] = E_2[\frac{1}{S_0(T)}X] = E_2[I_A] = P_2(A)$$

Thus $P_1 = P_2$.

Asset Pricing.

Suppose a financial contract guarantees the buyer to receive a claim X at the terminal date T. Now the question is : What is the arbitrage-free price of this contract ? How much should one pay for this contract? In an arbitrage-free complete market, the claim X is attainable. Let ϕ be a replicating strategy for X. Then $V_\phi(T) = X$. The value process $V_\phi(t)$ is called the arbitrage -free price of the contingent claim X at time t. For, if an investor sells the claim X at time t for $V_\phi(t)$, he can follow the strategy ϕ to replicate the claim at time T and clear the claim. A fundamental problem is to find the arbitrage price for a given claim X.

Let P^* be the equivalent martingale measure. Then the discounted value process $\tilde{V}_\phi(t) = \frac{1}{S_0(t)}V_\phi(t)$ is a martingale w.r.t. P^*. Hence for $t \leq T$,

$$\frac{1}{S_0(t)}V_\phi(t) = \tilde{V}_\phi(t) = E^*[\tilde{V}_\phi(T)|\mathcal{F}_t]$$

$$= E^*[\frac{1}{S_0(T)}X|\mathcal{F}_t]$$

Therefore,

$$V_\phi(t) = S_0(t)E^*[\frac{1}{S_0(T)}X|\mathcal{F}_t]$$

In particular, the arbitrage free price at time $t = 0$ is given by

$$\Pi_X(0) = S_0(0)E^*[\frac{1}{S_0(T)}X]$$

Let us summarize it as follows. To find the arbitrage free price of a claim X, one first determines a martingale probability measure P^* and the compute

$$\Pi_X(0) = S_0(0)E^*[\frac{1}{S_0(T)}X]$$

Next we re-visit Example 6.15 and answer the following two questions.

(3). Find the arbitrage-free price at time 0 for the claim $X = S_1(T)(\omega)$ if $r = \frac{1}{10}$.

(4). Determine a replicating strategy $\phi = (\phi_0, \phi_1)$ for the claim in (3).

Solution. Recall that the unique martingale probability measure P^* is given by

$$P^*(\{\omega_1\}) = p^* = \frac{2+5r}{3}, P^*(\{\omega_2\}) = q^* = \frac{1-5r}{3}$$

If $r = \frac{1}{10}$, then

$$P^*(\{\omega_1\}) = p^* = \frac{5}{6}, P^*(\{\omega_2\}) = q^* = \frac{1}{6}$$

(3). According to the above discussion, the price is given by

$$\Pi_X(0) = S_0(0)E^*[\frac{1}{S_0(T)}X]$$

$$= \frac{1}{1+r}E^*[X] = \frac{1}{1+r}E^*[S_1(T)]$$

$$= \frac{1}{1+\frac{1}{10}}[S_1(T)(\omega_1)p^* + S_1(T)(\omega_2)q^*]$$

$$= \frac{10}{11}[180 \times \frac{5}{6} + 90 \times \frac{1}{6}] = 150$$

(4). The replicating strategy $\phi = (\phi_0, \phi_1)$ for the claim X is determined by

$$V_\phi(T) = \phi_0 S_0(T) + \phi_1 S_1(T) = X$$

Put $\omega = \omega_1$ and $\omega = \omega_2$ in the above equation to get

$$\begin{cases} \phi_0(1+\frac{1}{10}) + \phi_1 \times 180 & = X(\omega_1) = 180 \\ \phi_0(1+\frac{1}{10}) + \phi_1 \times 90 & = X(\omega_2) = 90 \end{cases}$$

Solve the above equation to obtain $\phi_0 = 0, \phi_1 = 1$. So the strategy is to do nothing for the bank account and buy one share of the stock in order to replicate X.

Options. An option is a financial instrument (agreement) giving one the right but not the obligation to make a specified transaction at (or by) a specified date at a specified price. An European call (put) option gives one the right to buy (sell) on the specified date, the expiry date, at a specified price K, which is called the strike price.

Consider an European call option with strike price K and terminal date T. Let us find the claim (payoff) of this option. If at time T the asset price $S_1(T)$ is bigger than the strike price K, one exercises the right to get a profit $S_1(T) - K$. If $S_1(T) \leq K$, one does nothing. So the claim (the payoff) for the European call option is

$$X(\omega) = (S_1(T) - K)^+ = \begin{cases} S_1(T) - K & \text{if } S_1(T) > K, \\ 0 & \text{if } S_1(T) \leq K \end{cases}$$

The price of the European option is

$$\Pi_X(0) = S_0(0)E^*[\frac{1}{S_0(T)}X]$$

The contingent claim (payoff) of an European put option with strike price K is

$$X(\omega) = (K - S_1(T))^+ = \begin{cases} K - S_1(T) & \text{if } S_1(T) < K, \\ 0 & \text{if } S_1(T) \geq K. \end{cases}$$

Example 5.20 *Consider a single time period financial market, where we have the current time $t = 0$ and the terminal time $t = T$. Suppose that the sample space $\Omega = \{\omega_1, \omega_2\}$ consists of two outcomes. The market has two assets: a riskless bank account with price S_0 and a stock with price S_1. Suppose the price at $t = 0$ is given by*

$$S(0) = (S_0(0), S_1(0)) = (1, 10)$$

The prices at time $t = T$ are determined by the following rules. $S_0(T) = 1+r$, where r is the interest rate.

$$S_1(T) = \begin{cases} 15 & \text{if } w = w_1 \\ 5 & \text{if } w = w_2. \end{cases}$$

(1) Determine for which values of r the market is free of arbitrage.

(2) Determine whether the arbitrage-free market is complete or not.

(3). Consider an European put option (for the risky asset) at a strike price $K = 12$. Find the price for the European put option when $r = 0.1$.

(4). Find a replicating strategy for the claim in (3).

Solution.

(1).Recall that a market is arbitrage-free if and only if there exists an equivalent martingale probability measure P^*. Suppose $P^*(\{\omega_1\}) = p^*$, $P^*(\{\omega_2\}) = 1 - p^*$.. If P^* is an equivalent martingale probability measure, then the discounted price process $\tilde{S}(t)$ is a martingale. In particular,

$$E^*[\tilde{S}_1(T)] = E^*[\tilde{S}_1(0)]$$

That is

$$E^*[\frac{1}{S_0(T)}S_1(T)] = E^*[\frac{1}{S_0(0)}S_1(0)] = 10 \tag{5.3}$$

On the other hand,

$$E^*[\frac{1}{S_0(T)}S_1(T)] = \frac{1}{1+r}E^*[S_1(T)]$$

$$= \frac{1}{1+r}[S_1(T)(\omega_1)P^*(\{\omega_1\}) + S_1(T)(\omega_2)P^*(\{\omega_2\})]$$

$$= \frac{1}{1+r}[15p^* + 5(1-p^*)]$$

(6.3) becomes the following equation:

$$\frac{1}{1+r}[15p^* + 5(1-p^*)] = 10$$

Rearranging the terms,

$$(1+r)10 = 10p^* + 5$$

Thus,

$$p^* = \frac{10(1+r)-5}{10} = \frac{2r+1}{2}$$

To make sure P^* exists, we must have $0 < p^* = \frac{1+2r}{2} < 1$, which is equivalent to $0 \le r < \frac{1}{2}$.

(2). The market is complete since the martingale probability measure is uniquely given by

$$P^*(\{\omega_1\}) = p^* = \frac{2r+1}{2}, P^*(\{\omega_2\}) = 1 - p^*$$

(3).The European put option has the payoff

$$X = (12 - S_1(T))^+ = \begin{cases} 0 & \text{if } w = w_1 \\ 7 & \text{if } w = w_2. \end{cases}$$

If $r = 0.1$, then $p^* = \frac{1+2.1}{2} = 0.6$ and $1-p^* = 0.4$. The price of the European put option is

$$\Pi_X(0) = E^*[\frac{1}{S_0(T)}X] = E^*[\frac{1}{1+r}X]$$

$$= \frac{1}{1+0.1}[X(\omega_1)P^*(\{\omega_1\}) + X(\omega_2)P^*(\{\omega_2\})]$$

$$= \frac{1}{1+0.1}[0P^*(\{\omega_1\}) + 7 \times 0.4] = \frac{28}{11}$$

(4).The replicating strategy $\phi = (\phi_0, \phi_1)$ for the claim X is determined by

$$V_\phi(T) = \phi_0 S_0(T) + \phi_1 S_1(T) = X$$

Put $\omega = \omega_1$ and $\omega = \omega_2$ in the above equation to get

$$\begin{cases} \phi_0(1+\frac{1}{10}) + \phi_1 \times 15 & = X(\omega_1) = 0 \\ \phi_0(1+\frac{1}{10}) + \phi_1 \times 5 & = X(\omega_2) = 7 \end{cases}$$

Solve the above equation to obtain $\phi_0 = \frac{105}{11}, \phi_1 = -0.7$.

The Cox-Ross-Rubinstein Model

Suppose the financial market consists of two securities(assets). One of the assets is riskless (say bond or bank account) and the other is a risky asset. The time horizon is T and the set of dates is $t = 0, 1, ..., T$. Suppose that the price of the riskless asset is

$$S_0(t) = (1 + r)^t, t = 0, 1, ..., T,$$

implying that the riskless security yields a riskless rate of return r in each time interval $[t, t + 1]$. The price of the risky asset is determined by the following relation:

$$S_1(t+1) = \begin{cases} uS_1(t) & \text{with probability } p \\ dS_1(t) & \text{with probability } 1-p \end{cases} \quad t = 0, 1, ..., T-1$$

Let

$$Z(t+1) = \frac{S_1(t+1)}{S_1(t)}, t = 0, 1, ..., T-1$$

Then the price of the risky asset can be written as

$$S_1(t) = S_1(0)\Pi_{\tau=1}^{t} Z(\tau), t = 1, 2, ..., T,$$

where $Z(1), Z(2), ..., Z(T)$ are independent and identically distributed random variables with $P(Z(t) = u) = p, P(Z(t) = d) = 1 - p$. The probability space is chosen to be as follows. The sample space Ω is the collection of T-tuples $\omega = (\omega_1, \omega_2, ..., \omega_T)$ where $\omega_i \in \{d, u\}$. Define $\tilde{P}(\{u\}) = p$ and $\tilde{P}(\{d\}) = 1 - p$. The probability measure P on Ω is given by

$$P(\{\omega\}) = \tilde{P}(\{\omega_1\}) \times \tilde{P}(\{\omega_2\}) \times ... \times \tilde{P}(\{\omega_T\})$$

for $\omega = (\omega_1, \omega_2, ..., \omega_T)$ The family of σ-fields is taken as

$$\mathcal{F}_0 = \{\emptyset, \Omega\}$$

$$\mathcal{F}_t = \sigma(Z(1), ..., Z(t)) = \sigma(S_1(1), ..., S_1(t))$$

Let $\mathcal{P}$ be the class of probability measures Q on Ω such that $Z(1), Z(2), ..., Z(T)$ are independent and identically distributed random variables with $Q(Z(t) = u) = q, Q(Z(t) = d) = 1 - q$.

Theorem 5.21 *(i). A martingale probability measure $P^* \in \mathcal{P}$ exists if and only if*

$$d < 1 + r < u. \tag{5.4}$$

(ii) If equation (6.4) holds true, there is a unique such measure in $\mathcal{P}$ characterized by

$$P^*(Z(t) = u) = p^* = \frac{1 + r - d}{u - d}.$$

Proof. Since $S_1(t) = S_0(t)\tilde{S}_1(t) = (1+r)^t \tilde{S}_1(t)$, we have

$$Z(t+1) = \frac{S_1(t+1)}{S_1(t)} = \frac{\tilde{S}_1(t+1)}{\tilde{S}_1(t)}(1+r)$$

So the discounted price $\{\tilde{S}_1(t)\}$ is a P^*-martingale if and only if for $t = 0, 1, ..., T-1$

$$E^*[\tilde{S}_1(t+1)|\mathcal{F}_t] = \tilde{S}_1(t) \Leftrightarrow E^*[(\tilde{S}_1(t+1)/\tilde{S}_1(t)|\mathcal{F}_t] = 1$$

$$\Leftrightarrow E^*[Z(t+1)|\mathcal{F}_t] = 1+r$$

Since $Z(t+1)$ is independent of $\mathcal{F}_t$ we have

$$1+r = E^*[Z(t+1)|\mathcal{F}_t] = E^*[Z(t+1)] = uP^*(Z(t)=u) + dP^*(Z(t)=d) = up^* + d(1-p^*),$$

where $P^*(Z(t)=u) = p^*, P^*(Z(t)=d) = 1-p^*$. Solve the above equation for p^* to obtain

$$p^* = \frac{1+r-d}{u-d}$$

In order that P^* exists one must have

$$0 < p^* = \frac{1+r-d}{u-d} < 1$$

which is equivalent to $d < 1+r < u$. It is easy to see that p^* is uniquely determined.

Proposition 5.22 *If $d < 1+r < u$, then the Cox-Ross-Rubinstein model is arbitrage-free and complete.*

This is a consequence of Theorem.

Proposition 5.23 *The arbitrage -free price of a contingent claim in the Cox-Ross-Rubinstein model is given by*

$$\Pi_X(0) = S_0(0)E^{P^*}[\frac{X}{S_0(T)}]$$

where $p^* = \frac{1+r-d}{u-d}$.

Example 5.24 *Consider a European call option with expiry T and strike price K written on the stock S_1. The arbitrage price is given by*

$$C = (1+r)^{-T} \sum_{j=0}^{T} \binom{T}{j} (p^*)^j (1-p^*)^{T-j} (xu^j d^{T-j} - K)^+,$$

where $S_1(0) = x$.

Solution. Recall that

$$S_1(T) = S_1(0)\Pi_{\tau=1}^T Z(\tau)$$

The claim (payoff) for the European call option is $X = (S_1(T) - K)^+$. So the price is given by

$$C = S_0(0)E^{P^*}[\frac{1}{S_0(T)}X] = (1+r)^{-T}E^{P^*}[\big(S_1(0)\Pi_{\tau=1}^T Z(\tau) - K\big)^+]$$

$$= (1+r)^{-T}E^{P^*}[\big(x\Pi_{\tau=1}^T Z(\tau) - K\big)^+]$$

Now note that the random variable $\Pi_{\tau=1}^T Z(\tau)$ takes the values $u^j d^{T-j}, j = 0, 1, ..., T$ with

$$P^*(\Pi_{\tau=1}^T Z(\tau) = u^j d^{T-j})$$

$$= P^*(j \text{ of } Z(\tau) \text{ take the value } u \text{ and } T-j \text{ of them take the value } d)$$

$$= \begin{pmatrix} T \\ j \end{pmatrix} (p^*)^j (1-p^*)^{T-j}$$

Therefore,

$$C = (1+r)^{-T}\sum_{j=0}^{T}(xu^j d^{T-j} - K)^+ P^*(\Pi_{\tau=1}^T Z(\tau) = u^j d^{T-j})$$

$$= (1+r)^{-T}\sum_{j=0}^{T}\begin{pmatrix} T \\ j \end{pmatrix} (p^*)^j (1-p^*)^{T-j}(xu^j d^{T-j} - K)^+$$

Hedging. Since the COx-Ross-Rubinstein model is complete, we can hedge every claim. Here we look at one example: Hedging of the European call option. Set

$$C(t,x) = (1+r)^{-(T-t)}\sum_{j=0}^{T-t}\begin{pmatrix} T-t \\ j \end{pmatrix} (p^*)^j (1-p^*)^{T-t-j}(xu^j d^{T-j} - K)^+$$

Proposition 5.25 *The hedging (replicating) strategy* $\phi = (\phi_0(t), \phi_1(t))$ *of the European call option with time of expiry* T *and strike price* K *is given by*

$$\phi_1(t) = \frac{C(t, S_1(t-1)u) - C(t, S_1(t-1)d)}{S_1(t-1)(u-d)},$$

$$\phi_0(t) = \frac{uC(t, S_1(t-1)d) - dC(t, S_1(t-1)u)}{(1+r)^t(u-d)}.$$

Proof. Let $\phi = (\phi_0(t), \phi_1(t))$ be the replicating strategy for the European call option with claim $X = (S_1(T) - K)^+$. The discounted value process is a martingale. Then the value process $V_\phi(t)$ is given by

$$V_\phi(t) = S_0(t)\tilde{V}_\phi(t) = (1+r)^t E^*[\tilde{V}_\phi(T)|\mathcal{F}_t]$$

$$= (1+r)^{-(T-t)} E^*[(S_1(T)-K)^+|\mathcal{F}_t] = (1+r)^{-(T-t)} E^*[(S_1(t)\Pi_{i=t+1}^T Z(i)-K)^+|\mathcal{F}_t]$$

$$1+r)^{-(T-t)} E^*[(x\Pi_{i=t+1}^T Z(i) - K)^+|\mathcal{F}_t]|_{x=S_1(t)} = C(t, S_1(t))$$

Therefore, we have

$$\phi_0(t)(1+r)^t + \phi_1(t)S_1(t) = C(t, S_1(t))$$

Now $S_1(t) = S_1(t-1)u$ or $S_1(t-1)d$, so:

$$\phi_0(t)(1+r)^t + \phi_1(t)S_1(t-1)u = C(t, S_1(t-1)u)$$

$$\phi_0(t)(1+r)^t + \phi_1(t)S_1(t-1)d = C(t, S_1(t-1)d)$$

Subtract:

$$\phi_1(t)S_1(t-1)(u-d) = C(t, S_1(t-1)u) - C(t, S_1(t-1)d)$$

which yields

$$\phi_1(t) = \frac{C(t, S_1(t-1)u) - C(t, S_1(t-1)d)}{S_1(t-1)(u-d)}$$

Use any of the equations in the above system to obtain

$$\phi_0(t) = \frac{uC(t, S_1(t-1)d) - dC(t, S_1(t-1)u)}{(1+r)^t(u-d)}.$$