Martingale Theory for Finance

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1 Introduction

2 Probability spaces and σ-fields

3 Integration with respect to a probability measure.

4 Conditional expectation.

5 Martingales.

Continuous time martingales

So far the time index we consider is $T = \{0, 1, 2, ...\}$. Almost all the results for discrete time martingales also hold for continuous time martingales. We will list some of them without proofs. Let $X = (X_t, t \geq 0)$ be a stochastic process, i.e., a collection of random variables which describe systems that evolve randomly in time. Let $(\mathcal{F}_t, t \geq 0)$ be a family of increasing σ-fields, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Definition 5.1 $X = (X_t, t \geq 0)$ *is said to be a martingale (supermartingale, submartingale) with respect to* $(\mathcal{F}_t, t \geq 0)$ *if*

(i) X_t *is* $\mathcal{F}_t$ *-measurable (determined),*

(ii)X_t *is integrable, i.e.,* $E[|X_t|] < \infty$,

(iii) for every $s \leq t$,

$$E[X_t|\mathcal{F}_s] = X_s \quad (E[X_t|\mathcal{F}_s] \leq X_s, \quad E[X_t|\mathcal{F}_s] \geq X_s)$$

Next we introduce a popular stochastic process, the so called Brownian motion, that is widely used in applications, particularly in finance.

Definition 5.2 $X = (X_t, t \geq 0)$ *ia said to be a Brownian motion if*

(i) X_t *is continuous in* t *with* $X_0 = 0$,

(ii)X_t *has independent increments, i.e., for any* $s_1 < t_1 < s_2 < t_2 < ... < s_n < t_n$,

$$X_{t_n} - X_{s_n}, X_{t_{n-1}} - X_{s_{n-1}}, ..., X_{t_1} - X_{s_1}$$

are independent random variables

(iii) for every $s \leq t$, $X_t - X_s \sim N(0, t-s)$. *In particular,* $X_t \sim N(0, t)$.

Example 5.3 *Show a Brownian motion $X = (X_t, t \geq 0)$ is a martingale w.r.t. $\mathcal{F}_t = \sigma(X_u, u \leq t), t \geq 0$.*

Proof. We will check the three conditions in the definition of martingales. The first condition (i) is clear since the σ field $\mathcal{F}_t$ is generated by the process X. As $X_t \sim N(0, t)$, X_t is integrable. For the condition (iii), let $s < t$. We have

$$E[X_t|\mathcal{F}_s] = E[X_t - X_s + X_s|\mathcal{F}_s]$$
$$= E[X_t - X_s|\mathcal{F}_s] + E[X_s|\mathcal{F}_s] = X_s + E[X_t - X_s] = X_s$$

(iii) holds.

Example 5.4 *Let $X = (X_t, t \geq 0)$ be a Brownian motion. Show $M_t = X_t^2 - t, t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t = \sigma(X_u, u \leq t), t \geq 0$.*

Proof. We again need to check the three conditions in the definition of martingales. The first condition (i) is clear since M_t is a function of X_t. As $X_t \sim N(0, t)$, $X_t^2 - t = M_t$ is integrable. For the condition (iii), let $s < t$. We have

$$E[M_t|\mathcal{F}_s] = E[X_t^2 - t|\mathcal{F}_s]$$
$$= E[X_s^2 - s + X_t^2 - X_s^2 - t + s|\mathcal{F}_s] = X_s^2 - s + E[X_t^2 - X_s^2 - t + s|\mathcal{F}_s]$$
$$= M_s + E[X_t^2 - 2X_tX_s + X_s^2 - 2X_s^2 + 2X_tX_s - t + s|\mathcal{F}_s]$$
$$= M_s + E[(X_t - X_s)^2|\mathcal{F}_s] + E[-2X_s^2 + 2X_tX_s - t + s|\mathcal{F}_s]$$
$$= M_s + E[(X_t - X_s)^2] - 2X_s^2 + 2X_sE[X_t|\mathcal{F}_s] - (t - s)$$
$$= M_s + (t - s)^2 - 2X_s^2 + 2X_sX_s - (t - s) = M_s$$

(iii) is proved.

Definition 5.5 *$N = (N_t, t \geq 0)$ ia said to be a Poisson process of rate λ if*

(i) N_t is right continuous with left limits in t and $N_0 = 0$,

(ii)N_t has independent increments, i.e., for any $s_1 < t_1 < s_2 < t_2 < ... < s_n < t_n$,

$$N_{t_n} - N_{s_n}, N_{t_{n-1}} - N_{s_{n-1}}, ..., N_{t_1} - N_{s_1}$$

are independent random variables

(iii) for every $s \leq t$, $N_t - N_s \sim Poi(\lambda(t - s))$, i.e.,

$$P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t - s))^k}{k!}, k = 0, 1, ...,$$

Example 5.6 *Let $N = (N_t, t \geq 0)$ be a Poisson process of rate λ. Show $M_t = N_t - \lambda t, t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t = \sigma(N_u, u \leq t), t \geq 0$.*

Proof. We again need to check the three conditions in the definition of martingales. The first condition (i) is clear since M_t is a function of N_t. As N_t has a Poisson distribution, M_t is integrable. For the condition (iii), let $s < t$. We have

$$E[M_t|\mathcal{F}_s] = E[N_t - \lambda t|\mathcal{F}_s]$$
$$= E[M_s + (N_t - N_s) - \lambda(t-s)|\mathcal{F}_s] = M_s + E[(N_t - N_s) - \lambda(t-s)|\mathcal{F}_s]$$
$$= M_s + E[(N_t - N_s)] - \lambda(t-s)$$
$$= M_s + \lambda(t-s) - \lambda(t-s) = M_s$$

(iii) is proved.

Example 5.7 *Let $N = (N_t, t \geq 0)$ be a Poisson process of rate λ. For $\theta > 0$, show $V_t = exp(-\theta N_t + \lambda t(1 - e^{-\theta})), t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t = \sigma(N_u, u \leq t), t \geq 0$.*

Proof. We again need to check the three conditions in the definition of martingales. The first condition (i) is clear since V_t is a function of N_t. As N_t has a Poisson distribution, V_t is integrable. For the condition (iii), let $s < t$. We have

$$E[V_t|\mathcal{F}_s] = E[V_s \cdot exp(-\theta(N_t - N_s) + \lambda(t-s)(1-e^{-\theta}))|\mathcal{F}_s]$$
$$= V_s E[exp(-\theta(N_t - N_s) + \lambda(t-s)(1-e^{-\theta}))|\mathcal{F}_s]$$
$$= V_s E[exp(-\theta(N_t - N_s) + \lambda(t-s)(1-e^{-\theta}))]$$
$$= V_s E[exp(-\theta(N_t - N_s))] exp(\lambda(t-s)(1-e^{-\theta}))$$

Note that

$$E[exp(-\theta(N_t - N_s))] = \sum_{k=0}^{\infty} exp(-\theta k) P(N_t - N_s = k)$$
$$= \sum_{k=0}^{\infty} exp(-\theta k) e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$
$$= e^{-\lambda(t-s)} \sum_{k=0}^{\infty} (exp(-\theta))^k \frac{(\lambda(t-s))^k}{k!}$$
$$= e^{-\lambda(t-s)} exp(\lambda(t-s) exp(-\theta))$$

Substitute this back to the above equation to get

$$E[V_t|\mathcal{F}_s] = V_s$$

The proof is complete.

To state the Doob's optional stopping theorem, we introduce stopping times. A non-negative random variable σ is a stopping time with respect to $\mathcal{F}_t, t \geq 0$ if for any $t \geq 0$, $\{\sigma \leq t\} \in \mathcal{F}_t$.

Theorem 5.8 *(Doob's optional stopping theorem). Let $X = (X_t)_{t\geq 0}$ be a martingale and σ an almost surely finite stopping time. In each of the following two cases, we have $E[X_\sigma] = E[X_0]$.*

Case (i). σ is bounded, i.e., there is a constant T such that $\sigma \leq T$.

Case (ii). The sequence $X_{\sigma\wedge t}, t \geq 0$ is bounded by some integrable random variable Y i.e.,

$$|X_{\sigma\wedge t}| \leq Y$$

for all $t \geq 0$.

Doob's maximum inequality and martingale convergence theorem

Theorem 5.9 *Let $X = (X_t, t \geq 0)$ be a martingale such that $E[|X_t|^p] < \infty$, for some $p \geq 1$. Then for every $T > 0$,*

$$P(max_{0\leq t\leq T}|X_t| \geq \lambda) \leq \frac{E[|X_T|^p]}{\lambda^p}$$

and if $p > 1$

$$E[max_{0\leq t\leq T}|X_t|^p] \leq (\frac{p}{p-1})^p E[|X_T|^p]$$

Theorem 5.10 *Let $X = (X_t, t \geq 0)$ be a martingale such that $\sup_t E[|X_t|^p] < \infty, n = 0$ for some $p \geq 1$. Then*

$$X(\omega) = \lim_{t\to\infty} X_t(\omega)$$

almost surely.

Example 5.11 *Let $X = (X_t, t \geq 0)$ be a Brownian motion with $X_0 = 0$. Find the probability that the Brownian motion X leaves the interval $[-a, b]$ at the point b, where a, b are two positive numbers.*

Solution. We already knew that X is a martingale. Define

$$\sigma = \inf\{t \geq 0; X_t = -a \quad \text{or} \quad X_t = b\}$$

σ is the first time at which X hits $-a$ or b. The σ is a stopping time with respect to $\mathcal{F}_t = \sigma(X_u, u \leq t), t \geq 0$. Since $X_\sigma = -a$ or $X_\sigma = b$, we have

$$(1).P(X_\sigma = -a) + P(X_\sigma = b) = 1$$

Since $|X_{\sigma\wedge t}| \leq a + b$ for all $t \geq 0$, it follows from the Doob's theorem that $E[X_\sigma] = E[X_0] = 0$, which is

$$(2).E[X_\sigma] = (-a)P(X_\sigma = -a) + bP(X_\sigma = b) = 0$$

Solve (1), (2) together to get

$$P(X_\sigma = -a) = \frac{b}{a+b}, \qquad P(X_\sigma = -a) = \frac{a}{a+b}$$

In this lecture notes, all the processes considered are assumed to be right continuous with left limits.

Definition 5.12 *$X = (X_t, t \geq 0)$ is said to be a local martingale if there exists an increasing sequence $\{\tau_n, n \geq 1\}$ of stopping times such that*

(i) $\tau_n \to \infty$ almost surely as $n \to \infty$,

(ii) for every n, the stopped process $\{X_{t\wedge\tau_n}, t \geq 0\}$ is a martingale.

Let $f(t)$ be a real-valued function on $[0, \infty)$.

Definition 5.13 *We say that f is of bounded variation on the interval $[0, T]$ if*

$$\sup_{\tau^n} \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| < \infty,$$

where the sup is taken over all the possible partitions $\tau^n = \{t_0^n = 0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = T\}$ of the interval $[0, T]$.

Example 5.14 *If f is differentiable, say $f(t) = \int_0^t g(s)ds$, then f is of bounded variation on any finite interval $[0, T]$.*

Solution. Let $\tau^n = \{t_0^n = 0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = T\}$ be any partition of the interval $[0, T]$. We have

$$\begin{aligned} & \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| \\ = & \sum_{i=0}^{k_n-1} \left| \int_0^{t_{i+1}^n} g(s)ds - \int_0^{t_i^n} g(s)ds \right| = \sum_{i=0}^{k_n-1} \left| \int_{t_i^n}^{t_{i+1}^n} g(s)ds \right| \\ \leq & \sum_{i=0}^{k_n-1} \int_{t_i^n}^{t_{i+1}^n} |g(s)|ds = \int_0^T |g(s)|ds, \end{aligned} \tag{5.1}$$

which implies that

$$\sup_{\tau^n} \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| \leq \int_0^T |g(s)|ds < \infty.$$

Example 5.15 *If $f(t)$ is an increasing function, then f is of bounded variation on any finite interval $[0, T]$.*

Solution. Let $\tau^n = \{t_0^n = 0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = T\}$ be any partition of the interval $[0, T]$. We have

$$\begin{aligned} & \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| \\ = & \sum_{i=0}^{k_n-1} (f(t_{i+1}^n) - f(t_i^n)) = f(T) - f(0). \end{aligned} \tag{5.2}$$

Hence,

$$\sup_{\tau^n} \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| \leq f(T) - f(0) < \infty.$$

Definition 5.16 *We say that a process* $A = (A_t)_{t\geq 0}$ *is a bounded variation process if for almost all* ω*, the function* $t \to A_t(\omega)$ *is of bounded variation.*

Definition 5.17 *A process* $X = (X_t)_{t\geq 0}$ *is said to be a semimartingale if* $X_t = X_0 + M_t + A_t, t \geq 0$ *for some local martingale* M *and bounded variation process* A*.*

Let X, Y be two semimartingales.

Definition 5.18 *Let* X*,* Y *be two semimartingales. The quadratic covariation process of* X *and* Y*, denoted by* $[X, Y]_t, t \geq 0$*, is defined as*

$$[X, Y]_t = \lim_{k\to\infty} \sum_{i=1}^{n_k} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}),$$

where $\{0 = t_0 < t_1 < ... < t_{n_k-1} < t_{n_k} = t\}$ *is a sequence of partitions of the interval* $[0, t]$ *such that* $\Delta_k = max_{1\leq n_k}(t_i - t_{i-1}) \to 0$ *as* $k \to \infty$*.*

If $X = Y$, $[X, X]$ is also called the quadratic variation process of X.

Example 5.19 *If* A *is a continuous process of bounded variation, then* $[A, A] = 0$*.*

Solution. Let $\{0 = t_0 < t_1 < ... < t_{n_k-1} < t_{n_k} = t\}$ be a sequence of partitions of the interval $[0, t]$ such that $\Delta_k = max_{1\leq n_k}(t_i - t_{i-1}) \to 0$ as $k \to \infty$. We have

$$\begin{aligned} &\sum_{i=1}^{n_k} (A_{t_i} - A_{t_{i-1}})^2 \leq \sup_i |A_{t_i} - A_{t_{i-1}}| \cdot \sum_{i=1}^{n_k} |A_{t_i} - A_{t_{i-1}}| \\ \leq\ & C_t \sup_i |A_{t_i} - A_{t_{i-1}}|, \end{aligned} \tag{5.3}$$

where C_t is some constant because A is of bounded variation. Since A_t is continuous in t and since $\Delta_k = max_{1\leq n_k}(t_i - t_{i-1}) \to 0$, it follows that $\sup_i |A_{t_i} - A_{t_{i-1}}| \to 0$ as $k \to \infty$. Hence we deduce that

$$[A, A]_t = \lim_{k\to\infty} \sum_{i=1}^{n_k} (A_{t_i} - A_{t_{i-1}})^2 = 0$$

Example 5.20 *Let* B *be a standard Brownian motion. Let* $\{0 = t_0 < t_1 < ... < t_{n_k-1} < t_{n_k} = t\}$ *be a sequence of partitions of the interval* $[0, t]$ *such that* $\Delta_k = max_{1\leq n_k}(t_i - t_{i-1}) \to 0$ *as* $k \to \infty$*. Prove*

$$\lim_{k\to\infty} E[\left(\sum_{i=1}^{n_k} (B_{t_i} - B_{t_{i-1}})^2 - t\right)^2] = 0$$

Hence, $[B, B]_t = t$*.*

Proof. Noting that $t = \sum_{i=1}^{n_k}(t_i - t_{i-1})$, we have

$$\sum_{i=1}^{n_k}(B_{t_i} - B_{t_{i-1}})^2 - t = \sum_{i=1}^{n_k}\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\},$$

Hence,

$$\begin{aligned}
& E[(\sum_{i=1}^{n_k}(B_{t_i} - B_{t_{i-1}})^2 - t)^2] \\
= & \sum_{i=1,j=1}^{n_k} E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}\{(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})\}] \\
= & \sum_{i\neq j}^{n_k} E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}\{(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})\}] \\
& + \sum_{i=1}^{n_k} E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}^2] \qquad (5.4)
\end{aligned}$$

If $i \neq j$, by the independence we have

$$\begin{aligned}
& E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}\{(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})\}] \\
= & E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}]E[\{(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})\}] \\
& = 0 \qquad (5.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}^2] \\
= & E[(B_{t_i} - B_{t_{i-1}})^4] - 2E[(B_{t_i} - B_{t_{i-1}})^2](t_i - t_{i-1}) + (t_i - t_{i-1})^2 \\
= & E[(B_{t_i} - B_{t_{i-1}})^4] - (t_i - t_{i-1})^2 \\
\leq & C(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2. \qquad (5.6)
\end{aligned}$$

Combining the above calculations together, we obtain

$$\begin{aligned}
& E[(\sum_{i=1}^{n_k}(B_{t_i} - B_{t_{i-1}})^2 - t)^2] \\
\leq & \sum_{i=1}^{n_k}(C+1)(t_i - t_{i-1})^2 \leq (C+1)t \max_i (t_i - t_{i-1}) \\
& \to 0 \qquad (5.7)
\end{aligned}$$

as $k \to \infty$.

Finally we state the Doob-Meyer Decomposition Theorem without proof.

Theorem 5.21 *(Doob-Meyer Decomposition Theorem) Let Z be a supermartingale. Then Z has a decomposition $Z_t = Z_0 + M_t - A_t, t \geq 0$, where M is a local martingale and A is a predictable, increasing processes, and $M_0 = A_0 = 0$. Such a decomposition is unique.*