Martingale Theory for Finance

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1 Introduction

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5 Martingales.

For a family $\{X_1, X_2, ...X_n\}$ of random variables, denote by $\sigma(X_1, X_2, ..., X_n)$ the smallest σ-field containing the events of the form $\{\omega; a < X_k(\omega) < b\}, k = 1, ..., n$ for all choices of a, b. $\sigma(X_1, X_2, ..., X_n)$ is called the σ-field generated by $X_1, X_2, ..., X_n$. Random variables determined by $\sigma(X_1, X_2, ..., X_n)$ are functions of $X_1, X_2, ..., X_n$. To introduce the notion of martingales we begin with an example. Consider a series of games decided by the tosses of a coin, in which we either win £1 with probability p or lose £1 with probability $q = 1 - p$ in each round. Let X_i denote the net gain in the $i - th$ round. Then $X_i, i = 1, 2, ...$ are independent random variables with

$$P(X_i = 1) = p, \quad P(X_i = -1) = q$$

and so $E(X_i) = p - q$.

Our total net gain (possibly negative) after the $n - th$ round is given by $S_0 = 0$ and

$$S_n = X_1 + X_2 + ... + X_n, n = 1, 2, ...$$

Let $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ denote the σ-field generated by $X_1, X_2, ...X_n$. The $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\mathcal{F}_n$ can be regarded as the history of the games up to time n (the n-th round). Let us now compute the average gain after the $n + 1$-th round given the history up to time n. We have

$$E[S_{n+1}|\mathcal{F}_n]$$

$$= E[S_n + X_{n+1}|\mathcal{F}_n]$$

$$= E[S_n|\mathcal{F}_n] + E[X_{n+1}|\mathcal{F}_n]$$
$$= S_n + E[X_{n+1}],$$

where we used the fact that $S_n = X_1 + X_2 + ... + X_n$ is determined by $\mathcal{F}_n$ and X_{n+1} is independent of $\mathcal{F}_n$. Thus,

$$E[S_{n+1}|\mathcal{F}_n] = S_n + p - q$$

$$= \begin{cases} S_n & \text{if the game is fair, i.e. } p = q = \frac{1}{2} \\ > S_n, & \text{if } p > q, \\ < S_n, & \text{if } p < q. \end{cases}$$

In the first case, the game is fair, $\{S_n, n \geq 0\}$ is called a martingale. It is called a submartingale in the second case and a supermartingale in the third case.

Definition 5.1 *A sequence of random variables $Z_0, Z_1, Z_2, ...Z_n, ...$ is said to be a martingale (submartingale, or supermartingale) with respect to an increasing sequence $\mathcal{F}_0 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$ of σ-fields if*

(i) Z_n is determined by $\mathcal{F}_n$,

(ii) Z_n is integrable, i.e., $E[|Z_n|] < \infty$,

(iii) $E[Z_{n+1}|\mathcal{F}_n] = Z_n$. ($E[Z_{n+1}|\mathcal{F}_n] \geq Z_n$, $E[Z_{n+1}|\mathcal{F}_n] \leq Z_n$)

Remarks. 1. The notion of martingales is a mathematical formulation for fair games.

2.The typical choice of $\mathcal{F}_n$ is $\mathcal{F}_n = \sigma(Z_1, Z_2, ...Z_n)$, the σ-field generated by $Z_1, Z_2, ..., Z_n$.

3. If $Z_n, n \geq 0$ is a martingale, then

$$E[Z_n] = E[E[Z_{n+1}|\mathcal{F}_n]] = E[Z_{n+1}]$$

The expectation remains constant for all $n \geq 1$.

4. If $Z_n, n \geq 0$ is a martingale, then we have

$$E[Z_{n+2}|\mathcal{F}_n] = E[E[Z_{n+2}|\mathcal{F}_{n+1}]|\mathcal{F}_n]$$

$$= [Z_{n+1}|\mathcal{F}_n] = Z_n$$

More generally, it holds that for any $m > n$, $E[Z_m|\mathcal{F}_n] = Z_n$.

Examples.

(1). If $X_0, X_1, X_2, ...X_n, ...$ are independent, integrable random variables with $E[X_i] = 0$ for all i, then $S_n = X_1 + X_2 + ... + X_n, n \geq 1, S_0 = 0$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, X_2, ...X_n)$

Since S_n is a function of $X_0, X_1, X_2, ...X_n$, it is $\mathcal{F}_n$-determined. We are left to check the third condition (iii). In fact,

$$E[S_{n+1}|\mathcal{F}_n]$$

$$= E[S_n + X_{n+1}|\mathcal{F}_n]$$
$$= E[S_n|\mathcal{F}_n] + E[X_{n+1}|\mathcal{F}_n]$$
$$= S_n + E[X_{n+1}] = S_n$$

So (iii) is true and $S_n, n \geq 0$ is a martingale.

(2). Let Y be an integrable r.v. and $\{\mathcal{F}_n, n \geq 1\}$ be a sequence of increasing σ-fields. Define $Z_n = E[Y|\mathcal{F}_n], n \geq 1$. Then $\{Z_n, n \geq 1\}$ is a martingale w.r.t. $\{\mathcal{F}_n, n \geq 1\}$. We need to show that $\{Z_n, n \geq 1\}$ satisfies the definition of a martingale. By the definition of the conditional expectation, Z_n is $\mathcal{F}_n$-determined. For (ii), we notice that

$$|Z_n| \leq E[|Y||\mathcal{F}_n]$$

Hence $E[|Z_n|] \leq E[|Y|]$. Let us now check (iii). By the property of the conditional expectation,

$$E[Z_{n+1}|\mathcal{F}_n] = E[E[Y|\mathcal{F}_{n+1}]|\mathcal{F}_n]] = E[Y|\mathcal{F}_n] = Z_n$$

This proves (iii).

(3). If $Y_1, Y_2, ...Y_n, ...$ are independent, integrable r.v.'s with $a_i = E(Y_i) \neq 0$ for all i, then

$$Z_n = \frac{Y_1 Y_2 ... Y_n}{a_1 a_2 ... a_n}, \quad n = 1, 2, ...$$

is a martingale w.r.t. $\mathcal{F}_n = \sigma(Y_1, Y_2, ..., Y_n), n \geq 1$

Since Z_n is a function of $Y_1, Y_2, ..., Y_n$, Z_n is determined by $\mathcal{F}_n$. Z_n is integrable because Y_i are integrable and independent. We now check (iii). We have

$$E[Z_{n+1}|\mathcal{F}_n] = E[\frac{Y_1 Y_2 ... Y_n Y_{n+1}}{a_1 a_2 ... a_n a_{n+1}}|\mathcal{F}_n]$$
$$= \frac{Y_1 Y_2 ... Y_n}{a_1 a_2 ... a_n} E[\frac{Y_{n+1}}{a_{n+1}}|\mathcal{F}_n]$$
$$= \frac{Y_1 Y_2 ... Y_n}{a_1 a_2 ... a_n} E[\frac{Y_{n+1}}{a_{n+1}}] = \frac{Y_1 Y_2 ... Y_n}{a_1 a_2 ... a_n} = Z_n$$

Example 5.2 *Let $\{S_n\}_{n \geq 0}$ be the net gain process in a series of fair games. Then $S_0 = 0$, $S_n = X_1 + X_2 + ... + X_n, n \geq 1$, and $P(X_i = 1) = \frac{1}{2}$, $P(X_i = -1) = \frac{1}{2}$. We already know that $\{S_n\}_{n \geq 0}$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(Y_1, Y_2, ..., Y_n), n \geq 1$. Prove that $\{Y_n = S_n^2 - n, n \geq 0\}$ is also a martingale w.r.t. $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$.*

Proof. Since Y_n is a function of $X_1, X_2, ..., X_n$, Y_n is $\mathcal{F}_n$-measurable. As $|Y_n| \leq n + n^2$, Y_n is integrable. It remains to show that $E[Y_{n+1}|\mathcal{F}_n] = Y_n$. To this end, writing

$$Y_{n+1} = S_{n+1}^2 - (n+1) = (S_n + X_{n+1})^2 - (n+1)$$

$$= S_n^2 - n + 2S_nX_{n+1} + X_{n+1}^2 - 1$$
$$= Y_n + 2S_nX_{n+1} + X_{n+1}^2 - 1,$$

we have

$$E[Y_{n+1}|\mathcal{F}_n] = E[Y_n|\mathcal{F}_n] + E[2S_nX_{n+1}|\mathcal{F}_n]$$
$$+E[X_{n+1}^2|\mathcal{F}_n] - 1$$
$$= Y_n + 2S_nE[X_{n+1}|\mathcal{F}_n] + E[X_{n+1}^2] - 1$$
$$= Y_n + 2S_nE[X_{n+1}] + E[X_{n+1}^2] - 1 = Y_n + 0 + 1 - 1 = Y_n.$$

Example 5.3 *Let $X_1, X_2, ..., X_n, n \geq 1$ be a sequence of independent random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q$. Set $S_n = X_1 + X_2 + ... + X_n$ and define*

$$Z_n = \left(\frac{q}{p}\right)^{S_n}, n \geq 1.$$

Show $\{Z_n, n \geq 1\}$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$.

Proof. Since Z_n is a function of $X_1, X_2, ..., X_n$, Z_n is $\mathcal{F}_n$-measurable. Since $|Z_n| \leq \left(\frac{q}{p}\right)^n + \left(\frac{p}{q}\right)^n$, Z_n is integrable. Let us check (iii)

$$E[Z_{n+1}|\mathcal{F}_n] = E[\left(\frac{q}{p}\right)^{S_{n+1}}|\mathcal{F}_n]$$
$$= E[\left(\frac{q}{p}\right)^{S_n}\left(\frac{q}{p}\right)^{X_{n+1}}|\mathcal{F}_n]$$
$$= \left(\frac{q}{p}\right)^{S_n}E[\left(\frac{q}{p}\right)^{X_{n+1}}|\mathcal{F}_n]$$
$$= Z_nE[\left(\frac{q}{p}\right)^{X_{n+1}}]$$
$$= Z_n\{\left(\frac{q}{p}\right)P(X_{n+1} = 1) + \left(\frac{q}{p}\right)^{-1}P(X_{n+1} = -1)\}$$
$$= Z_n\{\left(\frac{q}{p}\right)p + \left(\frac{q}{p}\right)^{-1}q\} = Z_n(p + q) = Z_n.$$

Stopping times

Suppose we play a series of games by tossing a fair coin. As we know, the net gain at time n $S_n, n \geq 1$ is a martingale. If we quit at a fixed time n, then $E(S_n) = 0$. It is not very exciting. However, we can stop playing as soon as our net gain reaches 100. This amounts to saying that we will stop at the random time

$$T = min\{n; S_n = 100\}$$

Such random times are called stopping times. Here is the definition.

Definition 5.4 *Let $\{\mathcal{F}_n, n \geq 1\}$ be an increasing sequence of σ-fields. A random variable $T(\omega)$ taking values in the set $\{0, 1, 2, 3, ...\infty\}$ is called a stopping time (or optional time) w.r.t. $\{\mathcal{F}_n, n \geq 1\}$ if for every n, $\{T \leq n\} \in \mathcal{F}_n$.*

Example 5.5 *Let $S_n = X_1 + X_2 + ... + X_n, n \geq 1$ be the net gain process in a series of games. Set $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$. Define*

$$T = min\{n; S_n = 100\}$$

Then T is a stopping time w.r.t. $\{\mathcal{F}_n, n \geq 1\}$.

In fact, for every n,

$$\{T \leq n\} = \cup_{k=1}^{n} \{S_k = 100\} \in \mathcal{F}_n.$$

Stopped processes

Let T be a stopping time. Let $Z_n, n \geq 1$ be a sequence of random variables. Set $T \wedge n = min(T, n))$. Define

$$\hat{Z}_n = Z_{T\wedge n} = \begin{cases} Z_n & \text{if } n \leq T(\omega), \\ Z_{T(\omega)} & \text{if } n > T(\omega). \end{cases}$$

$Z_{T\wedge n}, n \geq 1$ is the stopped process of Z at the stoping time T.

Theorem 5.6 *If $Z_0, Z_1, ..., Z_n..$ is a martingale w.r.t. $\{\mathcal{F}_n, n \geq 1\}$, then the stopped process $Y_n = Z_{T\wedge n}$ is also a martingale w.r.t. $\{\mathcal{F}_n, n \geq 1\}$. In particular, $E[Z_{T\wedge n}] = E[Z_0]$.*

Proof. Observe that

$$Y_{n+1} = Z_{T\wedge(n+1)} = Z_{T\wedge n} + I_{\{T \geq n+1\}}(Z_{n+1} - Z_n)$$

$$= Y_n + I_{\{T \geq n+1\}}(Z_{n+1} - Z_n)$$

In fact, if $T \leq n$, the left is equal to the right, which is Z_T. While if $T \geq n+1$, then the left is also equal to the right, which is Z_{n+1}. Thus we have

$$E[Y_{n+1}|\mathcal{F}_n]$$

$$= E[Y_n + I_{\{T \geq n+1\}}(Z_{n+1} - Z_n)|\mathcal{F}_n]$$

Since $\{T \geq n+1\} = (\{T \leq n\})^c \in \mathcal{F}_n$, it follows that

$$E[Y_n + I_{\{T \geq n+1\}}(Z_{n+1} - Z_n)|\mathcal{F}_n]$$

$$= Y_n + I_{\{T \geq n+1\}} E[(Z_{n+1} - Z_n)|\mathcal{F}_n]$$

$$= Y_n + 0$$

which completes the proof.

If $Z_n, n \geq 0$ is a martingale, then we know that $E[Z_n] = E[Z_{n-1}] = ...E[Z_0]$. Now the question is what will happen if the deterministic time n is replaced by a stopping time T. Namely, will $E[Z_T] = E[Z_0]$ be still true? Before answering the question, let us look at one example. Let $S_n, n \geq 0$ denote the capital gain in a series of fair games. Define $T = min\{n; S_n = 10\}$, for example, the time at which the capital reaches 10. Then $E[S_T] = E[10] = 10 \neq E[S_0] = 0$. But the following theorem holds.

Theorem 5.7 *(Doob's optional stopping theorem). Let $Z_0, Z_1, ...$ be a martingale and T an almost surely finite stopping time. In each of the following three cases, we have $E[Z_T] = E[Z_0]$.*

Case (i). T is bounded, i.e., there is an integer m such that $T(\omega) \leq m$.

Case (ii). The sequence $Z_{T\wedge n}, n \geq 0$ is bounded in the sense that there is a constant K such that

$$|Z_{T\wedge n}| \leq K$$

for all n.

Case (iii): $E[T] < \infty$ and the step $Z_n - Z_{n-1}$ are bounded, i.e., there is a constant C such that $|Z_n - Z_{n-1}| \leq C$ for all n.

Proof. The proof is an application of the dominated convergence theorem. First of all, we note that $Z_{T\wedge n} \to Z_T$ as $n \to \infty$. Since the stopped process $Z_{T\wedge n}$ is a martingale, we always have

$$E[Z_{T\wedge n}] = E[Z_0]$$

Case (i): Since $T \leq m$, we have $T \wedge m = T$ and so

$$E[Z_T] = E[Z_{T\wedge m}] = E[Z_0]$$

Case (ii). Since $|Z_{T\wedge n}| \leq K$ is bounded for all n, it follows from the dominated convergence theorem that

$$E[Z_T] = \lim_{n\to\infty} E[Z_{T\wedge n}] = E[Z_0]$$

Case (iii): Write

$$Z_{T\wedge n} = Z_{T\wedge n} - Z_{T\wedge n-1} + Z_{T\wedge n-1} - Z_{T\wedge n-2}$$

$$\cdots + Z_3 - Z_2 + Z_2 - Z_1 + Z_1 - Z_0 + Z_0$$

Consequently,

$$|Z_{T\wedge n}| \leq |Z_{T\wedge n} - Z_{T\wedge n-1}| + |Z_{T\wedge n-1} - Z_{T\wedge n-2}|$$

$$\cdots + |Z_3 - Z_2| + |Z_2 - Z_1| + |Z_1 - Z_0| + |Z_0|$$

$$\leq C + C + \cdots + C + |Z_0| \leq (T \wedge n)C + |Z_0| \leq CT + |Z_0|.$$

By virtue of the dominated convergence theorem, we have

$$E[Z_T] = \lim_{n \to \infty} E[Z_{T \wedge n}] = E[Z_0]$$

Application of Doob's optimal stopping theorem: Gambler's ruin problem

Gambler A and B play a series of games against each other in which a fair coin is tossed repeatedly. In each game gambler A wins or loses £1 according as the toss results in a head or a tail respectively. The initial capital of gambler A is £a and that of gambler B is £b and they continue playing until one of them is ruined. Determine the probability that A will be ruined and also the expected number of games played.

Let $\hat{S}_n$ be the fortune of gambler A after the n-th game. Then

$$\hat{S}_n = a + X_1 + X_2 + \cdots + X_n = a + S_n,$$

where $X_i, i = 1, 2...$ are independent r.v.'s with $P(X_i = 1) = \frac{1}{2}$, $P(X_i = -1) = \frac{1}{2}$. The game will stop at the time the gambler A or B is ruined, i.e., the game stop at

$$T = min\{n; \hat{S}_n = 0 \quad \text{or} \quad \hat{S}_n = a + b\}$$

$$= min\{n; S_n = -a \quad \text{or} \quad S_n = b\}$$

As discussed before, $\{S_n, n \geq 0\}$ forms a martingale. Since $S_T = -a$ or $S_T = b$, we have

$$(1). P(S_T = -a) + P(S_T = b) = 1$$

Note that

$$\{\text{Gambler } A \text{ is ruined}\} = \{S_T = -a\}$$

Since $|S_{T \wedge n}| \leq a + b$ for all $n \geq 1$, it follows from the Doob's theorem that $E[S_T] = E[S_0] = 0$, which is

$$(2). E[S_T] = (-a)P(S_T = -a) + bP(S_T = b) = 0$$

Solve (1), (2) together to get

$$P(S_T = -a) = \frac{b}{a + b}, \qquad P(S_T = b) = \frac{a}{a + b}$$

Next we will find the expected number of games $E(T)$. To this end, define $Y_n = S_n^2 - n, n \geq 0$. Then in example 4.2, we have showed (check) that Y_n is a martingale. So the stopped process $Y_{T \wedge n} = S_{T \wedge n}^2 - (T \wedge n)$ is also a martingale. Thus we have

$$E[Y_{T \wedge n}] = E[S_{T \wedge n}^2] - E[(T \wedge n)] = 0$$

This yields by dominated convergence theorem that

$$E[T] = \lim_{n\to\infty} E[T \wedge n] = \lim_{n\to\infty} E[S^2_{T\wedge n}]$$

$$= E(S_T^2) = (-a)^2 P(S_T = -a) + b^2 P(S_T = b)$$

$$= a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = ab$$

Example 5.8 *Consider a simple random walk* $\{S_n, n \geq 0\}$ *with* $0 < S_0 = k < N$*, for which each step is rightwards with probability* $0 < p < 1$ *and leftwards with probability* $q = 1 - p$*, i.e.,* $S_n = S_0 + X_1 + ... + X_n$ *with* $P(X_i = 1) = p, \quad P(X_i = -1) = q$*. Assume* $p \neq q$*. Use Doob optional stopping theorem to find the probability that the random walk hits* 0 *before hitting* N*.*

Solution. Write $X_1, X_2, ...X_n, ...$ for the independent steps of the walk. Then $S_0 = k$, $S_n = k + X_1 + ... + X_n$ with

$$P(X_i = 1) = p, \quad P(X_i = -1) = q.$$

Define $Z_n = (\frac{q}{p})^{S_n}, n \geq 0$. Then as we have seen before $\{Z_n, n \geq 0\}$ is a martingale w.r.t. $\{\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 0\}$. Let $T = min\{n; S_n = 0 \quad \text{or} \quad S_n = N\}$ be the first time at which the r.w. hits 0 or N. Then $S_T = 0$ means that the random walk hits 0 before reaching N and $S_T = N$ implies that the random walk hits N before reaching 0. Since $|Z_{n\wedge T}| = |(\frac{q}{p})^{S_{n\wedge T}}| \leq max_{1\leq l\leq N}(\frac{q}{p})^l = M$, by the Doob's optional theorem,

$$E[Z_T] = E[Z_0] = (\frac{q}{p})^k$$

On the other hand,

$$E(Z_T) = (\frac{q}{p})^0 P(S_T = 0) + (\frac{q}{p})^N P(S_T = N)$$

$$= P(S_T = 0) + (\frac{q}{p})^N (1 - P(S_T = 0))$$

Thus we have

$$P(S_T = 0) + (\frac{q}{p})^N (1 - P(S_T = 0)) = (\frac{q}{p})^k$$

Solve this equation for $P(S_T = 0)$ to get

$$P(S_T = 0) = \frac{(\frac{q}{p})^k - (\frac{q}{p})^N}{1 - (\frac{q}{p})^N}$$

Example 5.9 *Let $X_1, ..., X_n...$ be independent, integrable non-negative random variables with the same mean $E(X_i) = \mu$. Let T be a stopping time w.r.t. $\{\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 0\}$such that $E[T] < \infty$. Show that*

$$E[\sum_{i=1}^{T} X_i] = E[T]\mu$$

Solution. Let $Y_0 = 0, \quad Y_n = \sum_{i=1}^{n} X_i - n\mu$. Then $\{Y_n, n \geq 0$ is a martingale. Indeed,

$$E[Y_{n+1}|\mathcal{F}_n] = E[\sum_{i=1}^{n} X_i - n\mu + X_{n+1} - \mu|\mathcal{F}_n]$$

$$= E[Y_n + X_{n+1} - \mu|\mathcal{F}_n] = Y_n + E(X_{n+1}) - \mu = Y_n$$

In particular, the stopped process $Y_{n\wedge T}$ is also a martingale. This yields that

$$E[Y_{n\wedge T}] = E[\sum_{i=1}^{n\wedge T} X_i - (n \wedge T)\mu] = 0$$

By the monotone convergence theorem we arrive at

$$E[T]\mu = \lim_{n\to\infty} E[(n \wedge T)\mu] = \lim_{n\to\infty} E[\sum_{i=1}^{n\wedge T} X_i]$$

$$= E[\sum_{i=1}^{T} X_i]$$

Example 5.10 *If $Z_n, n \geq 1$ is a martingale w.r.t. $\{\mathcal{F}_n, n \geq 1\}$, prove that $|Z|_n^p, n \geq 1$ is a submartingale for all $p \geq 1$.*

Solution. Since $Z_n, n \geq 1$ is a martingale, $E[Z_{n+1}|\mathcal{F}_n] = Z_n$. It follows that

$$|Z_n|^p = |E[Z_{n+1}|\mathcal{F}_n]|^p \leq E[|Z_{n+1}|^p|\mathcal{F}_n]$$

Therefore $|Z|_n^p, n \geq 1$ is a submartingale.

Doob's maximum inequality and martingale convergence theorem

Theorem 5.11 *Let $X = (X_n, n \geq 0)$ be a martingale such that $E[|X_n|^p] < \infty, n = 0, 1, ...$ for some $p \geq 1$. Then for every N,*

$$P(max_{0\leq n\leq N}|X_n| \geq \lambda) \leq \frac{E[|X_N|^p]}{\lambda^p}$$

and if $p > 1$

$$E[max_{0\leq n\leq N}|X_n|^p] \leq (\frac{p}{p-1})^p E[|X_N|^p]$$

Proof. Let

$$T = \inf\{n; |X_n| \geq \lambda\} \wedge N$$

T is a bounded stopping time. Since $|X_n|^p, n \geq 1$ is a submartingale, we particularly have $E[|X_T|^p] \leq E[|X_N|^p]$. Note that

$$\{max_{0\leq n\leq N}|X_n| \geq \lambda\} \subset \{|X_T| \geq \lambda\}$$

and

$$\{max_{0\leq n\leq N}|X_n| < \lambda\} \subset \{T = N\}$$

We have

$$E[|X_N|^p] \geq E[|X_T|^p] = \int_{\{max_{0\leq n\leq N}|X_n|\geq\lambda\}} |X_T|^p dP + \int_{\{max_{0\leq n\leq N}|X_n|<\lambda\}} |X_T|^p dP$$

$$\geq \lambda^p P(max_{0\leq n\leq N}|X_n| \geq \lambda)$$

This immediately gives

$$P(max_{0\leq n\leq N}|X_n| \geq \lambda) \leq \frac{E[|X_N|^p]}{\lambda^p}$$

For a real-valued $(\mathcal{F}_n)$-adapted process $(X_n)_{n\geq 0}$ and an interval $[a, b]$, define a sequence of stopping times τ_n as follows

$$\tau_1 = min\{n; X_n \leq a\},$$

$$\tau_2 = min\{n \geq \tau_1; X_n \geq b\},$$

$$...$$

$$\tau_{2k+1} = min\{n \geq \tau_{2k}; X_n \leq a\},$$

$$\tau_{2k+2} = min\{n \geq \tau_{2k+1}; X_n \geq b\},$$

$$...$$

We set

$$U_N^X(a, b)(\omega) = max\{k; \tau_{2k}(\omega) \leq N\}.$$

Then $U_N^X(a, b)$ is the number of upcrossings of $(X_n)_{n=0}^N$ for the interval $[a, b]$.

Theorem 5.12 *Let* (X_n) *be a supermartingale. We have*

$$P(U_N^X(a, b) > j) \leq \frac{1}{b-a} \int_{\{U_N^X(a,b)=j\}} (X_N - a)^- dP, \tag{5.1}$$

$$E[U_N^X(a, b)] \leq \frac{1}{b-a} E[(X_N - a)^-]. \tag{5.2}$$

Proof. We can assume that the process is stopped at N and also $a = 0$. Otherwise consider the process $(X_{n\wedge N} - a)$. Set

$$S = \tau_{2j+1} \wedge (N+1), \quad T = \tau_{2(j+1)} \wedge (N+1)$$

Then $\{S \le N\} = \{\tau_{2j+1} \le N\}$, and $X_S = X_{\tau_{2j+1}} \le 0$ on $\{S \le N\}$. Note that $\{\tau_{2j+2} \le N\}$ is the event that that the process $(X_n)_{n=0}^N$ has at least $j+1$ upcrossings for the interval $[a, b]$. Hence, we have

$$\{U_N^X(0,b) > j\} = \{\tau_{2j+2} \le N\} = \{S < N, X_T \ge b\}$$

$$\{S < N, X_T < b\} = \{S < N, T = N+1\} \subset \{U_N^X(0,b) = j\}$$

Thus,

$$bP(U_N^X(0,b) > j) = \int_{\{U_N^X(0,b)>j\}} bdP = \int_{\{S<N,X_T\ge b\}} bdP$$

$$\le \int_{\{S<N,X_T\ge b\}} X_T dP = \int_{\{S<N\}} X_T dP - \int_{\{S<N,X_T<b\}} X_T dP$$

$$\le \int_{\{S<N\}} X_S dP - \int_{\{S<N,T=N+1\}} X_T dP \le 0 - \int_{\{S<N,T=N+1\}} X_{N+1} dP = -\int_{\{S<N,T=N+1\}} X_N dP$$

$$\le \int_{\{S<N,T=N+1\}} X_N^- dP \le \int_{\{U_N^X(0,b)=j\}} X_N^- dP$$

Adding the above inequality from $j = 0$ to ∞ we obtain

$$E[U_N^X(0,b)] \le \frac{1}{b} E[(X_N)^-]. \tag{5.3}$$

Theorem 5.13 *Let* $X = (X_n, n \ge 0)$ *be a martingale such that* $\sup_n E[|X_n|^p] < \infty, n = 0, 1, ...$ *for some* $p \ge 1$. *Then*

$$X(\omega) = \lim_{n\to\infty} X_n(\omega)$$

almost surely.

Proof. For $a < b$, set

$$U^X(a,b) = \lim_{N\to\infty} U_N^X(a,b).$$

$U^X(a,b)$ is the number of upcrossings of the process $(X_n)_{n\ge 0}$. By Theorem, we have

$$E[U^X(a,b)] \le \lim_{N\to\infty} E[U_N^X(a,b)] \le \sup_N E[(X_N - a)^-] < \infty$$

This yields that $P(W_{a,b}) = 0$, where $W_{a,b} = \{U^X(a,b) = \infty\}$. Set

$$V_{a,b} = \{\liminf_{n\to\infty} X_n < a, \limsup_{n\to\infty} X_n > b\}$$

Then $V_{a,b} \subset W_{a,b}$, and hence $P(V_{a,b}) = 0$. Since

$$\{\liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n\} = \cup_{a<b,a,b\in Q} V_{a,b},$$

where Q denotes the set of rational numbers, we deduce that

$$P(\{\liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n\}) = 0$$

Hence

$$P(\{\lim_{n\to\infty} X_n \quad \text{exists} \quad \}) = 1.$$

Example 5.14 *Let $\{\mathcal{F}_n, n \geq 0\}$ be a sequence of increasing σ-fields. Let Z be an integrable random variable. Set $X_n = E[Z|\mathcal{F}_n], n \geq 0$. Explain why the limit*

$$X = \lim_{n\to\infty} X_n$$

exists.

Solution. We knew from previous examples that $X_n, n \geq 0$ is a martingale. Moreover, we have

$$\sup_n E[|X_n|] \leq \sup_n E[E[|Z||\mathcal{F}_n]] = E[|Z|] < \infty$$

By Theorem 4.11, $X = \lim_{n\to\infty} X_n$ exists.

Example 5.15 *Let $X_1, X_2, ..., X_n, ...$ be independent random variables with $E[X_i] = \mu_i, Var(X_i) = \sigma_i^2$. If $\sum_{i=1}^{\infty} \mu_i < \infty$ and $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, show $\sum_{i=1}^{\infty} X_i(\omega)$ converges almost surely.*

Solution. Set $Y_n = \sum_{i=1}^{n}(X_i - \mu_i)$, $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$. It is easy to verify that $Y_n, n \geq 1$ is a martingale. Furthermore,

$$\sup_n E[Y_n^2] = \sup_n \sum_{i=1}^{n} E[(X_i - \mu_i)^2] = \sum_{i=1}^{\infty} \sigma_i^2 < \infty$$

It follows from the martingale convergence theorem that

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \sum_{i=1}^{n}(X_i - \mu_i)$$

exists. Since by assumption,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mu_i$$

exists, we conclude that

$$\sum_{i=1}^{\infty} X_i(\omega) = \lim_{n\to\infty} \sum_{i=1}^{n} X_i$$

exists.