Martingale theory for finance

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1 Financial markets in continuous time (MSc)

In this extra part for MSc, we will introduce financial market in continuous time, in particular the Black-Scholes model. For simplicity, we consider a market that consists of one riskless asset, whose price is denoted by S_t^0 and a single risky asset whose price will be denoted by S_t^1. (S_t^0, S_t^1) are stochastic processes on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that $S_t = (S_t^0, S_t^1)$ is $\mathcal{F}_t$ measurable.

Definition 1.1 *A trading strategy (or a portfolio) is a pair of stochastic processes $H_t = (H_t^0, H_t^1)$, where H_t^0 represents the amount of asset 0 held at time t and H_t^1 is the amount of asset 1 held at time t. We assume H_t is $\mathcal{F}_t$ measurable.*

Definition 1.2 *The value of a portfolio H at time t is*

$$V_t(H) = H_t^0 S_t^0 + H_t^1 S_t^1.$$

Definition 1.3 *A trading strategy $H = (H_t^0, H_t^1)$ is said to be self-financing if $dV_t = H_t \cdot dS_t := H_t^0 dS_t^0 + H_t^1 dS_t^1$. This means that the change in value is purely due to the change in prices.*

Suppose that the price S_t^0 of the riskless asset satisfies the ordinary differential equation

$$dS_t^0 = rS_t^0 dt,$$

where r is the interest rate. Set $S_0^0 = 1$, so that $S_t^0 = e^{rt}$. The discounted price process $\tilde{S}_t$ is defined as $\tilde{S}_t = (S_t^0)^{-1} S_t = e^{-rt} S_t$, so that $\tilde{S}_t^0 = 1, \tilde{S}_t^1 = e^{-rt} S_t^1$. The corresponding discounted value process is given by

$$\tilde{V}_t(H) = H_t^0 \tilde{S}_t^0 + H_t^1 \tilde{S}_t^1.$$

It is easy to show that a trading strategy $H = (H_t^0, H_t^1)$ is self-financing if and only if $d\tilde{V}_t(H) = H_t \cdot d\tilde{S}_t$. In fact, suppose H is self-financing, then

$$\begin{aligned} d\tilde{V}_t(H) &= d(e^{-rt} V_t(H)) \\ &= -re^{-rt} V_t(H) dt + e^{-rt} dV_t(H)) \\ &= -re^{-rt} H_t \cdot S_t dt + e^{-rt} H_t \cdot dS_t \\ &= H_t \cdot (-re^{-rt} \cdot S_t dt + e^{-rt} dS_t) = H_t \cdot d\tilde{S}_t \end{aligned}$$

The other direction is similar.

Proposition 1.4 *Let the trading strategy H_t^1 for the risky asset and the initial value v_0 be given. One can always choose H_t^0 so that $H = (H_t^0, H_t^1)$ is a self-financing portfolio.*

Proof. According to the above discussion, $H = (H_t^0, H_t^1)$ is a self-financing portfolio if and only if

$$d\tilde{V}_t(H) = H_t \cdot d\tilde{S}_t = H_t^0 d\tilde{S}_t^0 + H_t^1 d\tilde{S}_t^1$$

Namely,

$$\begin{aligned}\tilde{V}_t(H) &= H_t^0 \tilde{S}_t^0 + H_t^1 \tilde{S}_t^1 = H_t^0 + H_t^1 \tilde{S}_t^1 \\ &= v_0 + \int_0^t H_s^1 d\tilde{S}_t^1\end{aligned}$$

This yields

$$H_t^0 = -H_t^1 \tilde{S}_t^1 + v_0 + \int_0^t H_s^1 d\tilde{S}_t^1$$

Hence, H_t^0 is determined by v_0 and H_t^1.

Definition 1.5 *We say that a trading strategy $H = (H_t^0, H_t^1)$ is an arbitrage opportunity if $V_0(H) = 0$, $V_T(H) \geq 0$ and*

$$P(V_T(H) > 0) > 0$$

This says that with an arbitrage strategy H one can make something out of nothing. We say that a market is arbitrage free if no arbitrage strategy exists.

Stochastic integration against Brownian motion

Let us briefly recall the definition of the stochastic integral against a Brownian motion and some of the important properties. Let $\mathcal{L}$ denote the class of $\mathcal{F}_t$ adapted processes $\theta_t, t \geq 0$ with $E[\int_0^T \theta_t^2 dt] < \infty$. We can define the stochastic integral of θ_t against a Brownian motion, denoted by $\int_0^t \theta_s dB_s$, in the two steps:

Step 1. Assume $\theta \in \mathcal{L}$ is a simple process of the form:

$$\theta_t = \sum_{i=0}^{n} \xi_i I_{(t_i, t_{i+1}]}(t),$$

where ξ_i is bounded, $\mathcal{F}_{t_i}$-measurable random variables and $t_0 = 0 < t_1 < t_2 ... < t_n$. For $t_k < t \leq t_{k+1}$, define

$$\int_0^t \theta_s dB_s = \sum_{i=0}^{k-1} \xi_i (B_{t_{i+1}} - B_{t_i}) + \xi_k (B_t - B_{t_k})$$

Step 2. For $\theta \in \mathcal{L}$, choose a sequence θ_t^n of simple processes such that

$$E[\int_0^T (\theta_t^n - \theta_t)^2 dt] \to 0$$

as $n \to \infty$. Define

$$\int_0^t \theta_s dB_s = \lim_{n \to \infty} \int_0^t \theta_s^n dB_s$$

It can be shown that $M_t = \int_0^t \theta_s dB_s, t \geq 0$ is a martingale and the following isometry holds:

$$E[(\int_0^t \theta_s dB_s)^2] = E[\int_0^t \theta_s^2 ds]$$

Ito's formula.

If $f(t,x) \in C^2(R_+ \times R)$, then it holds that

$$df(t,B_t) = \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt$$

Theorem 1.6 *A market is arbitrage free if and only if there exists a probability measure P^* , equivalent to P, under which the discounted prices $\tilde{S}_t$ is a martingale.*

P^* is often referred as an equivalent martingale measure.

Theorem 1.7 *(***Girsanov's Theorem***) Suppose that B_t is a Brownian motion with the natural filtration $\mathcal{F}_t$. Suppose that θ_t is an $\mathcal{F}_t$-adapted process such that*

$$E\left[exp(\frac{1}{2}\int_0^T \theta_t^2 dt)\right] < \infty$$

Define

$$L_t = exp(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds)$$

and a probability measure P^* by

$$P^*(A) = \int_A L_T(\omega)dP$$

The under the new probability measure P^*, the process W_t defined by

$$W_t = B_t - \int_0^t \theta_s ds$$

a again a Brownian motion.

Example 1.8 *Let X_t be a drifting Brownian motion process defined by*

$$X_t = B_t + \mu t,$$

where B_t is a Brownian motion and μ is a constant. Construct a probability measure P^* such that X_t is a Brownian motion under P^*.

Solution. Taking $\theta = -\mu$ and defining

$$L_t = exp(-\mu B_t - \frac{1}{2}\mu^2 t)$$

and

$$P^*(A) = \int_A L_T(\omega) dP$$

, the Girsanov Theorem implies that under P^* X_t is a Brownian motion.

Example 1.9 *Let X_t be defined by*

$$X_t = B_t - f(t),$$

where B_t is a Brownian motion and $f \in C^1(R)$. Construct a probability measure P^* such that X_t is a Brownian motion under P^*.

Solution. Taking $\theta_s = f'(s)$ and defining

$$L_t = exp(\int_0^t f'(s) dB_s - \frac{1}{2}\int_0^t (f')^2(s) ds)$$

and

$$P^*(A) = \int_A L_T(\omega) dP$$

, the Girsanov Theorem implies that under P^* X_t is a Brownian motion.

The Black-Schoes model. The Black-Scholes model is a continuous time model in which the price S_t^0 of the riskless asset satisfies the ordinary differential equation:

$$dS_t^0 = rS_t^0 dt$$

and the price S_t^1 of the risky asset follows the stochastic differential equation:

$$dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB_t,$$

where μ, σ are constants.

Lemma 1.10 *There exists a probability measure P^*, equivalent to P, under which the discounted share price $\tilde{S}_t$ in the Black-Scholes model is a martingale. In particular, the Black-Scholes market is arbitrage free.*

Proof. Recall $S_t^0 = e^{rt}$. Thus

$$\begin{aligned} d\tilde{S}_t^1 &= d(e^{-rt}S_t^1) = -re^{-rt}S_t^1 dt + e^{-rt}dS_t^1 \\ &= -r\tilde{S}_t^1 dt + e^{-rt}(\mu S_t^1 dt + \sigma S_t^1 dB_t) \\ &= \tilde{S}_t^1(-rdt + \mu dt + \sigma dB_t). \end{aligned}$$

If $W_t = B_t + \frac{\mu - r}{\sigma}t$, then

$$d\tilde{S}_t^1 = \sigma \tilde{S}_t^1 dW_t$$

If we can construct a probability measure P^* under which W is a Brownian motion, then $\tilde{S}_t^1$ is a martingale. Define

$$L_t = exp(-\frac{\mu - r}{\sigma}B_t - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 t)$$

and

$$P^*(A) = \int_A L_T(\omega)dP$$

According to the Girsanov Theorem, $W - t$ is a Brownian motion under P^*. Consequently,

$$\tilde{S}_t^1 = \tilde{S}_0^1 + \int_0^t \sigma \tilde{S}_s^1 dW_s$$

is a martingale. The proof is complete.

Recall that a claim is simply a $\mathcal{F}_T$-measurable random variable X representing a cash flow.

Definition 1.11 *We say that a T claim is attainable if there exists a portfolio $H = (H_t^0, H_t^1)$ and a real number x such that $V_0(H) = x$ and $V_T(H) = X$.*

Such a strategy H is called a replicating or hedging portfolio for X.

Definition 1.12 *A market is called complete if every T-claim is attainable.*

Theorem 1.13 *Let $W_t, t \geq 0$ be a Brownian motion with a natural filtration $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$. If $\{M_t, t \geq 0\}$ is a square integrable $\mathcal{F}_t$-martingale, then there exists an $\mathcal{F}_t$-adapted process ϕ_s such that*

$$M_t = M_0 + \int_0^t \phi_s dW_s$$

Theorem 1.14 *The Black-Scholes market is complete*

Before proving the theorem, let us see how we can replicate a claim X at time T. Suppose that somehow we can find an adapted process H_t^1 and a real number v_0 such that the discounted claim $\tilde{X} = e^{-rT}X$ has the representation:

$$\tilde{X} = v_0 + \int_0^T H_u^1 d\tilde{S}_u^1$$

Now choose H_t^0 so that

$$\tilde{V}_t(H) = H_t^1 \tilde{S}_t^1 + H_t^0 = v_0 + \int_0^t H_u^1 d\tilde{S}_u^1$$

Then $H = (H_t^0, H_t^1)$ is self-financing. Moreover

$$e^{-rT} X = \tilde{V}_T(H) = v_0 + \int_0^T H_u^1 d\tilde{S}_u^1$$

Therefore, $V_T(H) = X$. So the claim is attainable.

Proof of Theorem . According to the above discussion, it is sufficient to show that for every bounded $T-$ claim X, there exists an adapted process $H_t^1, 0 \leq t \leq T$ such that

$$e^{-rT} X = v_0 + \int_0^T H_u^1 d\tilde{S}_u^1 \tag{1.1}$$

If $W_t = B_t + \frac{\mu - r}{\sigma} t$, then by the Girsanov Theorem W_t is a Brownian motion under a new probability P^* defined by

$$P^*(A) = \int_A L_T(\omega) dP,$$

where

$$L_t = exp(-\frac{\mu - r}{\sigma} B_t - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 t)$$

Observe that

$$\mathcal{F}_t^W = \sigma(W_s, 0 \leq s \leq t) = \sigma(B_s, 0 \leq s \leq t) = \mathcal{F}_t^B$$

Consider

$$M_t = E^{P^*}[e^{-rT} X | \mathcal{F}_t^W], 0 \leq t \leq T.$$

M_t is a martingale w.r.t. $\mathcal{F}_t^W$. By the martingale representation theorem, there exists an $\mathcal{F}_t^W = \mathcal{F}_t^B$-adapted process ϕ_t such that

$$M_t = M_0 + \int_0^t \phi_s dW_s$$

In particular,

$$e^{-rT} X = M_0 + \int_0^T \phi_s dW_s$$

On the other hand,

$$d\tilde{S}_t^1 = \sigma \tilde{S}_t^1 dW_t$$

(see the proof of theorem) Consequently,

$$dW_s = \frac{1}{\sigma} \frac{1}{\tilde{S}_s^1} d\tilde{S}_s^1$$

Combing this with we obtain

$$e^{-rT}X = M_0 + \int_0^T \phi_s \frac{1}{\sigma}\frac{1}{\tilde{S}_s^1} d\tilde{S}_s^1$$

This proves () with $H_t^1 = \phi_t \frac{1}{\sigma}\frac{1}{\tilde{S}_t^1}$.

Fair price

Given a T-claim X. We say that v_0 is a fair price of X at time zero if there exists a self-financing strategy $H = (H_t^0, H_t^1)$ such that

$$V_0(H) = v_0, \qquad V_T(H) = X$$

How do we compute v_0? Suppose that the market is arbitrage-free and P^* is an equivalent martingale measure under which the discounted price $\tilde{S}_t$ is a martingale. Then the discounted value process $\tilde{V}_t(H)$ satisfies

$$\tilde{V}_t(H) = v_0 + \int_0^t H_u^1 d\tilde{S}_u^1$$

In particular,

$$e^{-rT}X = \tilde{V}_T(H) = v_0 + \int_0^T H_u^1 d\tilde{S}_u^1$$

Taking expectation w.r.t. P^*, we obtain the following formula for the fair price

$$v_0 = E^{P^*}[e^{-rT}X] \tag{1.2}$$

Two steps to find the fair price of a claim X

1. Find a probability measure P^* under which the discounted price $\tilde{S}_t$ is a martingale.

2. Determine the fair price $v_0 = E^{P^*}[e^{-rT}X]$

Proposition 1.15 *Consider the Black-Schoes market:*

$$dS_t^0 = rS_t^0 dt$$

$$dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB_t$$

The price of an European option whose payoff at maturity T is $X = f(S_T^1)$ is given by

$$F(x) = e^{-rT} \int_{-\infty}^{\infty} f\Big(xexp\big((r - \frac{\sigma^2}{2})T + \sigma y\sqrt{T}\big)\Big) \frac{1}{\sqrt{2\pi}} exp(-\frac{y^2}{2}) dy$$

Proof. The price is $F(x) = E^{P^*}[e^{-rT}f(S_T^1)]$. Using Ito's formula, we see easily that

$$S_T^1 = xexp(\sigma B_T - \frac{1}{2}\sigma^2 T + \mu T)$$

We also know from Lemma 7.10 that under the martingale measure P^*, $W_t = B_t + \frac{\mu - r}{\sigma}t$ is a Brownian motion. Thus,

$$E^{P^*}[e^{-rT}f(S_T^1)] = E^{P^*}[e^{-rT}f(xexp(\sigma B_T - \frac{1}{2}\sigma^2 T + \mu T))]$$

$$== E^{P^*}[e^{-rT}f(xexp(\sigma W_T - \frac{1}{2}\sigma^2 T + rT))]$$

$$= e^{-rT}\int_{-\infty}^{\infty} f\Big(xexp\big((r - \frac{\sigma^2}{2})T + \sigma y\big)\Big)\frac{1}{\sqrt{2\pi T}}exp(-\frac{y^2}{2T})dy$$

$$= \int_{-\infty}^{\infty} f\Big(xexp\big((r - \frac{\sigma^2}{2})T + \sigma y\sqrt{T}\big)\Big)\frac{1}{\sqrt{2\pi}}exp(-\frac{y^2}{2})dy.$$