Three hours

THE UNIVERSITY OF MANCHESTER

MARTINGALE THEORY FOR FINANCE

26 January 2017

14:00 – 17:00

Answer **ALL** questions in Section A and Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** questions

A1. (i) Let $A_n, n \geq 1$ be a sequence of events. Set $A = \cup_{n=1}^{\infty} A_n$. Show that $P(A) \leq \sum_{n=1}^{\infty} P(A_n)$.

[5 marks]

(ii) Let $A_1, A_2, ...$ be a sequence of events with $P(A_n) = (\frac{1}{8})^n$ for $n \geq 1$. Set $A = \cup_{n=1}^{\infty} A_n$. Prove that $P(A) \leq \frac{1}{7}$.

[5 marks]

A2.

(i) Let X be a random variable such that $E[|X|^{\alpha}] < \infty$ for $\alpha > 0$. For $\varepsilon > 0$, prove the Chebyshev inequality:

$$P(|X| \geq \varepsilon) \leq \frac{E[|X|^{\alpha}]}{\varepsilon^{\alpha}}.$$

[5 marks]

(ii) Let $X_n, n \geq 1$ be a sequence of random variables such that $E|X_n|] \leq \frac{1}{n}$. Prove that for $0 < \beta < 1$,

$$\lim_{n \to \infty} P(|X_n| \geq \frac{1}{n^{\beta}}) = 0.$$

[5 marks]

A3. (i) State the definition of $\{Z_n, n \geq 0\}$ being a martingale with respect to a family of increasing σ-fields $\{\mathcal{F}_n, n \geq 0\}$.

[5 marks]

(ii) Suppose that $\{Z_n, n \geq 0\}$ is a martingale with respect to the family of σ-fields $\{\mathcal{F}_n, n \geq 0\}$ such that $E[|Z_n|^2] < \infty$, $n \geq 0$. Prove

(a) $E[Z_{n+1}Z_n] = E[Z_n^2]$.

[3 marks]

(b) $E[Z_{n+2}Z_n] = E[Z_{n+1}Z_n]$.

[2 marks]

A4. Let $X_k, k \geq 1$ be a sequence of independent random variables with identical mean $E[X_k] = \mu$ and finite variance $var(X_k) < \infty$, $k \geq 1$. Put $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$.

(i) Define $Z_n = \sum_{k=1}^{n}(X_k - \mu)$. Prove that $\{Z_n, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[5 marks]

(ii) Set $M_n = \left(\sum_{k=1}^{n}(X_k - \mu)\right)^2 - \sum_{k=1}^{n}(X_k - \mu)^2$. Prove that $\{M_n, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[5 marks]

SECTION B

Answer **ALL** questions

B5. Let $\{S_n, n \geq 0\}$ be a simple random walk with $S_0 = 0$ and

$$S_n = X_1 + X_2 + \cdots + X_n,$$

for $n \geq 1$, where $X_i, i = 1, 2, ...$ are independent random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q = 1 - p$ for $i \geq 1$. Assume $p \neq q$. Put

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \quad \mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1.$$

Let b, a be two fixed positive integers. Define

$$T = min\{n; S_n = -a \quad \text{or} \quad S_n = b\}.$$

T is the first time at which the random walk reaches the position $-a$ or b.

(i) Show that T is a stopping time with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[4 marks]

(ii) Define $Z_n = (\frac{q}{p})^{S_n}, n \geq 0$. Prove $\{Z_n, n \geq 0\}$ is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[6 marks]

(iii) Consider $Z_{T \wedge n} = (\frac{q}{p})^{S_{T \wedge n}}, n \geq 0$. Prove $\{Z_{T \wedge n}, n \geq 0\}$ is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[6 marks]

(iv) Determine the value of $E[Z_T]$ and explain why.

[4 marks]

B6. Consider a multi-period financial market consisting of a bank account $S_0(t)$ and a stock $S_1(t)$ modeled on a probability space $(\Omega, \mathcal{F}, P)$, $t = 0, 1, ..., T$.

(i) Write down the definition of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t))$ and explain what it means.

[4 marks]

(ii) Suppose a bounded portfolio $\phi = (\phi_0(t), \phi_1(t))$ is self-financing. Denote by $\tilde{S} = (\tilde{S}_1, \tilde{S}_2)$ the discounted price process and by $\tilde{V}_t(\phi)$ the discounted value process. Prove

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \sum_{k=1}^{t} \phi(k) \cdot (\tilde{S}(k) - \tilde{S}(k-1)),$$

where $\cdot$ stands for the scalar product of two vectors.

[6 marks]

In the remaining part of the question, consider now a one period market, i.e, $T = 1$. Suppose $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F}$ being the collection of all events and P a probability measure such that $P(\{\omega_1\}) > 0$, $P(\{\omega_2\}) > 0$. Suppose that the current asset prices (time $t = 0$) are $S_0(0) = 4$, and $S_1(0) = 6$ and the terminal prices (time $t = 1$) are $S_0(1, \omega_1) = S_0(1, \omega_2) = 4(1 + r)$, and $S_1(1, \omega_1) = 5, S_1(1, \omega_2) = 10$. r is the interest rate.

(iii) Find the condition on the interest rate r under which the market is free of arbitrage and determine the corresponding equivalent martingale probability measure Q.

[10 marks]

B7.

(i) Let X be a Poisson random variable with mean λ. For a constant a, prove

$$E[exp(aX)] = e^{-\lambda} exp(\lambda e^a).$$

[5 marks]

(ii) Let $\{N_t : t \geq 0\}$ be a Poisson process of rate λ. Let $\mathcal{F}_t = \sigma(N_u, 0 \leq u \leq t)$ be the σ-field generated by the Poisson process and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. For $\theta > 0$, show that $V_t = exp(-\theta N_t + \lambda t(1 - e^{-\theta})), t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t, t \geq 0$.

[7 marks]

(iii) Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. Set $\mathcal{F}_t = \sigma(B_u, 0 \leq u \leq t)$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Prove that $Z_t = B_t^3 - 3\int_0^t B_u du, t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t, t \geq 0$.

[8 marks]

(Hint: If $X \sim N(0, \sigma^2)$, $E[X^3] = 0$.)

END OF EXAMINATION PAPER