Two hours

THE UNIVERSITY OF MANCHESTER

MARTINGALES AND APPLICATIONS TO FINANCE

26 January 2017

14:00 – 16:00

Answer **ALL** questions in Section A and Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** questions

A1. (i) Let $A_n, n \geq 1$ be a sequence of events. Set $A = \cup_{n=1}^{\infty} A_n$. Show that $P(A) \leq \sum_{n=1}^{\infty} P(A_n)$.

[5 marks]

(ii) Let $A_1, A_2, ...$ be a sequence of events with $P(A_n) = (\frac{1}{8})^n$ for $n \geq 1$. Set $A = \cup_{n=1}^{\infty} A_n$. Prove that $P(A) \leq \frac{1}{7}$.

[5 marks]

A2. (i) Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ a sub-σ-field of $\mathcal{F}$. State the definition of the conditional expectation $E[X|\mathcal{G}]$ of X given $\mathcal{G}$.

[5 marks]

(ii) Let A be an event with $0 < P(A) < 1$. Define the σ-field $\mathcal{G} = \{\Omega, \emptyset, A, A^c\}$. For any event B, prove that

$$E[I_B|\mathcal{G}] = P(B|A)I_A + P(B|A^c)I_{A^c}.$$

[5 marks]

A3. (i) State the definition of $\{Z_n, n \geq 0\}$ being a martingale with respect to a family of increasing σ-fields $\{\mathcal{F}_n, n \geq 0\}$.

[5 marks]

(ii) Suppose that $\{Z_n, n \geq 0\}$ is a martingale with respect to the family of σ-fields $\{\mathcal{F}_n, n \geq 0\}$ such that $E[|Z_n|^2] < \infty$, $n \geq 0$. Prove

(a) $E[Z_{n+1}Z_n] = E[Z_n^2]$.

[3 marks]

(b) $E[Z_{n+2}Z_n] = E[Z_{n+1}Z_n]$.

[2 marks]

A4. Let $X_k, k \geq 1$ be a sequence of independent random variables with identical mean $E[X_k] = \mu$ and finite variance $var(X_k) < \infty$, $k \geq 1$. Put $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$.

(i) Define $Z_n = \sum_{k=1}^{n}(X_k - \mu)$. Prove that $\{Z_n, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[5 marks]

(ii) Set $M_n = \left(\sum_{k=1}^{n}(X_k - \mu)\right)^2 - \sum_{k=1}^{n}(X_k - \mu)^2$. Prove that $\{M_n, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[5 marks]

SECTION B

Answer **ALL** questions

B5. Let $\{S_n, n \geq 0\}$ be a simple random walk with $S_0 = 0$ and

$$S_n = X_1 + X_2 + \cdots + X_n,$$

for $n \geq 1$, where $X_i, i = 1, 2, ...$ are independent random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q = 1 - p$ for $i \geq 1$. Assume $p \neq q$. Put

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \quad \mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1.$$

Let b, a be two fixed positive integers. Define

$$T = min\{n; S_n = -a \quad \text{or} \quad S_n = b\}.$$

T is the first time at which the random walk reaches the position $-a$ or b.

(i) Show that T is a stopping time with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[5 marks]

(ii) Define $Z_n = (\frac{q}{p})^{S_n}, n \geq 0$. Prove $\{Z_n, n \geq 0\}$ is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[10 marks]

(iii) Explain why one can use Doob's Optional Stopping Theorem to conclude that $E[Z_T] = 1$.

[5 marks]

(iv) Determine the probabilities $P(S_T = -a)$ and $P(S_T = b)$.

[10 marks]

B6. Consider a multi-period financial market consisting of a bank account $S_0(t)$ and a stock $S_1(t)$ modeled on a probability space $(\Omega, \mathcal{F}, P)$, $t = 0, 1, ..., T$.

(i) Write down the definition of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t))$ and explain what it means.

[5 marks]

(ii) Suppose a bounded portfolio $\phi = (\phi_0(t), \phi_1(t))$ is self-financing. Denote by $\tilde{S} = (\tilde{S}_1, \tilde{S}_2)$ the discounted price process and by $\tilde{V}_t(\phi)$ the discounted value process. Prove

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \sum_{k=1}^{t} \phi(k) \cdot (\tilde{S}(k) - \tilde{S}(k-1)),$$

where $\cdot$ stands for the scalar product of two vectors.

[10 marks]

(iii) Write down the definition of an equivalent martingale probability measure.

[5 marks]

In the remaining part of the question, consider a one period market, i.e, $T = 1$. Suppose now $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F}$ being the collection of all events and P a probability measure such that $P(\{\omega_1\}) > 0$, $P(\{\omega_2\}) > 0$. Suppose that the current asset prices (time $t = 0$) are $S_0(0) = 4$, and $S_1(0) = 6$ and the terminal prices (time $t = 1$) are $S_0(1, \omega_1) = S_0(1, \omega_2) = 4(1 + r)$, and $S_1(1, \omega_1) = 5, S_1(1, \omega_2) = 10$. r is the interest rate.

(iv) Find the condition on the interest rate r under which the market is free of arbitrage and determine the corresponding equivalent martingale probability measure Q.

[10 marks]

END OF EXAMINATION PAPER