Two hours

THE UNIVERSITY OF MANCHESTER

MARTINGALES AND APPLICATIONS TO FINANCE

20 January 2014

09:45 – 11:45

Answer **ALL** questions in Section A and Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** questions

A1. (i) Let $B_n, n \geq 1$ be a sequence of events. Determine which of the following statements is correct:

[5 marks]

(a)

$$P(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n);$$

(b)

$$P(\cup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P(B_n).$$

(ii) Let $A_n, n \geq 1$ be a sequence of events with $P(A_n) = 1$. Set $A = \cap_{n=1}^{\infty} A_n$. Show that $P(A) = 1$.

[5 marks]

A2. (i) State the definition of $\{Z_n, n \geq 0\}$ being a martingale with respect to a family of increasing σ-fields $\{\mathcal{F}_n, n \geq 0\}$.

[4 marks]

(ii) Suppose that $\{Z_n, n \geq 0\}$ is a martingale with finite second moment with respect to $\{\mathcal{F}_n, n \geq 0\}$. Deduce that

(a) $E[Z_j Z_i] = E[Z_i^2]$ for $j > i$.

[2 marks]

(b) $E[(Z_{n+1} - Z_n)^2 | \mathcal{F}_n] = E[Z_{n+1}^2 | \mathcal{F}_n] - Z_n^2$.

[4 marks]

A3. Let $X_n, n \geq 1$ be independent, identically distributed random variables with normal distribution $N(\mu, \sigma^2)$. Let $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ be the σ-field generated by $X_1, X_2, ..., X_n$. Let $S_n = \sum_{i=1}^{n} X_i$.

(i) Show that $\{Z_n = S_n - n\mu, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[4 marks]

(ii) Show that $Y_n = exp\{S_n - n\mu - \frac{1}{2}\sigma^2 n\}, n \geq 1$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[6 marks]

(Hint: If $X \sim N(\mu, \sigma^2)$ then $E[e^X] = e^{\frac{1}{2}\sigma^2 + \mu}$.)

A4. Consider a financial market with one risk-free asset labeled 0 and d risky assets labeled $1, 2, 3, ..., d$. The terminal time is T. The prices of the assets at time t ($t = 0, 1, ..., T$) are random variables $S_0(t), S_1(t), ..., S_d(t)$ on a probability space $(\Omega, \mathcal{F}, P)$.

(i) Write down the mathematical definition of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ and explain what it means.

[5 marks]

(ii) Write down the value process of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ and explain what it means to say that ϕ is an arbitrage opportunity.

[5 marks]

SECTION B

Answer **ALL** questions

B5.

Let $X_n, n \geq 1$ be independent random variables with $P(X_n = 1) = p$ and $P(X_n = -1) = q = 1-p$. Let $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ be the σ-field generated by $X_1, X_2, ..., X_n$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Consider the random walk $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$ with $S_0 = 0$. Suppose $0 < p < 1$.

(i) Prove that $\{Z_n = (\frac{1-p}{p})^{S_n}, n \geq 0\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 0$.

[7 marks]

For any integer x, define the stopping time $T_x = min\{n; S_n = x\}$, the first time at which the random walk hits the position x. For integers $a < 0 < b$, let $T = T_a \wedge T_b = min(T_a, T_b)$.

(ii) Determine the value of $E[Z_T]$ and give your reason.

[7 marks]

(iii) Use the result in (ii) to show that

$$P(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)},$$

where $\varphi(x) = (\frac{1-p}{p})^x$.

[10 marks]

(iv) Suppose $\frac{1}{2} < p < 1$. If $a < 0 < b$, determine respectively the probabilities that the random walk hits a and b, i.e.,

$$P(T_a < \infty) = \lim_{b \to \infty} P(T_a < T_b), \qquad P(T_b < \infty).$$

[6 marks]

B6. Consider a financial market consisting of a bank account $S_0(t)$ and a stock $S_1(t)$ modeled on a probability space $(\Omega, \mathcal{F}, P)$ with the time indices $t = 0, 1, 2, ..., T$. Fix two positive numbers l, and u such that $l < 1 < u$. Let $Z(t), t = 1, 2, ..., T$ be independent, identically distributed random variables with

$$P(Z(t) = u) = p > 0, \quad P(Z(t) = l) = q = 1 - p > 0.$$

Suppose the price processes are given as follows:

$$S_0(0) = 1, \quad S_0(t) = (1 + r)^t, t = 1, 2, ..., T.$$

$$S_1(0) = 1, \quad S_1(t) = S_1(0)\Pi_{m=1}^{t} Z(m) = Z(1)Z(2)...Z(t), t \geq 1.$$

(i) It is a fact from the course notes that the discounted value process of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t))$ is a martingale under a martingale probability measure. Use this to show that a market is free of arbitrage if a martingale probability measure exists.

[8 marks]

(ii) Determine conditions (in terms of r, p) under which the discounted price process $\tilde{S}(t) = (\tilde{S}_0(t), \tilde{S}_1(t)), t \geq 0$ is a martingale under P with respect to $\mathcal{F}_t = \sigma(Z(1), ..., Z(t))$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$.

[14 marks]

(iii) Find the price of an option with payoff $X = exp\{S_1(T)\}$.

[8 marks]

END OF EXAMINATION PAPER