**Two hours**

**THE UNIVERSITY OF MANCHESTER**

MARTINGALES WITH APPLICATIONS TO FINANCE

? January 2013

09:45 – 11:45

Answer **ALL** questions in Section A and Section B.

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Electronic calculators may be used, provided that they cannot store text.

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# SECTION A

Answer **ALL** questions

**A1.** (i) State the Monotone Convergence Theorem.

[4 marks]

(ii) Let $B_n, n \geq 1$, be an increasing sequence of events. Define $X_n(\omega) = I_{B_n}(\omega)$, the indicator of $B_n$. Set $X(\omega) = I_B(\omega)$, where $B = \cup_{n=1}^{\infty} B_n$.

a) Explain why $X_n(\omega) \leq X_{n+1}(\omega)$ and $X_n(\omega) \to X(\omega)$ as $n \to \infty$.

[3 marks]

b) Deduce that

$$\lim_{n\to\infty} P(B_n) = P(B).$$

[3 marks]

**A2.** (i) Let $X$ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ a $\sigma$-field. State the definition of the conditional expectation $E[X|\mathcal{G}]$ of $X$ given $\mathcal{G}$.

[4 marks]

(ii) Let $A, B \in \mathcal{F}$. Define $\mathcal{G} = \{\Omega, \emptyset, A, A^c\}$. Let $X = I_B$ be the indicator of $B$. Prove that

$$E[X|\mathcal{G}] = P(B|A)I_A + P(B|A^c)I_{A^c}.$$

[6 marks]

**A3.** Let $X_i, i = 1, 2, ...,$ be independent random variables with $P(X_i = 1) = \frac{1}{2}$, $P(X_i = -1) = \frac{1}{2}$. Put

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \quad \mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1.$$

Define $S_n = \sum_{i=1}^{n} X_i, n \geq 1$, and $Z_n = S_n^2 - n, n \geq 1$.

(i) Prove that $S_n, n \geq 1$, is a martingale with respect to $\{\mathcal{F}_n, n \geq 1\}$.

[4 marks]

(ii) Show that $Z_n, n \geq 1$, is a martingale with respect to $\{\mathcal{F}_n, n \geq 1\}$.

[6 marks]

**A4.** Consider a financial market with one risk-free asset labeled 0 and $d$ risky assets labeled $1, 2, 3, ..., d$. The terminal time is $T$. The prices of the assets at time $t$ ($t = 0, 1, ..., T$) are random variables $S_0(t), S_1(t), ..., S_d(t)$ on a probability space $(\Omega, \mathcal{F}, P)$.

(i) Write down the definition of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ and explain what it means.

[5 marks]

(ii) Explain what it means to say that a financial market is complete and state the condition under which a market is complete.

[5 marks]

## SECTION B

Answer **ALL** questions

**B5.** (i) State the definition of $\{Z_n, n \geq 0\}$ being a martingale with respect to a family of increasing $\sigma$-fields $\{\mathcal{F}_n, n \geq 0\}$.

[5 marks]

Let $X_i, i = 1, 2, ...,$ be independent random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q = 1 - p$. Define $S_0 = 0$,

$$S_n = X_1 + X_2 + \cdots + X_n$$

for $n \geq 1$. Then $\{S_n, n \geq 0\}$ forms a random walk. Let $a$ and $b$ be two positive integers. The first time at which the random walk leaves the interval $[-a, b]$ is given by

$$T = min\{n; S_n = -a \quad \text{or} \quad S_n = b\}.$$

For $\theta \geq 0$, introduce

$$Z_n = \phi(\theta)^{-n} e^{\theta S_n}, \quad n \geq 0,$$

where $\phi(\theta) = E[e^{\theta X_1}] = e^{\theta} p + e^{-\theta} q$.

(ii) Prove that $\{Z_n, n \geq 0\}$ is a martingale with respect to the $\sigma$-fields $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 0$.

[5 marks]

(iii) Explain why $T$ is a stopping time with respect to the increasing family of $\sigma$-fields $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$.

[5 marks]

(iv) Use Doob's optional stopping theorem to explain why

$$E[\phi(\theta)^{-T} e^{\theta S_T}] = 1$$

for any $\theta \geq 0$ satisfying $\phi(\theta) \geq 1$.

[5 marks]

(v). Suppose there exists $\theta_0 > 0$ for which $\phi(\theta_0) = 1$. Determine the probability that the random walk $\{S_n, n \geq 0\}$ hits $-a$ before visiting $b$.

[10 marks]

**B6.** Consider a one period financial market model consisting of a bank account $S_0$ and a stock $S_1$ modeled on a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F}$ being the collection of all events and $P$ a probability measure such that $P(\{\omega_1\}) > 0$, $P(\{\omega_2\}) > 0$. Suppose that the current asset prices (time $t = 0$) are $S_0(0) = 10$, and $S_1(0) = 15$ and the terminal prices (time $t = T$) are $S_0(T, \omega_1) = S_0(T, \omega_2) = 10(1 + r)$, and $S_1(T, \omega_1) = 12, S_1(T, \omega_2) = 30$.

(i) Write down the definition of the value process $V_\phi(t)$ of a portfolio $\phi = (\phi_0(t), \phi_1(t))$. Explain what it means to say that a portfolio $\phi = (\phi_0(t), \phi_1(t))$ is an arbitrage opportunity.

[5 marks]

(ii) Use the fact that the discounted value process $\tilde{V}_\phi(t)$ is a martingale under an equivalent martingale probability measure to show that if an equivalent martingale probability measure exists, then the market is free of arbitrage.

[5 marks]

(iii) Find the condition on the interest rate $r$ under which the market is free of arbitrage and determine the corresponding equivalent martingale probability measure $Q$.

[10 marks]

(iv) Explain what an European put option with strike price $K$ means and write down the payoff of the option.

[5 marks]

(v) Assume $r = \frac{3}{5}$, $K = 18$. Find the price of the European put option.

[5 marks]

**END OF EXAMINATION PAPER**