Two hours

THE UNIVERSITY OF MANCHESTER

MARTINGALES AND APPLICATIONS TO FINANCE

16 January 2012

09:45 – 11:45

Answer **ALL** questions in Section A and Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** questions

A1. (i) Let $B_n, n \geq 1$ be a sequence of events with $P(B_n) = \frac{1}{8^n}, n \geq 1$. Set $A = \cup_{n=1}^{\infty} B_n$. Show that $P(A) \leq \frac{1}{7}$.

[5 marks]

(ii) Let $A_1, A_2, ...,$ be a sequence of events. Denote by I_{A_j} the indicator random variable of the event A_j. Set

$$Y = \sum_{j=1}^{\infty} I_{A_j}.$$

Y is the (random) number of $A_1, A_2, ...$ that occur. Find $E[Y]$ and explain your reasoning.

[5 marks]

A2.

(i) Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ a σ-field. State the definition of the conditional expectation $E[X|\mathcal{G}]$ of X given $\mathcal{G}$.

[4 marks]

(ii) Let X be a random variable taking three values:

$$P(X = a) = p_1, \quad P(X = b) = p_2 \quad P(X = c) = p_3,$$

where $0 < p_i < 1$ and $p_1 + p_2 + p_3 = 1$. Let $A = \{X = a\}$. Define $\mathcal{G} = \{\Omega, \emptyset, A, A^c\}$. Prove that

$$E[X^2|\mathcal{G}] = a^2 I_A + \frac{b^2 p_2 + c^2 p_3}{p_2 + p_3} I_{A^c}.$$

(Hint: $A^c = \{X = b\} \cup \{X = c\}$.)

[6 marks]

A3. (i) State the definition of $\{Z_n, n \geq 0\}$ being a martingale with respect to a family of increasing σ-fields $\{\mathcal{F}_n, n \geq 0\}$.

[4 marks]

(ii) Suppose that $\{Z_n, n \geq 0\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 0$. Deduce
(a) $E[Z_{n+2}|\mathcal{F}_n] = Z_n$ for $n \geq 1$.

[2 marks]

(b) $E[(Z_{j+1} - Z_j)X] = 0$ for $j \geq 1$ and any bounded $\mathcal{F}_j$-measurable random variable X.

[4 marks]

A4. Consider a financial market with one risk-free asset labeled 0 and d risky assets labeled $1, 2, 3, ..., d$. The terminal time is T. The prices of the assets at time t ($t = 0, 1, ..., T$) are random variables $S_0(t), S_1(t), ..., S_d(t)$ on a probability space $(\Omega, \mathcal{F}, P)$.

(i) Write down the value process of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ and explain what it means to say that ϕ is an arbitrage opportunity.

[4 marks]

(ii) Explain what it means to say that P^* is an equivalent martingale probability measure. Use the fact that the discounted value process $\tilde{V}_\phi(t)$ is a martingale under an equivalent martingale probability measure to show that if an equivalent martingale probability P^* exists, then the financial market is arbitrage-free.

[6 marks]

SECTION B

Answer **ALL** questions

B5. Let $\{S_n, n \geq 0\}$ be a simple symmetric random walk with $S_0 = k > 0$ and

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n,$$

for $n \geq 1$, where $X_i, i = 1, 2...$ are independent random variables with $P(X_i = 1) = \frac{1}{2}$, $P(X_i = -1) = \frac{1}{2}$. Put

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \quad \mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1.$$

Let $N > k$ be a fixed positive integer. Define a stopping time

$$T = min\{n; S_n = 0 \quad \text{or} \quad S_n = N\}.$$

(i) Show that $S_n, n \geq 0$, is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[5 marks]

(ii) Show that $Y_n = S_n^2 - n, n \geq 0$, is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[5 marks]

(iii) Use Doob's Optional Stopping Theorem to find the values of $P(S_T = 0)$, $P(S_T = N)$ and $E[T]$.

[8 marks]

(iv) For $\lambda > 0$, define

$$Z_n = \frac{e^{\lambda S_n}}{(cosh(\lambda))^n}, n \geq 0,$$

where $cosh(\lambda) = \frac{1}{2}(e^{\lambda} + e^{-\lambda}) > 1$. Show that $Z_n, n \geq 0$, is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[5 marks]

(v) Use Doob's Optional Stopping Theorem to find $E[(cosh(\lambda))^{-T_N}]$, where $T_N = min\{n; S_n = N\}$.

[7 marks]

B6. Consider a financial market consisting of a bank account $S_0(t)$ and a stock $S_1(t)$ modeled on a probability space $(\Omega, \mathcal{F}, P)$ with the time indices $t = 0, 1, 2, ..., T$. Fix three positive numbers l, x and u such that $l < x < u$. Let $Z(t), t = 1, 2, ..., T$ be independent, identically distributed random variables with

$$P(Z(t) = u) = p > 0, \quad P(Z(t) = l) = q = 1 - p > 0.$$

Define the price processes as follows:

$$S_0(0) = 1, \quad S_0(t) = (1 + r)^t, t = 1, 2, ..., T.$$

$$S_1(0) = x, \quad S_1(t) = S_1(0)\Pi_{m=1}^{t} Z(m) = xZ(1)Z(2)...Z(t), t \geq 1.$$

(i) Write down the mathematical definition of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t))$ and explain what it means.

[5 marks]

(ii) Explain what it means to say that a market is complete. Give conditions under which an arbitrage free market is complete.

[5 marks]

(iii) Determine conditions (in terms of r, l, u) under which the discounted price process $\tilde{S}(t) = (\tilde{S}_0(t), \tilde{S}_1(t)), t \geq 0$ is a martingale under P with respect to $\mathcal{F}_t = \sigma(Z(1), ..., Z(t))$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$.

[12 marks]

(iv) Find the price of an option with payoff $X = (S_1(T))^2$.

[8 marks]

END OF EXAMINATION PAPER