Two hours

THE UNIVERSITY OF MANCHESTER

MARTINGALES AND APPLICATIONS TO FINANCE

27 January 2010

14:00 – 16:00

Answer **ALL** questions in Section A and TWO questions in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer **ALL** the four questions

A1. (i) Let $B_n, n \geq 1$ be a sequence of events. Determine which of the following statements is correct:

[4 marks]

(a)

$$P(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n)$$

(b)

$$P(\cup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P(B_n)$$

(ii) Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ a σ-field. State the definition of the conditional expectation $E[X|\mathcal{G}]$ of X given $\mathcal{G}$ and show that $E[E[X|\mathcal{G}]] = E[X]$.

[6 marks]

A2. (i) State the definition of $Z_n, n \geq 0$ being a martingale with respect to a family of increasing σ-fields $\{\mathcal{F}_n, n \geq 0\}$.

[4 marks]

(ii) Suppose that $\{Z_n, n \geq 0\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 0$. Deduce
(a) $E[Z_{n+3}|\mathcal{F}_n] = Z_n$.

[2 marks]

(b) Let $\mathcal{G}_n, n \geq 1$ be another sequence of increasing σ-fields such that $\mathcal{G}_n \subset \mathcal{F}_n$ for every $n \geq 0$. If Z_n is also $\mathcal{G}_n$-determined, prove that $\{Z_n, n \geq 0\}$ is a martingale with respect to $\mathcal{G}_n, n \geq 0$.

[4 marks]

A3. Let $X_n, n \geq 1$ be independent random variables with $P(X_n = 1) = p$ and $P(X_n = -1) = q = 1 - p$. Let $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$ be the σ-field generated by $X_1, X_2, ..., X_n$. Put $S_n = \sum_{i=1}^{n} X_i$.
(i) If $p = q = \frac{1}{2}$, show that $\{Z_n = S_n^2 - n, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[5 marks]

(ii) If $p \neq q$, show that $\{Z_n = (\frac{q}{p})^{S_n}, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 1$.

[5 marks]

A4. Consider a financial market with one risk-free asset labeled 0 and d risky assets labeled $1, 2, 3, ..., d$. The terminal time is T. The prices of the assets at time t ($t = 0, 1, ..., T$) are random variables $S_0(t), S_1(t), ..., S_d(t)$ on a probability space $(\Omega, \mathcal{F}, P)$.
(i) Write down the mathematical definition of a self-financing portfolio $\phi = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ and explain what it means.

[5 marks]

(ii) Let $\tilde{V}_\phi(t)$ denote the discounted value process of a portfolio $\phi = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ and $\tilde{G}_\phi(t)$ denote the discounted gain process:

$$\tilde{G}_\phi(t) = \sum_{\tau=1}^{t} \phi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1)),$$

where $\tilde{S}$ is the discounted price. If

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \tilde{G}_\phi(t) \qquad \text{for all} \quad t \geq 1,$$

prove that ϕ is self-financing.

[5 marks]

SECTION B

Answer **TWO** of the three questions

B5. Let $\{S_n, n \geq 0\}$ be a simple random walk, i.e., $S_0 = 0$ and

$$S_n = X_1 + X_2 + \cdots + X_n,$$

for $n \geq 1$, where $X_i, i = 1, 2...$ are independent random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q = 1 - p$ and $p \neq q$. Put

$$\mathcal{F}_0 = \{\Omega, \emptyset\}, \quad \mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1.$$

Let a, b be two fixed positive integers. Define

$$T = min\{n; S_n = -a \quad \text{or} \quad S_n = b\}.$$

(i) Prove that T is a stopping time with respect to the increasing family of σ-fields $\mathcal{F}_n, n \geq 1$.

[5 marks]

(ii) Show that $Y_n = S_n - n(p - q), \quad n \geq 0$, is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$.

[5 marks]

(iii) It is known from A3 (ii) that $Z_n = \left(\frac{q}{p}\right)^{S_n}, \quad n \geq 0$, is a martingale with respect to the σ-fields $\mathcal{F}_n, n \geq 0$. Moreover it is known that $E[T] < \infty$. Use the appropriate form of the Doob's Optional Theorem to deduce $E[Y_T] = 0$ and $E[Z_T] = 1$.

[8 marks]

(iv) Use (iii) to find the values of $P(S_T = -a)$, $P(S_T = b)$ and $E[T]$.

[12 marks]

B6. Recall that the Poisson process of rate λ, $\{N(t) : t \geq 0\}$, has the following properties:

(a) $N(0) = 0$.

(b) Independent increments. For any $0 \leq s_1 < t_1 < s_2 < t_2 < ... < s_n < t_n$, $N(t_n) - N(s_n)$, ...,$N(t_1) - N(s_1)$ are independent.

(c) For $s < t$, $N(t) - N(s)$ has the Poisson distribution with parameter $\lambda(t - s)$.

Let $\mathcal{F}_t = \sigma(N(u), 0 \leq u \leq t)$ be the σ-field generated by the Poisson process.

(i) Show that $U(t) = N(t) - \lambda t, t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t, t \geq 0$.

[6 marks]

(ii) Show that $V(t) = U^2(t) - \lambda t, t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t, t \geq 0$.

[12 marks]

(iii) Show that $W(t) = exp[-\theta N(t) + \lambda t(1 - e^{-\theta})], t \geq 0$ is a martingale w.r.t. $\mathcal{F}_t, t \geq 0$.

[12 marks]

B7. Consider a one period financial market model consisting of a bank account S_0 and a stock S_1 modeled on a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F}$ being the collection of all events and P a probability measure such that $p = P(\{\omega_1\}) > 0$, $q = P(\{\omega_2\}) = 1 - p > 0$. Suppose that the current asset prices (time $t = 0$) are $S_0(0) = 5$, and $S_1(0) = d$ and the terminal prices (time $t = 1$) are $S_0(1, \omega_1) = S_0(1, \omega_2) = 5(1 + r)$, and $S_1(1, \omega_1) = d_1, S_1(1, \omega_2) = d_2$, where $0 < d_1 < d_2$.

(i) Explain what it means to say that a portfolio $\phi = (\phi_0(t), \phi_1(t))$ is an arbitrage opportunity.

[5 marks]

(ii) Find the condition (in terms of r, d, d_1, d_2) under which the market is free of arbitrage and determine the corresponding equivalent martingale probability Q.

[10 marks]

(iii) Explain what an European call option with strike price K means and write down the payoff of the option.

[5 marks]

(iv) Assume $d(1 + r) = 5$, $d_1 = 3$, $d_2 = 8$, $K = 7$.

(a) Find the price of the European call option.

[5 marks]

(b) Construct a strategy (portfolio) to replicate the claim X in (iii).

[5 marks]

END OF EXAMINATION PAPER