Solutions to sheet 6 for Math67201/47201

September 25, 2015

1 Sheet 6

Problem 1

(i). The value process of the portfolio ϕ is given by

$$V_\phi(t) = \sum_{i=0}^{d} S_i(t)\phi_i(t) = S_0(t)\phi_0(t) + S_1(t)\phi_1(t) + ... + S_d(t)\phi_d(t)$$

(ii). ϕ is self-financing if

$$\phi(t) \cdot S(t) = \sum_{i=0}^{d} S_i(t)\phi_i(t)$$

$$= \phi(t+1) \cdot S(t) = \sum_{i=0}^{d} S_i(t)\phi_i(t+1)$$

for $t = 1, 2, ..., T-1$. This means that the investor adjusts his strategy from $\phi(t)$ to $\phi(t+1)$ without bringing in or consuming any wealth.

(iii). A self-financing portfolio $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_{d-1}(t), \phi_d(t))$ is called an arbitrage opportunity or arbitrage strategy if $V_\phi(0) = 0$ and the terminal wealth of ϕ satisfies

(1) $V_\phi(T) \geq 0$,

(2) $P(V_\phi(T) > 0) > 0$.

A market is arbitrage-free if and only if there exists an equivalent martingale probability measure P^*, i.e., the discounted price process $\tilde{S}(t)$ is a martingale under P^*.

(iv). A market is complete if every contingent claim X is attainable, i.e., there exists a replicating self-financing portfolio $\phi(t) = (\phi_0(t), ..., \phi_d(t))$ such that $V_\phi(T) = X$.

(v). An arbitrage-free market is complete if and only if there exists a unique probability measure P^* equivalently to P under which the discounted price process $\tilde{S}(t)$ is a martingale.

(vi). Let P^* be the unique equivalent martingale measure. The price at time zero is given by

$$\Pi_X(0) = E^*[\frac{X}{S_0(T)}],$$

where E^* stands for the expectation under P^*.

Problem 2.

(1). We need to choose $r \geq 0$ so that there exists an equivalent martingale probability measure P^*. Suppose $P^*(\{\omega_1\}) = p^*, P^*(\{\omega_2\}) = 1 - p^*$.. If P^* is an equivalent martingale probability measure, then the discounted price process $\tilde{S}(t)$ is a martingale. In particular,

$$E^*[\tilde{S}_1(1)] = E^*[\tilde{S}_1(0)]$$

That is

$$E^*[\frac{1}{S_0(1)}S_1(1)] = E^*[\frac{1}{S_0(0)}S_1(0)] = 80$$

On the other hand,

$$E^*[\frac{1}{S_0(1)}S_1(1)] = \frac{1}{1+r}E^*[S_1(1)]$$

$$= \frac{1}{1+r}[S_1(1)(\omega_1)P^*(\{\omega_1\}) + S_1(1)(\omega_2)P^*(\{\omega_2\})]$$

$$= \frac{1}{1+r}[120p^* + 40(1-p^*)]$$

So P^* is an equivalent martingale probability if and only if

$$\frac{1}{1+r}[120p^* + 40(1-p^*)] = 80, \quad 0 < p^* < 1$$

Solve the equation to obtain

$$p^* = \frac{2r+1}{2}$$

To make sure P^* exists, we must have $0 < p^* = \frac{1+2r}{2} < 1$, which is equivalent to $0 \leq r < \frac{1}{2}$.

(2). The market is complete since the martingale probability measure is uniquely given by

$$P^*(\{\omega_1\}) = p^* = \frac{2r+1}{2}, P^*(\{\omega_2\}) = 1 - p^*$$

(3). The price of the claim X at time zero is given by

$$\Pi_X(0) = E^*[\frac{1}{S_0(1)}X] = E^*[\frac{1}{1+r}X]$$

$$= \frac{1}{1+r}E^*[S_0(1)^2 + S_1(1)^2]$$

$$= 1 + r + \frac{1}{1+r}[S_1(1)(\omega_1)^2P^*(\{\omega_1\}) + S_1(1)(\omega_2)^2P^*(\{\omega_2\})]$$

$$= 1 + r + \frac{1}{1+r}[120^2 \frac{2r+1}{2} + 40^2 \frac{1-2r}{2}]$$

$$= \frac{1}{1+r}[(1+r)^2 + 12800r + 8000]$$

(4).The replicating strategy $\phi = (\phi_0, \phi_1)$ for the claim X is determined by

$$V_\phi(1) = \phi_0 S_0(1) + \phi_1 S_1(1) = X$$

Put $\omega = \omega_1$ and $\omega = \omega_2$ in the above equation to get

$$\begin{cases} \phi_0(1+r) + \phi_1 \times 120 & = (1+r)^2 + 120^2 \\ \phi_0(1+r) + \phi_1 \times 40 & = (1+r)^2 + 40^2 \end{cases}$$

Solve the above equations to obtain

$$\phi_1 = 160, \quad \phi_0 = \frac{(1+r)^2 - 3 \times 40^2}{1+r}$$

Problem 3.

(1).Note that $\{\tilde{S}_0(t) = 1, t = 0, 1, ..., T\}$ is always a martingale. So we need to choose r, p so that the discounted price $\{\tilde{S}_1(t), t = 0, 1, ...\}$ is a P-martingale , i. e.,

$$E^*[\tilde{S}_1(t+1)|\mathcal{F}_t] = \tilde{S}_1(t)$$

Now,

$$\tilde{S}_1(t+1) = \frac{S_1(t+1)}{S_0(t+1)} = \frac{1}{(1+r)^{t+1}} S_1(0) \Pi_{m=1}^{t+1} Z(m)$$

$$= \frac{1}{(1+r)^t} S_1(0) \Pi_{m=1}^{t} Z(m) \frac{1}{(1+r)} Z(t+1) = \tilde{S}_1(t) \frac{1}{(1+r)} Z(t+1)$$

It follows that

$$E[\tilde{S}_1(t+1)|\mathcal{F}_t] = E[\tilde{S}_1(t) \frac{1}{(1+r)} Z(t+1)|\mathcal{F}_t]$$

$$= \tilde{S}_1(t) E[\frac{1}{(1+r)} Z(t+1)|\mathcal{F}_t] = \tilde{S}_1(t) \frac{1}{(1+r)} E[Z(t+1)]$$

In order that $\{\tilde{S}_1(t), t = 0, 1, ...\}$ is a P-martingale, we must have

$$1 + r = E[Z(t+1)], t = 0, 1, 2, ..., T-1$$

But

$$E[Z(t+1)] = uP(Z(t) = u) + lP(Z(t) = l) = up + l(1-p).$$

Therefore we must have

$$up + l(1-p) = 1 + r$$

This yields

$$p = \frac{1+r-l}{u-l}$$

In order that P exists one must have

$$0 < p = \frac{1+r-l}{u-l} < 1$$

which is equivalent to $l < 1+r < u$. It is also seen that p is uniquely determined.

(2).The price of the claim is given by

$$\Pi_X(0) = E[\frac{X}{S_0(T)}] = \frac{1}{(1+r)^T} E[X]$$

$$= (1+r)^{-T} E[\big(\Pi_{\tau=1}^T Z(\tau) - K\big)^+]$$

Now note that the random variable $\Pi_{\tau=1}^T Z(\tau)$ takes the values $u^j l^{T-j}, j = 0, 1, ..., T$ with

$$P(\Pi_{\tau=1}^T Z(\tau) = u^j l^{T-j})$$

$$= P(j \text{ of } Z(\tau) \text{ take the value } u \text{ and } T-j \text{ of them take the value } l)$$

$$= \left(\begin{array}{c} T \\ j \end{array} \right) (p)^j (1-p)^{T-j}$$

Therefore,

$$\Pi_X(0) = (1+r)^{-T} \sum_{j=0}^{T} (u^j l^{T-j} - K)^+ P(\Pi_{\tau=1}^T Z(\tau) = u^j l^{T-j})$$

$$= (1+r)^{-T} \sum_{j=0}^{T} \left(\begin{array}{c} T \\ j \end{array} \right) (p)^j (1-p)^{T-j} (u^j l^{T-j} - K)^+$$