Solutions to sheet 5 for Math67201/47201

November 7, 2016

1 Sheet 5

Problem 1. As $|Z_n| \leq E[|X||\mathcal{F}_n]$, we have

$$E[|Z_n|] \leq E[E[|X||\mathcal{F}_n]] = E[|X|]$$

Hence, $\sup_n E[|Z_n|] < +\infty$. Applying the martingale convergence theorem, it follows that $\lim_{n\to\infty} Z_n$ exists.

Problem 2. Put $M_n = \sum_{i=1}^n (X_i - \mu_i)$ for $n \geq 1$. we can verify (exercise) that $\{M_n, n \geq 1\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, ..., X_n), n \geq 1$. As $E[M_n] = 0$, we have

$$E[M_n^2] = var(M_n) = \sum_{i=1}^n \sigma_i^2 \leq \sum_{i=1}^\infty \sigma_i^2 < \infty$$

Note that $|x| \leq x^2 + 1$. It follows that

$$\sup_n E[|M_n|] \leq 1 + \sup_n E[M_n^2] < \infty$$

By the Martingale Convergence Theorem, we conclude that

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} (\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i)$$

exists. But $\lim_{n\to\infty} \sum_{i=1}^n \mu_i = \sum_{i=1}^\infty \mu_i$ exists. Thus we see that

$$\lim_{n\to\infty} \sum_{i=1}^n X_i = \sum_{i=1}^\infty \mu_i + \lim_{n\to\infty} M_n$$

exists too. Hence, $\sum_{i=1}^\infty X_i(\omega)$ converges almost surely.

Problem 3. Proof.

(1) From the definition of $\mathcal{F}_t$, we see that B_t is $\mathcal{F}_t$-determined. Also we have $[|B_t|] < \infty$. Now for $s < t$,

$$E[B_t|\mathcal{F}_s] = E[B_t - B_s + B_s|\mathcal{F}_s]$$

$$= E[B_s|\mathcal{F}_s] + E[B_t - B_s|\mathcal{F}_s]$$

$$= B_s + E[B_t - B_s] = B_s$$

Thus the three conditions in the definition of martingales are met. Hence, $B_t, t \geq 0$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$.

(2). Since Z_t is a function of B_t, Z_t is $\mathcal{F}_t$-determined. We also have

$$E[|Z_t|] \leq E[B_t^2] + t = 2t < \infty.$$

Let $s < t$. We have

$$E[Z_t|\mathcal{F}_s] = E[B_t^2 - t|\mathcal{F}_s] = E[(B_t - B_s + B_s)^2 - t|\mathcal{F}_s]$$

$$= E[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 - s - (t-s)|\mathcal{F}_s]$$

$$= E[(B_t - B_s)^2 - (t-s)|\mathcal{F}_s] + 2E[(B_t - B_s)B_s|\mathcal{F}_s] + E[B_s^2 - s|\mathcal{F}_s]$$

$$= E[(B_t - B_s)^2] - (t-s) + 2B_sE[(B_t - B_s)] + B_s^2 - s$$

$$= (t-s) - (t-s) + 2 \times 0 + Z_s = Z_s$$

We conclude that $Z_t = B_t^2 - t, t \geq 0$, is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$.

(3). We only check

$$E[Z_t|\mathcal{F}_s] = Z_s$$

for $s < t$, and leave other details to you as exercise. Recall if $X \sim N(0, \sigma^2)$, then

$$E[e^X] = e^{\frac{1}{2}\sigma^2}$$

We proceed as follows:

$$E[Z_t|\mathcal{F}_s] = E[exp(B_t - B_s - \frac{1}{2}(t-s))exp(B_s - \frac{1}{2}s)|\mathcal{F}_s]$$

$$= exp(B_s - \frac{1}{2}s)E[exp(B_t - B_s - \frac{1}{2}(t-s))|\mathcal{F}_s]$$

$$= Z_sE[exp(B_t - B_s)|\mathcal{F}_s]exp(-\frac{1}{2}(t-s)) = Z_sE[exp(B_t - B_s)]exp(-\frac{1}{2}(t-s))$$

$$= Z_s exp(\frac{1}{2}(t-s))exp(-\frac{1}{2}(t-s)) = Z_s$$

(4). From (1) we know that $B_t, t \geq 0$ is a martingale. As $|B_{t\wedge T}| \leq a + b$, by Doob's Optimal Stopping Theorem we have $E[B_T] = E[B_0] = 0$. Noticing that

$$E[B_T] = (-a)P(B_T = -a) + bP(B_T = b)$$

it follows that

$$(-a)P(B_T = -a) + bP(B_T = b) = 0$$

On the other hand,

$$P(B_T = -a) + P(B_T = b) = 1$$

Solve the above two equations to obtain

$$P(B_T=-a)=\frac{b}{a+b}, \quad P(B_T=b)=\frac{a}{a+b}$$

(5). It is known from (2) that $Z_t = B_t^2 - t, t \geq 0$, is a martingale. Consequently, the stopped process $Z_{t\wedge T} = B_{t\wedge T}^2 - (t\wedge T), t \geq 0$ is also a martingale. Thus

$$E[Z_{t\wedge T}] = E[B_{t\wedge T}^2 - (t\wedge T)] = 0$$

i.e.,

$$E[B_{t\wedge T}^2] = E[(t\wedge T)]$$

By Monotone Convergence Theorem,

$$\lim_{t\to\infty} E[(t\wedge T)] = E[T]$$

Note that $B_{t\wedge T}^2 \leq (a+b)^2$. The dominated Convergence Theorem yields

$$\lim_{t\to\infty} E[B_{t\wedge T}^2] = E[B_T^2]$$

Combining the above two convergence we get

$$E[T] = E[B_T^2]$$

$$= (-a)^2 P(B_T=-a) + b^2 P(B_T=b) = a^2\frac{b}{a+b} + b^2\frac{a}{a+b} = ab$$

Problem 4. (a). Show $U(t) = N(t) - \lambda t$ is a martingale.

(1). $U(t)$ is $\mathcal{F}_t$-measurable by the definition of $\mathcal{F}_t$.

(2). $U(t)$ is integrable because $N(t)$ has a Poisson distribution with parameter λt.

(3). For $s < t$, we have

$$E[U(t)|\mathcal{F}_s] = E[U(t)-U(s)|\mathcal{F}_s] + E[U(s)|\mathcal{F}_s]$$

$$= U(s) + E[N(t)-N(s)|\mathcal{F}_s] - \lambda(t-s)$$

$$= U(s) + E[N(t)-N(s)] - \lambda(t-s) = U(s)$$

(b). Show $V(t) = U^2(t) - \lambda t$ is a martingale.

(1). $V(t)$ is $\mathcal{F}_t$-measurable by the definition of $\mathcal{F}_t$.

(2). $V(t)$ is integrable because $N(t)$ has a Poisson distribution with parameter λt.

(3). For $s < t$, we have

$$E[V(t)|\mathcal{F}_s] = E[U^2(t) - \lambda t|\mathcal{F}_s]$$

$$= E[(U(t)-U(s)+U(s))^2|\mathcal{F}_s] - \lambda t$$

$$= E[(U(t)-U(s))^2|\mathcal{F}_s] + 2E[(U(t)-U(s))U(s)|\mathcal{F}_s] + E[U^2(s)|\mathcal{F}_s] - \lambda t$$

$$= E[(N(t) - N(s) - \lambda(t-s))^2] + 2U(s)E[(U(t) - U(s))|\mathcal{F}_s] + U^2(s) - \lambda t$$

$$= \lambda(t-s) + 2U(s) \times 0 + U^2(s) - \lambda t = U^2(s) - \lambda s = V(s),$$

where the fact $N(t) - N(s) \sim Poisson(\lambda(t-s))$ has been used.

(c). Show $W(t) = exp[-\theta N(t) + \lambda t(1 - e^{-\theta})]$ is a martingale.

(1). $W(t)$ is $\mathcal{F}_t$-measurable by the definition of $\mathcal{F}_t$.

(2). $W(t)$ is integrable because $N(t)$ has a Poisson distribution with parameter λt.

(3). For $s < t$, we have

$$E[W(t)|\mathcal{F}_s] = E[W(s)W^{-1}(s)W(t)|\mathcal{F}_s]$$

$$= E[W(s)exp[-\theta(N(t) - N(s)) + \lambda(t-s)(1 - e^{-\theta})]|\mathcal{F}_s]$$

$$= W(s)exp[\lambda(t-s)(1 - e^{-\theta})]E[exp[-\theta(N(t) - N(s))]|\mathcal{F}_s]$$

$$= W(s)exp[\lambda(t-s)(1 - e^{-\theta})]E[exp[-\theta(N(t) - N(s))]]$$

But

$$E[exp[-\theta(N(t) - N(s))]] = \sum_{n=0}^{\infty} exp(-\theta n)exp(-\lambda(t-s))\frac{(\lambda(t-s))^n}{n!}$$

$$= exp(-\lambda(t-s))\sum_{n=0}^{\infty}\frac{(exp(-\theta)\lambda(t-s))^n}{n!} = exp[-\lambda(t-s)(1 - e^{-\theta})]$$

Combine the above equations to get

$$E[W(t)|\mathcal{F}_s] = W(s)$$