Solutions to sheet 4 for Math67201/47201

November 7, 2016

1 Sheet 4

Problem 1.

(1) For $k > j$, we have

$$E[Y_k|\mathcal{F}_j] = E[E[Y_k|\mathcal{F}_{k-1}]|\mathcal{F}_j]$$

.

$$= E[Y_{k-1}|\mathcal{F}_j] = E[E[Y_{k-1}|\mathcal{F}_{k-2}]|\mathcal{F}_j]$$
$$= E[Y_{k-2}|\mathcal{F}_j] = \cdots = E[Y_{j+1}|\mathcal{F}_j] = Y_j$$

(2).

$$E[(Y_k - Y_j)^2] = E[(Y_k)^2] + E[(Y_j)^2] - 2E[Y_k Y_j]$$
$$= E[(Y_k)^2] + E[(Y_j)^2] - 2E[E[Y_k Y_j|\mathcal{F}_j]]$$
$$= E[(Y_k)^2] + E[(Y_j)^2] - 2E[Y_j E[Y_k|\mathcal{F}_j]]$$
$$= E[(Y_k)^2] + E[(Y_j)^2] - 2E[(Y_j)^2] = E[(Y_k)^2] - E[(Y_j)^2]$$

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Problem 2. (1). From the definition, we see that M_n is $\mathcal{F}_n$-measurable.

(2). M_n is integrable if for example, $S_k, k \geq 1$ is bounded.

(3).

$$E[M_{n+1}|\mathcal{F}_n] = E[M_n + S_{n+1}(Y_{n+1} - Y_n)|\mathcal{F}_n]$$
$$= E[M_n|\mathcal{F}_n] + E[S_{n+1}(Y_{n+1} - Y_n)|\mathcal{F}_n]$$
$$= M_n + S_{n+1}E[Y_{n+1} - Y_n|\mathcal{F}_n] = M_n + S_{n+1} \times 0 = M_n,$$

we have used the fact that S_{n+1} is $\mathcal{F}_n$-measurable.

Problem 3. Measurability and integrability of Z_n follow from the fact that X and Y are martingales. Let us show

$$E[Z_{n+1}|\mathcal{F}_n] = Z_n$$

Write

$$Z_n = I_{\{n<T\}}X_n + I_{\{T \leq n\}}Y_n.$$

Note that

$$I_{\{n+1<T\}}X_{n+1} = X_{n+1} - I_{\{n+1 \geq T\}}X_{n+1}$$

$$= X_{n+1} - I_{\{n+1>T\}} X_{n+1} - I_{\{n+1=T\}} X_{n+1}$$
$$= X_{n+1} - I_{\{n\geq T\}} X_{n+1} - I_{\{n+1=T\}} X_T$$

and

$$I_{\{T\leq n+1\}} Y_{n+1} = I_{\{T\leq n\}} Y_{n+1} + I_{\{T=n+1\}} Y_T$$

We have

$$E[Z_{n+1}|\mathcal{F}_n]$$
$$= E[I_{\{n+1<T\}} X_{n+1} + I_{\{T\leq n+1\}} Y_{n+1}|\mathcal{F}_n]$$
$$= E[X_{n+1} - I_{\{T\leq n\}} X_{n+1} - I_{\{n+1=T\}} X_T|\mathcal{F}_n]$$
$$+E[I_{\{T\leq n\}} Y_{n+1} + I_{\{T=n+1\}} Y_T|\mathcal{F}_n]$$
$$= E[X_{n+1}|\mathcal{F}_n] - E[I_{\{T\leq n\}} X_{n+1}|\mathcal{F}_n] - E[I_{\{n+1=T\}} X_T|\mathcal{F}_n]$$
$$+E[I_{\{T\leq n\}} Y_{n+1}|\mathcal{F}_n] + E[I_{\{T=n+1\}} Y_T|\mathcal{F}_n]$$
$$= X_n - I_{\{T\leq n\}} E[X_{n+1}|\mathcal{F}_n] - E[I_{\{n+1=T\}} Y_T|\mathcal{F}_n]$$
$$+I_{\{T\leq n\}} E[Y_{n+1}|\mathcal{F}_n] + E[I_{\{T=n+1\}} Y_T|\mathcal{F}_n]$$
$$= X_n - I_{\{T\leq n\}} X_n + I_{\{T\leq n\}} Y_n$$
$$= I_{\{T>n\}} X_n + I_{\{T\leq n\}} Y_n = Z_n$$

Problem 4. Set $S_n = X_1 + X_2 + ... + X_n, n \geq 1$, and $S_0 = 0$. As in problem 1 we see that $\{S_n, n \geq 0\}$ forms a martingale. Then the stopped process $\{Z_n = S_{n\wedge T}, n \geq 0\}$ is also a martingale. In particular,

$$E[Z_n] = E[S_{n\wedge T}] = E[S_0] = 0$$

Remark that

$$\lim_{n\to\infty} S_{n\wedge T} = X_1 + X_2 + ... + X_T$$

and

$$|S_{n\wedge T}| \leq |X_1| + |X_2| + ... + |X_T| \leq TM$$

Since $E(T) < \infty$, by dominated convergence theorem we have

$$E[S_T] = \lim_{n\to\infty} E[S_{n\wedge T}] = 0$$