Solutions to sheet 3 for Math67201/47201

November 7, 2016

1 Sheet 3

Problem 1.

(1). We have

$$E[S_{n+1}|\mathcal{F}_n] = E[S_n + X_{n+1}|\mathcal{F}_n]$$
$$= E[S_n|\mathcal{F}_n] + E[X_{n+1}|\mathcal{F}_n] = S_n + E[X_{n+1}]$$

as S_n is determined by $\mathcal{F}_n$ and X_{n+1} is independent of $\mathcal{F}_n$. $\{S_n, n \geq 0\}$ is a martingale with respect to $\mathcal{F}_n, n \geq 0$ if and only if $E[S_{n+1}|\mathcal{F}_n] = S_n$, and this is the case if and only if $E[X_n] = 0$ for all n.

(2) Since $\{S_n, n \geq 0\}$ is a martingale, $E[X_n] = 0$ for all n by (1). M_n is a function of $X_1, X_2, ...X_n$. So M_n is determined by $\mathcal{F}_n$. As $|M_n| \leq C(\sum_{k=1}^n X_k^2 + n)$, we see that $E[M_n] < \infty$. Write

$$M_{n+1} = S_{n+1}^2 - (n+1) = (S_n + X_{n+1})^2 - (n+1)$$
$$= S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - n - 1$$
$$= M_n + 2S_nX_{n+1} + X_{n+1}^2 - 1$$

Taking into account the independence, it follows that

$$E[M_{n+1}|\mathcal{F}_n] = E[M_n + 2S_nX_{n+1} + X_{n+1}^2 - 1|\mathcal{F}_n]$$
$$= E[M_n|\mathcal{F}_n] + E[2S_nX_{n+1}|\mathcal{F}_n] + E[X_{n+1}^2 - 1|\mathcal{F}_n]$$
$$= M_n + 2S_nE[X_{n+1}] + E[X_{n+1}^2] - 1 = M_n$$

This shows that $\{M_n = S_n^2 - n, n \geq 0\}$ is a martingale.

Problem 2.

(i). Since Z_n is a function of $X_1, X_2, ..., X_n$, Z_n is $\mathcal{F}_n$-determined.

(ii).

$$E[|Z_n|] = e^{-na}E[e^{S_n}] = e^{-na}E[e^{X_1+X_2+...+X_n}]$$
$$= e^{-na}(E[e^{X_1}])^n < \infty$$

(iii).

$$E[Z_{n+1}|\mathcal{F}_n] = E[exp(S_{n+1} - (n+1)a)|\mathcal{F}_n]$$
$$= E[exp(S_n - na)exp(X_{n+1} - a)|\mathcal{F}_n] = exp(S_n - na)E[exp(X_{n+1} - a)|\mathcal{F}_n]$$
$$= Z_nE[exp(X_{n+1} - a)] = Z_nE[exp(X_{n+1})]e^{-a} = Z_ne^ae^{-a} = Z_n$$

Combining (i), (ii) and (iii) we see that $\{Z_n = exp(S_n - na), n \geq 1\}$ is a martingale w.r.t. $\mathcal{F}_n$.

Problem 3. Y_n is a function of $X_1, X_2, ...X_n$. So Y_n is determined by $\mathcal{F}_n$. As $X_k, k \geq 1$ are normal random variables, $E[|Y_n|] < \infty$. Write

$$Y_{n+1} = Y_n exp(X_{n+1} - \frac{1}{2}\sigma^2)$$

By the independence, we have

$$E[Y_{n+1}|\mathcal{F}_n] = E[Y_n exp(X_{n+1} - \frac{1}{2}\sigma^2)|\mathcal{F}_n]$$

$$= Y_n E[exp(X_{n+1} - \frac{1}{2}\sigma^2)|\mathcal{F}_n] = Y_n E[exp(X_{n+1} - \frac{1}{2}\sigma^2)]$$

$$= Y_n E[exp(X_{n+1})]exp(-\frac{1}{2}\sigma^2) = Y_n exp(\frac{1}{2}\sigma^2)exp(-\frac{1}{2}\sigma^2) = Y_n,$$

where we used the fact that $E[exp(X_{n+1})] = exp(\frac{1}{2}\sigma^2)$, which is true for Gaussian random variables with mean 0 and variance σ^2.

Problem 4. The martingale property requires that

$$E[Y_{n+1}|\mathcal{F}_n] = Y_n$$

Now,

$$E[Y_{n+1}|\mathcal{F}_n] = E[aX_{n+1} + X_n|\mathcal{F}_n]$$
$$= aE[X_{n+1}|\mathcal{F}_n] + X_n = a(\alpha X_n + \beta X_{n-1}) + X_n$$
$$= X_n(\alpha a + 1) + \beta a X_{n-1}$$

In order that the martingale property holds, one needs

$$X_n(\alpha a + 1) + \beta a X_{n-1} = Y_n = aX_n + X_{n-1}$$

This will hold if $\alpha a + 1 = a$ and $\beta a = 1$. Using $\alpha = 1 - \beta$, we see that both $\alpha a + 1 = a$ and $\beta a = 1$ lead to $a = \frac{1}{\beta}$.

Problem 5. For $i \neq j$, let us assume without loss of generality that $i < j$. Since $\{S_n, n \geq 1\}$ is a martingale, we have

$$E[S_j|\mathcal{F}_{j-1}] = S_{j-1}$$

Equivalently,

$$E[X_j|\mathcal{F}_{j-1}] = E[S_j - S_{j-1}|\mathcal{F}_{j-1}] = 0$$

As $i \leq j - 1$, X_i is also $\mathcal{F}_{j-1} \supset \mathcal{F}_i$-determined. Multiplying the above equation by X_i, we get

$$0 = X_i E[X_j|\mathcal{F}_{j-1}] = E[X_i X_j|\mathcal{F}_{j-1}]$$

Taking expectation gives

$$E[X_i X_j] = E[E[X_i X_j|\mathcal{F}_{j-1}]] = 0$$