Solutions to sheet 2 for Math67201/47201

September 25, 2015

1 Sheet 2

Problem 1.

(i). Let $A \in \mathcal{F}$. The indicator of A is given by

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c \end{cases}$$

(ii). $\int_\Omega I_A(\omega) dP = P(A)$.

(iii). In particular, take $X = I_A(\omega)$ to get

$$P(A) = \int_\Omega I_A(\omega) dP = \int_\Omega I_A(\omega) dQ = Q(A)$$

for any event A.

Problem 2. If $\omega \in [0, 1]$ and $\omega > 0$, then there exists an integer N such that for $n \geq N$, $\frac{2}{n} < \omega$. Thus by definition $X_n(\omega) = 0$ for $n \geq N$. Therefore, $\lim_{n\to\infty} X_n(\omega) = 0$. If $\omega = 0$, by definition $X_n(\omega) = 0$ for all $n \geq 1$. In conclusion, $\lim_{n\to\infty} X_n(\omega) = 0$ for every $\omega \in [0, 1]$. Now,

$$E[X_n] = \int_\Omega X_n(\omega) dP = \int_0^1 X_n(\omega) dw$$

$$= \int_{\frac{1}{n}}^{\frac{2}{n}} n^2 dw = n^2 (\frac{2}{n} - \frac{1}{n}) = n$$

This yields that

$$\lim_{n\to\infty} E[X_n] = \lim_{n\to\infty} n = +\infty \neq 0$$

Problem 3.

(i). Put $Y_n = \sum_{j=1}^n X_j$. Since $X_j, j = 1, 2, ...$ are non-negative random variables, $Y_n, n \geq 1$ ia an increasing sequence of random variables and $Y_n \uparrow \sum_{j=1}^\infty X_j$ as $n \to \infty$. By the monotone convergence theorem,

$$E(\sum_{j=1}^\infty X_j) = \lim_{n\to\infty} E[Y_n]$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{\infty} E(X_j)$$

(ii) By definition, $I_{A_j} = 1$ if A_j occurs and $I_{A_j} = 0$ if A_j does not occur. Thus, we see that

$$N = I_{A_1} + I_{A_2} + I_{A_3} + ... + I_{A_n} + ...$$

Using (i) with $X_j = I_{A_j}$ we get

$$E[N] = E[\sum_{j=1}^{\infty} I_{A_j}] = \sum_{j=1}^{\infty} E[I_{A_j}] = \sum_{j=1}^{\infty} P(A_j)$$

Problem 4. Set $Y_m(t) = \sum_{n=1}^{m} \frac{1}{n^2} cos(nt)$. Then

$$\lim_{m\to\infty} Y_m(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} cos(nt).$$

On the other hand,

$$|Y_m(t)| \leq \sum_{n=1}^{m} \frac{1}{n^2} |cos(nt)|$$

$$\leq \sum_{n=1}^{m} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = M < +\infty.$$

By the Dominated Convergence Theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} (\sum_{n=1}^{\infty} \frac{1}{n^2} cos(nt))dt$$

$$= \lim_{m\to\infty} \frac{1}{2\pi} \int_0^{2\pi} Y_m(t)dt = \lim_{m\to\infty} \frac{1}{2\pi} \sum_{n=1}^{m} \int_0^{2\pi} \frac{1}{n^2} cos(nt)dt$$

$$= \lim_{m\to\infty} \frac{1}{2\pi} \sum_{n=1}^{m} \frac{1}{n^2} [\frac{1}{n} sin(nt)]_0^{2\pi}$$

$$= \lim_{m\to\infty} \frac{1}{2\pi} \sum_{n=1}^{m} 0 = 0$$

Problem 5.

(1). Proof.

$$E[|X|^\alpha] = \int_\Omega |X|^\alpha dP \geq \int_{\{|X|\geq\varepsilon\}} |X|^\alpha dP$$

$$\geq \int_{\{|X|\geq\varepsilon\}} \varepsilon^\alpha dP = \varepsilon^\alpha P(|X| \geq \varepsilon)$$

This gives the Chebyshev inequality:

$$P(|X| \geq \varepsilon) \leq \frac{E[|X|^\alpha]}{\varepsilon^\alpha}, \quad \alpha > 0, \varepsilon > 0.$$

(2) Use (1) to obtain

$$P(|X| > n) \leq \frac{E[|X|]}{n}$$

Hence,

$$\lim_{n\to\infty} P(|X| > n) \leq \lim_{n\to\infty} \frac{E[|X|]}{n} = 0$$

(3) Using (1) for $\alpha = 2$,

$$P(\left|\frac{X_1 + X_2 + ... + X_n}{n}\right| > \delta) \leq \frac{1}{\delta^2} E\big[(\frac{X_1 + X_2 + ... + X_n}{n})^2\big]$$

$$= \frac{1}{\delta^2}\frac{1}{n^2} E\big[(X_1 + X_2 + ... + X_n)^2\big] = \frac{1}{\delta^2}\frac{1}{n^2}\big[E(X_1^2) + E(X_2^2) + ... + E(X_n^2)\big]$$

$$= \frac{1}{\delta^2}\frac{1}{n^2} n = \frac{1}{\delta^2}\frac{1}{n}$$

Consequently,

$$\lim_{n\to\infty} P(\left|\frac{X_1 + X_2 + ... + X_n}{n}\right| > \delta) = 0$$