Solutions to sheet 1 for Math67201/47201

September 25, 2015

1 Sheet 1

Problem 1.

(i). We have

$$P(A) = P(\cup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} P(B_n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

$$= \frac{1}{4} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}$$

So

$$P(A^c) = 1 - P(A) \geq 1 - \frac{1}{2} = \frac{1}{2}$$

(ii).

$$P(A_m) \leq \sum_{n=m}^{\infty} P(B_n) = \sum_{n=m}^{\infty} \frac{1}{2^{n+1}}$$

$$= \frac{1}{2^{m+1}} \sum_{n=m}^{\infty} \frac{1}{2^{n-m}} = \frac{1}{2^{m+1}} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^m}$$

(iii). Note for any $n \geq 1$, $\cap_{m=1}^{\infty} A_m \subset A_n$. Thus,

$$P(\cap_{m=1}^{\infty} A_m) \leq P(A_n) = \frac{1}{2^n}$$

for all $n \geq 1$. hence,

$$P(\cap_{m=1}^{\infty} A_m) \leq \lim_{n \to \infty} \frac{1}{2^n} = 0$$

Problem 2. Put $B = \cap_{n=1}^{\infty} B_n$. Then

$$P(B^c) = P(\cup_{n=1}^{\infty} B_n^c) \leq \sum_{n=1}^{\infty} P(B_n^c)$$

$$= \sum_{n=1}^{\infty} (1 - P(B_n)) = 0$$

So $P(B) = 1$.

Problem 3. **Proof**. Assume $B_n \uparrow B$. Introduce $A_1 = B_1$, $A_2 = B_2 \setminus B_1$, ..., $A_n = B_n \setminus B_{n-1}$,...Then, $B_n = \cup_{k=1}^n A_k$ and

$$\cup_{n=1}^\infty B_n = \cup_{n=1}^\infty A_n = B$$

Since $A_i' s$ are disjoint, we have

$$P(B) = P(\cup_{n=1}^\infty A_k) = \sum_{k=1}^\infty P(A_k)$$

Hence,

$$\sum_{k=1}^n P(A_k) \to P(B)$$

But,

$$\sum_{k=1}^n P(A_k) = \sum_{k=1}^n [P(B_k) - P(B_{k-1})] = P(B_n)$$

The result follows.

Problem 4. $\mathcal{G}$ is a σ-field if (1) $\Omega \in \mathcal{G}$. (2). $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$. (3). If $A_1, A_2, ..., A_n, .. \in \mathcal{G}$, then $\cup_{n=1}^\infty A_n \in \mathcal{G}$.

(i) As $\Omega \in \mathcal{G}$ and $\Omega^c = \emptyset \in \mathcal{G}$, (1) and (2) are satisfied in the definition. Since $\Omega \cup \emptyset = \Omega \in \mathcal{G}$, (3) is satisfied too.

(ii). The smallest σ-field $\mathcal{G}$ that contains an event $A \subset \Omega$ is given by

$$\mathcal{G} = \{\Omega, \emptyset, A, A^c\}$$

(iii). Proof.

(1). Since $\Omega \in \mathcal{G}_1$, $\Omega \in \mathcal{G}_2$, $\Omega \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$.

(2) If $A \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$, then $A \in \mathcal{G}_1$ and $A \in \mathcal{G}_2$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are both σ-fields, we conclude that $A^c \in \mathcal{G}_1$ and $A^c \in \mathcal{G}_2$. Hence, $A^c \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$.

(3). If $A_i \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$ for $i = 1, 2, ...$, then $A_i \in \mathcal{G}_1$ and $A_i \in \mathcal{G}_2$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are both σ-fields, we conclude that $\cup_{i=1}^\infty A_i \in \mathcal{G}_1$ and $\cup_{i=1}^\infty A_i \in \mathcal{G}_2$. Hence, $\cup_{i=1}^\infty A_i \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$.