# Problem sheets for Math67201/47201: part II

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## 1 Sheet 4

**Problem 1**. Let $\{Y_n, n \geq 0\}$ is a martingale w.r.t. $\mathcal{F}_n$. For $k > j$, show that

(1) $E[Y_k|\mathcal{F}_j] = Y_j$.

(2). $E[(Y_k - Y_j)^2] = E[(Y_k)^2] - E[(Y_j)^2]$.

**Problem 2**. Let $\{Y_n, n \geq 0\}$ be a martingale w.r.t. $\mathcal{F}_n$. Let $\{S_n, n \geq 1\}$ be a sequence of bounded random variables such that $S_n$ is $\mathcal{F}_{n-1}$-measurable. Define

$$M_n = Y_0 + \sum_{k=1}^{n} S_k(Y_k - Y_{k-1}), n \geq 1$$

Prove that $M_n, n \geq 1$ is a martingale.

**Problem 3**. Let $\{Y_n, n \geq 0\}$ and $\{X_n, n \geq 0\}$ be two martingales w.r.t. $\mathcal{F}_n, n \geq 1$. Let $T$ be a stopping time w.r.t. $\mathcal{F}_n, n \geq 1$. Suppose $X_T = Y_T$ on the event $\{\omega; T(\omega) < \infty\}$. Define

$$Z_n = \begin{cases} X_n & \text{if } n < T \\ Y_n & \text{if } n \geq T \end{cases}$$

Show that $\{Z_n, n \geq 1\}$ is a martingale w.r.t. $\mathcal{F}_n, n \geq 1$.

**Problem 4**. Let $X_1, X_2, ..., X_n...$ be independent random variables. Define $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$. Let $T$ be a stopping time w.r.t. $\mathcal{F}_n$. Assume $|X_n| \leq M$ for all $n \geq 1$, where $M$ is a constant and $E(T) < \infty$. Show that if $E(X_n) = 0$ for all $n \geq 1$, then $E(X_1 + X_2 + ... + X_T) = 0$.

## 2 Sheet 5

**Problem 1**. Let $X$ be a random variable with $E[|X|] < \infty$ and $\{\mathcal{F}_n\}_{n\geq 1}$ an increasing sequence of $\sigma$-fields. Set

$$Z_n = E[X|\mathcal{F}_n], n \geq 1$$

Show that $\lim_{n\to\infty} Z_n$ exists.

**Problem 2**. Let $X_1, X_2, ..., X_n...$ be independent random variables with $E[X_i] = \mu_i$, $Var(X_i) = \sigma_i^2$. If $\sum_{i=1}^{\infty} \mu_i < \infty$ and $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, show $\sum_{i=1}^{\infty} X_n(\omega)$ converges almost surely.

**Problem 3**. Let $B_t, t \geq 0$ be a Brownian motion. Set $\mathcal{F}_t = \sigma(B_u, 0 \leq u \leq t)$. Show

(1) $B_t, t \geq 0$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$.

(2). $Z_t = B_t^2 - t, t \geq 0$, is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$.

(3) $Z_t = exp(B_t - \frac{1}{2}t), t \geq 0$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$.

(4) Define

$$T = \inf\{t \geq 0; B_t = -a, \quad \text{or} \quad b\}$$

Use Doob's optimal stopping theorem to find the probability that the Brownian motion $B_t, t \geq 0$ leaves the interval $[-a, b]$ at the point $-a$, i.e., $P(B_T = -a)$, where $a, b$ are positive numbers.

(5) Let $T$ be defined as in (4). Find $E(T)$.

**Problem 4**. Let $\{N(t) : t \geq 0\}$ be a Poisson process of rate $\lambda$ and let $\mathcal{F}_t = \sigma(N(u), 0 \leq u \leq t)$ be the $\sigma$-field generated by the Poisson process. Show that $U(t) = N(t) - \lambda t$, $V(t) = U^2(t) - \lambda t$ and $W(t) = exp[-\theta N(t) + \lambda t(1 - e^{-\theta})]$ are all martingales w.r.t. $\mathcal{F}_t, t \geq 0$

# 3 Sheet 6

**Problem 1**. Consider a financial market consisting of one risk-free asset whose price at time $t$ is $S_0(t)$ and $d$ risky assets whose prices at time $t$ are random variables $S_1(t), ..., S_d(t)$ on a probability space $(\Omega, \mathcal{F}, P)$. Let $\phi(t) = (\phi_0(t), \phi_1(t), ..., \phi_d(t))$ denote the portfolio of an agent at time $t$.

(i) Write down the value process of the portfolio $\phi$.

(ii) Explain what it means to say that $\phi$ is self-financing.

(iii) Explain what it means to say that the strategy $\phi$ is an arbitrage opportunity and state the condition under which the market is free of arbitrage.

(iv) Explain what it means to say that the market is complete.

(v) State the condition under which an arbitrage-free market is complete.

(vi) Let $X$ be a contingent claim. Write down the formula for the price of $X$ at time zero provided that the market is arbitrage-free and complete.

**Problem 2**. Consider the model in problem 1 with $\Omega = \{\omega_1, \omega_2\}$ and $d = 1$. Assume $P(\{\omega_1\}) = p$ and $P(\{\omega_2\}) = 1 - p$. The time indices are $t = 0$ and $t = 1$. Suppose the price at time zero is

$$S(0) = (S_0(0), S_1(0)) = (1, 80)$$

The price $S(1) = (S_0(1), S_1(1))$ at time 1 is determined by $S_0(1) = 1 + r$, $S_1(1)(\omega_1) = 120$ and $S_1(1)(\omega_2) = 40$

(1) Choose $r$ so that the market is free of arbitrage.

(2) Determine whether the market is complete.

(3). Find the price at time zero for the contingent claim $X = S_0(1)^2 + S_1(1)^2$.

(4). Construct a trading strategy $\phi = (\phi_0, \phi_1)$ to replicate the claim $X$.

**Problem 3**. Consider the situation in problem 1. The time indices are assumed to be $t = 0, 1, 2, ..., T$. Suppose $d = 1$. Fix two positive numbers $l$ and $u$ such that $l < 1$, $u > 1$. Let $Z(t), t = 1, 2, ..., T$ be independent, identically distributed random variables with

$$P(Z(t) = u) = p > 0, \quad P(Z(t) = l) = q = 1 - p > 0,$$

where $P$ is the probability measure. Define the price process as follows:

$$S_0(0) = 1, \quad S_0(t) = (1 + r)^t, t = 1, 2, ..., T.$$

$$S_1(t) = S_1(0)\Pi_{m=1}^t Z(m) = S_1(0)Z(1)Z(2)...Z(t).$$

(1) Choose $r$ and $p$ so that the discounted price process $\tilde{S}(t) = (\tilde{S}_0(t), \tilde{S}_1(t))$ is a martingale under $P$ with respect to $\mathcal{F}_t = \sigma(Z(1), ..., Z(t))$.

(2) Find the price of the European call option with payoff $X(\omega) = (S_1(T) - K)^+$. Here we assume $S_1(0) = 1$.