Problem sheets for Math67201/47201: part I

November 7, 2016

1 Sheet 1

Problem 1. Let $\{B_n\}_{n\geq 1}$ be a sequence of events with $P(B_n) = \frac{1}{2^{n+1}}, n \geq 1$.
(i). Let $A = \cup_{n=1}^{\infty} B_n$. Show that $P(A^c) \geq \frac{1}{2}$.
(ii). Let $A_m = \cup_{n=m}^{\infty} B_n$. Show that $P(A_m) \leq \frac{1}{2^m}$.
(iii). Deduce the probability of $\cap_{m=1}^{\infty} A_m$.

Problem 2. Show that if $P(B_n) = 1$, for $n \geq 1$, then $P(\cap_{n=1}^{\infty} B_n) = 1$.

Problem 3. We write $B_n \uparrow B$ if $B_1 \subset B_2 \subset B_3 ...$ and $B = \cup_{n=1}^{\infty} B_n$. Likewise, write $B_n \downarrow B$ if $B_1 \supset B_2 \supset B_3 ...$ and $B = \cap_{n=1}^{\infty} B_n$.

If $B_n \uparrow B (B_n \downarrow B)$, show that $P(B_n) \uparrow P(B)(P(B_n) \downarrow P(B))$.

Problem 4. State the definition of a σ-field $\mathcal{G}$.
(i) Verify that $\mathcal{G} = \{\Omega, \emptyset\}$ is a σ-field.
(ii). Write down the smallest σ-field that contains an event $A \subset \Omega$.
(iii). If $\mathcal{G}_1$, $\mathcal{G}_2$ are two σ-fields, show that $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$ is also a σ-field.

2 Sheet 2

Problem 1.
(i). Let $A \in \mathcal{F}$. Write down the indicator $I_A(\omega)$ of A.
(ii). Write down the value of $\int_\Omega I_A(\omega) dP$.
(iii). If P and Q are two probability measures on $(\Omega, \mathcal{F})$ such that

$$\int_\Omega X(\omega) dP = \int_\Omega X(\omega) dQ$$

for all non-negative random variables, then $P = Q$, i.e., $P(A) = Q(A)$ for all $A \in \mathcal{F}$.

Problem 2. Let $\Omega = [0, 1]$, $\mathcal{F}$ is the Borel σ-field that contains intervals. The probability P is the generalized length. Define

$$X_n(\omega) = \begin{cases} n^2, & \text{if } \frac{1}{n} \leq \omega \leq \frac{2}{n} \\ 0 & \text{otherwise.} \end{cases}$$

Show that for every $\omega \in \Omega$ we have $\lim_{n\to\infty} X_n(\omega) = 0$. Is $\lim_{n\to\infty} E(X_n) = 0$ true ?

Problem 3.

(i). If $X_1, X_2, ...,$ are non-negative random variables on a probability space, apply the monotone convergence theorem to the sequence $Y_n = \sum_{j=1}^{n} X_j$ to show that

$$E(\sum_{j=1}^{\infty} X_j) = \sum_{j=1}^{\infty} E(X_j)$$

(ii) By choosing $X_j = I_{A_j}$ in (i), show that if N is the (random) number of $A_1, A_2, ...$ which occur, then

$$E(N) = P(A_1) + P(A_2) + ... + P(A_n) + ...$$

Problem 4. Use the Dominated Convergence Theorem to prove that

$$\int_0^{2\pi} (\sum_{n=1}^{\infty} \frac{1}{n^2} cos(nt))dt = 0$$

Problem 5.

(1). Prove the Chebyshev inequality:

$$P(|X| \geq \varepsilon) \leq \frac{E[|X|^{\alpha}]}{\varepsilon^{\alpha}}, \quad \alpha > 0, \varepsilon > 0.$$

(2) Use (1) to show that if $E[|X|] < \infty$, then

$$\lim_{n\to\infty} P(|X| > n) = 0$$

(3) Let $X_1, X_2, ..., X_n ...$ be independent random variables with mean 0 and variance 1. Show that for any $\delta > 0$,

$$\lim_{n\to\infty} P(\left|\frac{X_1 + X_2 + ... + X_n}{n}\right| > \delta) = 0$$

3 Sheet 3

Problem 1. Let $X_1, X_2, ..., X_n ...$ be independent random variables. Define $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$. Set $S_0 = 0$, $S_n = X_1 + X_2 + ... + X_n$, $n \geq 1$.

(1) Prove that $\{S_n, n \geq 0\}$ is a martingale w.r.t. $\mathcal{F}_n$ if and only if $E(X_n) = 0$ for $n \geq 1$.

(2) If $\{S_n, n \geq 0\}$ is a martingale and if $E(X_n^2) = 1$ for $n \geq 1$, show that $\{M_n = S_n^2 - n, n \geq 0\}$ is also a martingale.

Problem 2. Let $X_1, X_2, ..., X_n ...$ be independent, identically distributed random variables. Define $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$. Put $a = log(E[e^{X_1}])$ and $S_n = X_1 + X_2 + ... + X_n$, $n \geq 1$. Prove that $\{Z_n = exp(S_n - na), n \geq 1\}$ is a martingale w.r.t. $\mathcal{F}_n$.

Problem 3. Suppose that $X_1, X_2, ..., X_n...$ are independent and normally distributed with mean zero and variance σ^2. Define $Y_0 = 1$ and

$$Y_n = exp\{(X_1 + X_2 + ... + X_n) - \frac{1}{2}n\sigma^2\}, \quad n \geq 1$$

Show that $Y_n, n \geq 0$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$.

Problem 4. Consider a family of random variables $\{X_n, n \geq 0\}$, each having finite absolute expectation and satisfying

$$E(X_{n+1}|\mathcal{F}_n) = \alpha X_n + \beta X_{n-1}, n \geq 1,$$

with $\alpha > 0, \beta > 0$, and $\alpha + \beta = 1$. Here $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$. Find an appropriate value of a such that the sequence $Y_n = aX_n + X_{n-1}$, $n \geq 1$, $Y_0 = X_0$ constitutes a martingale with respect to $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$.

Problem 5. Let $X_1, X_2, ..., X_n...$ be a sequence of random variables such that the partial sums $S_n = X_0 + X_1 + ... + X_n$, $n \geq 1$ determine a martingale w.r.t. $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$. Show that the summands are mutually uncorrelated, i.e.,

$$E(X_i X_j) = 0$$

for $i \neq j$.