Solutions to extra problems for Math67201-47201

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Problem 1.

Proof. Let $\tau^n = \{t_0^n = 0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = T\}$ be any partition of the interval $[0, T]$. We have

$$\begin{aligned}
& \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| \\
\leq & \sum_{i=0}^{k_n-1} |(f_1(t_{i+1}^n) - f_1(t_i^n))| + \sum_{i=0}^{k_n-1} |(f_2(t_{i+1}^n) - f_2(t_i^n))| \\
= & \sum_{i=0}^{k_n-1} (f_1(t_{i+1}^n) - f_1(t_i^n)) + \sum_{i=0}^{k_n-1} (f_2(t_{i+1}^n) - f_2(t_i^n)) \\
= & f_1(T) - f_1(0) + f_2(T) - f_2(0). \qquad (0.1)
\end{aligned}$$

Hence,

$$\sup_{\tau^n} \sum_{i=0}^{k_n-1} |f(t_{i+1}^n) - f(t_i^n)| \leq f_1(T) - f_1(0) + f_2(T) - f_2(0) < \infty.$$

This proves the statement.

Problem 2.

Proof. Let $\{0 = t_0 < t_1 < ... < t_{n_k-1} < t_{n_k} = t\}$ be a sequence of partitions of the interval $[0, t]$ such that $\Delta_k = max_{1 \leq n_k}(t_i - t_{i-1}) \to 0$ as $k \to \infty$. It suffices to prove

$$\lim_{k\to\infty} E[\left(\sum_{i=1}^{n_k} (B_{t_i} - B_{t_{i-1}})^2 - t\right)^2] = 0$$

Noting that $t = \sum_{i=1}^{n_k}(t_i - t_{i-1})$, we have

$$\sum_{i=1}^{n_k} (B_{t_i} - B_{t_{i-1}})^2 - t = \sum_{i=1}^{n_k} \{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\},$$

Hence,

$$
\begin{aligned}
& E[(\sum_{i=1}^{n_k}(B_{t_i}-B_{t_{i-1}})^2-t)^2] \\
= & \sum_{i=1,j=1}^{n_k} E[\{(B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\}\{(B_{t_j}-B_{t_{j-1}})^2-(t_j-t_{j-1})\}] \\
= & \sum_{i\neq j}^{n_k} E[\{(B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\}\{(B_{t_j}-B_{t_{j-1}})^2-(t_j-t_{j-1})\}] \\
& +\sum_{i=1}^{n_k} E[\{(B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\}^2] \qquad (0.2)
\end{aligned}
$$

If $i \neq j$, by the independence we have

$$
\begin{aligned}
& E[\{(B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\}\{(B_{t_j}-B_{t_{j-1}})^2-(t_j-t_{j-1})\}] \\
= & E[\{(B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\}]E[\{(B_{t_j}-B_{t_{j-1}})^2-(t_j-t_{j-1})\}] \\
& =0 \qquad (0.3)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& E[\{(B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\}^2] \\
= & E[(B_{t_i}-B_{t_{i-1}})^4]-2E[(B_{t_i}-B_{t_{i-1}})^2](t_i-t_{i-1})+(t_i-t_{i-1})^2 \\
= & E[(B_{t_i}-B_{t_{i-1}})^4]-(t_i-t_{i-1})^2 \\
\leq & C(t_i-t_{i-1})^2+(t_i-t_{i-1})^2. \qquad (0.4)
\end{aligned}
$$

Combining the above calculations together, we obtain

$$
\begin{aligned}
& E[(\sum_{i=1}^{n_k}(B_{t_i}-B_{t_{i-1}})^2-t)^2] \\
\leq & \sum_{i=1}^{n_k}(C+1)(t_i-t_{i-1})^2 \leq (C+1)t\max_i(t_i-t_{i-1}) \\
& \to 0 \qquad (0.5)
\end{aligned}
$$

as $k \to \infty$.

Problem 3.

Proof. Let $\{0=t_0<t_1<...<t_{n_k-1}<t_{n_k}=t\}$ be a sequence of partitions of the interval $[0,t]$ such that $\Delta_k = max_{1\leq n_k}(t_i-t_{i-1}) \to 0$ as $k\to\infty$. We need to show that

$$[B,A]_t = \lim_{k\to\infty}\sum_{i=1}^{n_k}(B_{t_i}-B_{t_{i-1}})(A_{t_i}-A_{t_{i-1}})=0.$$

By Holder inequality, we have

$$\begin{aligned} & \left(\sum_{i=1}^{n_k}(B_{t_i}-B_{t_{i-1}})(A_{t_i}-A_{t_{i-1}})\right)^2 \\ \leq & \ [\sum_{i=1}^{n_k}(B_{t_i}-B_{t_{i-1}})^2][\sum_{i=1}^{n_k}(A_{t_i}-A_{t_{i-1}})^2] \end{aligned} \tag{0.6}$$

We already know that

$$\lim_{k\to\infty}\sum_{i=1}^{n_k}(B_{t_i}-B_{t_{i-1}})^2 = t.$$

To complete the proof, it is sufficient to show that

$$\lim_{k\to\infty}\sum_{i=1}^{n_k}(A_{t_i}-A_{t_{i-1}})^2 = 0.$$

In deed, we have

$$\begin{aligned} & \sum_{i=1}^{n_k}(A_{t_i}-A_{t_{i-1}})^2 \leq \sup_i |A_{t_i}-A_{t_{i-1}}| \cdot \sum_{i=1}^{n_k}|A_{t_i}-A_{t_{i-1}}| \\ \leq & \ C_t \sup_i |A_{t_i}-A_{t_{i-1}}|, \end{aligned} \tag{0.7}$$

where C_t is some constant because A is of bounded variation. Since A_t is continuous in t and since $\Delta_k = max_{1\leq n_k}(t_i - t_{i-1}) \to 0$, it follows that $\sup_i |A_{t_i}-A_{t_{i-1}}| \to 0$ as $k\to\infty$. Hence we deduce that

$$\lim_{k\to\infty}\sum_{i=1}^{n_k}(A_{t_i}-A_{t_{i-1}})^2 = 0$$

which finishes the proof.

Problem 4.

Proof. (a). Since Z_t is a functional of $B_u, 0\leq u\leq t$, Z_t is $\mathcal{F}_t$-measurable.

(b). As $B_t \sim N(0,t)$, we have

$$E[|Z_t|] \leq E[|B_t|^3] + 3\int_0^t E[|B_u|]du < \infty.$$

(c). For $s < t$,

$$
\begin{aligned}
& E[Z_t|\mathcal{F}_s] = E[B_t^3 - 3\int_0^t B_u du|\mathcal{F}_s] \\
= \ & E[(B_t - B_s + B_s)^3 - 3\int_0^s B_u du - 3\int_s^t B_u du|\mathcal{F}_s] \\
= \ & E[(B_t - B_s)^3 + 3(B_t - B_s)^2 B_s + 3(B_t - B_s)B_s^2 + B_s^3|\mathcal{F}_s] \\
& -3\int_0^s B_u du - 3E[\int_s^t B_u du|\mathcal{F}_s] \\
= \ & E[(B_t - B_s)^3] + 3E[(B_t - B_s)^2]B_s + 3E[(B_t - B_s)]B_s^2 + B_s^3 \\
& -3\int_0^s B_u du - 3(t-s)B_s - 3E[\int_s^t (B_u - B_s)du|\mathcal{F}_s] \\
= \ & Z_s - 3E[\int_s^t (B_u - B_s)du] = Z_s, \qquad (0.8)
\end{aligned}
$$

where the property of independent increments of the Brownian motion was used. The proof is complete.