LARGE DEVIATIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS ON S^d ASSOCIATED WITH THE CRITICAL SOBOLEV BROWNIAN VECTOR FIELDS

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Abstract: In this paper, we obtain a large deviation principle for the stochastic differential equations on the sphere S^d associated with the critical Sobolev Brownian vector fields.

1. Introduction

The purpose of our paper is to prove a large deviation principle on the asymptotic behavior of the stochastic differential equations on the sphere S^d associated with a critical Sobolev Brownian vector field which was constructed by Shizan Fang and Tusheng Zhang [6].

Recall that Schilder theorem states that if B is the real Brownian motion and $C_0[0,1]$ the space of real continuous functions defined on $[0,1]$, null at 0, and endowed with the uniform norm, then for any open set $G \subset C_0[0,1]$ and closed set $F \subset C_0[0,1]$,

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log P(\varepsilon B \in G) \geq -\inf_{f\in G} I_0(f),$$

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log P(\varepsilon B \in F) \leq -\inf_{f\in F} I_0(f),$$

with

$$I_0(f) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{f}|^2 ds, & f \text{ absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

This result was then generalized by Freidlin and Wentzell in their famous paper [7] by considering the Ito equation

$$\begin{cases} dx_t^\varepsilon = & \varepsilon\sigma(x_t^\varepsilon)dW(t) + b(x_t^\varepsilon)dt, \\ x^\varepsilon(0) = & x. \end{cases}$$

They proved a large deviation principle for the above equation under usual Lipschitz conditions.

In this paper, we consider the critical Sobolev isotropic Brownian flows on the sphere S^d. It is defined by the following sde,

$$\begin{aligned} dx_t &= \sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(x_t)\circ dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(x_t)\circ dB^2_{\ell,k}(t)\Big\}, \\ x_0 &= x, \end{aligned}$$

where $A^i_{\ell,k}$ are eigenvector fields of Δ. Please refer to Section 2 for background and detailed description of the equation. We aim to prove a large deviation principle for solutions of small noise perturbations of above equation. The main new features of this equation are

(1) The coefficients are non-Lipschitz.

(2) Infinitely many driving Brownian motions are involved.

Because of the complex structure of the equation and particularly the non-Lipschtiz coefficients, existing results in literature do not apply. We hope that the techniques we used in this paper to treat the non-Lipschtiz coefficients are also useful in some other occasions. Our strategy is as follows.

We first work with the solution $x^{n,\varepsilon}$ of an equation (5.1) (below) driven by finitely many Brownian motions. This equation has smooth coefficients, so the large deviation principle for this equation is known. Next, we show that $x^{n,\varepsilon} \to x^\varepsilon$ exponentially fast, which together with the special relation of rate functions guaranties that the large deviation estimate of $x^{n,\varepsilon}$ can be transferred to x^ε, where x^ε is the solution of the small perturbed system (see (3.1)).

The rest of the paper is organized as follows:

In Section 2, we recall the critical Sobolev isotropic Brownian flows on the sphere S^d. In Section 3, we introduce the main result. Section 4 is devoted to the study of the rate function. The large deviation principle is proved in Section 5.

2. Framework

Let Δ be the Laplace operator on S^d, acting on vector fields. The spectrum of Δ is given by spectrum$(\Delta) = \{-c_{\ell,d};\ \ell \geq 1\} \cup \{-c_{\ell,\delta};\ \ell \geq 1\}$, where $c_{\ell,d} = \ell(\ell+d-1),\ c_{\ell,\delta} =$

$(\ell+1)(\ell+d-2)$. Let $\mathcal{G}_\ell$ be the eigenspace associated to $c_{\ell,d}$ and $\mathcal{D}_\ell$ the eigenspace associated to $c_{\ell,\delta}$. Their dimensions will be denoted by $D_{\ell,1} = \text{dim}\mathcal{G}_\ell, \quad D_{\ell,2} = \text{dim}\mathcal{D}_\ell$. It is known (see [9]) that

$$D_{\ell,1} = O(\ell^{d-1}), \quad D_{\ell,2} = O(\ell^{d-1}) \quad \text{as} \quad \ell \to +\infty. \tag{2.1}$$

Denote by $\{A^i_{\ell,k};\ k = 1, \cdots, D_{\ell,i},\ \ell \geq 1\}$ for $i = 1, 2$, the orthonormal basis of $\mathcal{G}_\ell$ and $\mathcal{D}_\ell$ in L^2, i.e,

$$\int_{S^d} \langle A^i_{\ell,k}(x), A^j_{\alpha,\beta}(x)\rangle\, dx = \delta_{ij}\delta_{\ell\alpha}\delta_{k\beta},$$

where δ_{ij} is the Kronecker symbol and dx is the normalized Riemannian measure on S^d, which is the unique one invariant by actions of $g \in \text{SO}(d+1)$. By Weyl theorem, the vector fields $\{A^i_{\ell,k}\}$ are smooth. For more detailed properties of the eigenvector fields, we refer the reader to the appendix at the end of this paper.

Let $s > 0$ and $H^s(S^d)$ be the Sobolev space of vector fields on S^d, which is the completion of smooth vector fields with respect to the norm

$$||V||^2_{H^s} = \int_{S^d} \langle (-\Delta + 1)^s V, V\rangle\, dx. \tag{2.2}$$

Then $\Big\{A^1_{\ell,k}/(1 + c_{\ell,d})^{s/2}, A^2_{\ell,\beta}/(1 + c_{\ell,\delta})^{s/2};\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,1}\,, 1 \leq \beta \leq D_{\ell,2}\Big\}$ is an orthonormal basis of H^s. If we consider

$$a_\ell = \frac{a}{(\ell-1)^{1+\alpha}},\ b_\ell = \frac{b}{(\ell-1)^{1+\alpha}}, \quad \alpha > 0,\ a, b > 0,\ \ell \geq 2, \tag{2.3}$$

then

$$\sqrt{\frac{a_\ell}{D_{\ell,1}}} = O(\frac{1}{\ell^{(\alpha+d)/2}}), \quad \sqrt{\frac{b_\ell}{D_{\ell,2}}} = O(\frac{1}{\ell^{(\alpha+d)/2}}). \tag{2.4}$$

Let $\{B^i_{\ell,k}(t);\ \ell \geq 1,\ 1 \leq k \leq D_{\ell,i}\}$ for $i = 1, 2$ be two family of independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the series

$$W_t(\omega) = \sum_{\ell \geq 1} \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} B^1_{\ell,k}(t) A^1_{\ell,k} + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} B^2_{\ell,k}(t) A^2_{\ell,k} \Big\} \tag{2.5}$$

which converges in L^2, uniformly with respect to t in any compact subset of $[0, +\infty[$. According to (2.4), $(W_t)_{t\geq 0}$ is a *cylindrical* Brownian motion in the Sobolev space $H^{(\alpha+d)/2}$. Moreover, W_t takes values in the space $H^s(S^d)$ for any $0 < s < \alpha/2$. By Sobolev embedding theorem, in order to ensure that W_t takes values in the space of C^2 vector fields, α must be *large* than $d+2$. In this later case, the classical Kunita's framework ([8]) can be applied to integrate the vector field W_t so that we obtain a flow of diffeomorphisms. For the case of small α, the notion of statistical solutions was introduced in [9] and the phenomenon of phase transition appears. It was also shown in [9] that the statistical solutions give rise to a flow of maps if $\alpha > 2$, and the solution is not a flow of maps if $0 < \alpha < 2$. The critical case $\alpha = 2$ was studied in [6]. Instead of introducing $(W_t)_{t\geq 0}$ as in (2.5), the authors in [6] consider first the stochastic differential equations on S^d

$$(2.6)\quad dx_t^n = \sum_{\ell=1}^{2^n}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(x_t^n)\circ dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(x_t^n)\circ dB^2_{\ell,k}(t)\Big\},\quad x_0^n = x.$$

Using the specific properties of eigenvector fields, it was proved that $x_t^n(x)$ converges uniformly in $(t,x) \in [0,T]\times S^d$ to a solution of the sde (2.7) below. We quote the following result from [6].

Theorem A[6] *Let $\alpha = 2$ in definition (2.3). Then the stochastic differential equation on S^d:*

$$(2.7)\quad dx_t = \sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(x_t)\circ dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(x_t)\circ dB^2_{\ell,k}(t)\Big\},\quad x_0 = x$$

admits a unique strong solution $(x_t(x))_{t\geq 0}$, which gives rise to a flow of homeomorphisms.

In the case of the circle S^1, this property of flows of homeomorphisms was discovered in [10].

3. Statement of The Result

Consider the small perturbation of (2.7),

$$(3.1)\quad dx_t^\varepsilon = \varepsilon\sum_{\ell=1}^{\infty}\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(x_t^\varepsilon)\circ dB^1_{\ell,k}(t) + \varepsilon\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(x_t^\varepsilon)\circ dB^2_{\ell,k}(t)\Big\},\quad x_0^\varepsilon = x.$$

(3.1) has a unique strong solution $(x_t^\varepsilon(x))_{t\geq 0}$ according to Theorem A, denoted by x_t^ε.

We consider the abstract Wiener space $(\Omega, \mathcal{H}, \mathcal{F}, P)$ associated with Wiener processes $W(s) = \{B^i_{\ell,k}(t); \ell \geq 1, 1 \leq k \leq D_{\ell,i}, i = 1, 2\}$. P is the Wiener measure and

$$\mathcal{H} = \{h = \{h^i_{\ell,k}(t)\}, \quad \ell \geq 1,\ 1 \leq k \leq D_{\ell,i}, i = 1, 2, \|h\|^2_{\mathcal{H}} < \infty\}$$

is the Comeron-Martin space associated with W, where

$$\|h\|^2_{\mathcal{H}} = \sum_{\ell \geq 1} \Big\{ \sum_{k=1}^{D_{\ell,1}} \int_0^T |\dot{h}^1_{\ell,k}(t)|^2 dt + \sum_{k=1}^{D_{\ell,2}} \int_0^T |\dot{h}^2_{\ell,k}(t)|^2 dt \Big\}.$$

The purpose of this paper is to prove a large deviation principle for the family $\{x^\varepsilon, \varepsilon > 0\}$ in the space $C_x([0,T], S^d)$, the collection of continuous functions f from $[0,T]$ into S^d with $f(0) = x$. To state the result, let us introduce the rate function. For any $h \in \mathcal{H}$, let $\{S^h(t), t \in [0,T]\}$ be the solution of the following equation

(3.3)
$$dS^h(t) = \sum_{\ell=1}^{\infty} \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(S^h(t))\, \dot{h}^1_{\ell,k}(t) dt + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(S^h(t)) \dot{h}^2_{\ell,k}(t) dt \Big\}, S^h(0) = x.$$

And for any $f \in C_x([0,T], S^d)$, let

$$I(f) = \inf\{\frac{1}{2}\|h\|^2_{\mathcal{H}} : f = S^h, h \in \mathcal{H}\}.$$

We recall the definition of the good rate function. Let E be a metric space.

Definition 3.1 A function I mapping E into $[0, \infty]$ is called a good rate function if for each $a < \infty$, the level set $\{f \in E : I(f) \leq a\}$ is compact.

Our main result reads as

Theorem 3.2 Let x^ε_t be the solution of (3.1) on $C_x([0,T], S^d)$, then $\{x^\varepsilon_t, \varepsilon > 0\}$ satisfies a large deviation principle with a good rate function $I(f)$, $f \in C_x([0,T], S^d)$,i.e.,

(i) For any closed subset $C \subset C_x([0,T], S^d)$,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log P(x^\varepsilon \in C) \leq - \inf_{f \in C} I(f).$$

(ii) For any open set $G \subset C_x([0,T], S^d)$,

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \log P(x^\varepsilon \in G) \geq - \inf_{f \in G} I(f).$$

4. Skeleton Equation And The Rate Function

Theorem 4.1 For any $h \in \mathcal{H}$, the equation (3.3) has a unique solution, denoted by $S^h(t)$.

We now introduce

$$G_n(\theta) = \sum_{\ell=1}^{2^n} \frac{\gamma_{\ell+1}(\cos\theta)}{\ell^3} = \sum_{\ell=1}^{2^n} \frac{\tilde{\gamma}_\ell(\theta)}{\ell^3},$$

$$G(\theta) = \sum_{\ell=1}^{\infty} \frac{\gamma_{\ell+1}(\cos\theta)}{\ell^3} = \sum_{\ell=1}^{\infty} \frac{\tilde{\gamma}_\ell(\theta)}{\ell^3},$$

where

$$\tilde{\gamma}_\ell(\theta) = \int_0^\pi (\cos\theta - \sqrt{-1}\sin\theta\cos\varphi)^\ell \sin^d\varphi \frac{d\varphi}{c_d}$$

with $c_d = \int_0^\pi sin^d\varphi d\varphi$.

We have $\gamma'_{\ell+1}(\cos\theta) = -\frac{\tilde{\gamma}'_\ell(\theta)}{\sin\theta}$. Define

$$\Xi_n(\theta) = \sum_{\ell=2^n+1}^{2^{n+1}} \frac{\tilde{\gamma}_\ell(\theta)}{\ell^3}.$$

Set

$$\begin{aligned} V_n(\theta) &= a\Big[(d-\sin^2\theta)G_n(\theta) + \cos\theta\sin\theta G'_n(\theta)\Big] \\ &+ b\Big[d\cos\theta G_n(\theta) + \sin\theta G'_n(\theta)\Big] - d(a+b)\,\cos\theta\, G_n(0) \\ &+ \frac{1}{2}\Big\{a\Big[(d-\sin^2\theta)\Xi_n(\theta) + \cos\theta\sin\theta\Xi'_n(\theta)\Big] \\ &+ b\Big[d\cos\theta\Xi_n(\theta) + \sin\theta\Xi'_n(\theta)\Big] - d(a+b)\,\cos\theta\,\Xi_n(0)\Big\}, \end{aligned}$$

and

$$\begin{aligned} U_n(\theta) &= 2\sin^2\theta\Big\{a\Big[G_n(0) - \cos\theta\, G_n(\theta) - \sin\theta\, G'_n(\theta)\Big] \\ &+ b\big(G_n(0) - G_n(\theta)\big)\Big\} + \sin^2\theta\,(a+b)\Xi_n(0). \end{aligned}$$

The following estimates are proved in [6].

Lemma 4.2[6] *Let*

$$\sigma_n(\theta) = -\frac{\sqrt{U_n(\theta)}}{\sin\theta},$$

$$B_n(\theta) = \frac{V_n(\theta)}{\sin\theta} + \frac{1}{2}\frac{\cos\theta}{\sin^3\theta}U_n(\theta).$$

There exist some constants $N > 0$, $C > 0$, such that for any $n > N$,

$$\sigma_n^2(\theta) \le C\theta^2 \log \frac{2\pi}{\theta} + 2^{-n},$$

$$-B_n(\theta) \le C\theta \log \frac{2\pi}{\theta} + 2^{-n}.$$

Proof of Theorem 4.1. Let $S^{n,h}$ be the solution of the following system:

(4.1)

$$dS^{n,h}(t) = \sum_{\ell=1}^{2^n} \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(S^{n,h}(t))\, \dot{h}^1_{\ell,k}(t)dt + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(S^{n,h}(t))\dot{h}^2_{\ell,k}(t)dt \Big\}, S^h(0) = x.$$

Since $A^i_{\ell,k}$ are smooth, the solution of (4.1) exists.

For $x, y \in S^d$, consider the Riemannian distance $d(x,y)$ defined by

$$\cos d(x,y) =< x, y >,$$

where $<,>$ denotes the inner product in R^{d+1}. Let $|\cdot|$ denote the Euclidean distance . We have the relation

$$|x-y| \le d(x,y) \le \frac{\pi}{2}|x-y|. \tag{4.2}$$

Our aim is to show that $S^{n,h}$ converges to a solution of equation (3.3). By the chain rule,

$$\begin{aligned} d\big\langle S^{n,h}(t), S^{n+1,h}(t)\big\rangle &= \big\langle dS^{n,h}(t), S^{n+1,h}(t)\big\rangle + \big\langle S^{n,h}(t), dS^{n+1,h}(t)\big\rangle \\ &= \sum_{\ell=1}^{2^n}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle S^{n+1,h}(t), A^1_{\ell,k}(S^{n,h}(t))\big\rangle \dot{h}^1_{\ell,k}(t) \\ &\quad + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle S^{n+1,h}(t), A^2_{\ell,k}(S^{n,h}(t))\big\rangle \dot{h}^2_{\ell,k}(t)\Big]dt \\ &\quad + \sum_{\ell=1}^{2^{n+1}}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle S^{n,h}(t), A^1_{\ell,k}(S^{n+1,h}(t))\big\rangle \dot{h}^1_{\ell,k}(t) \\ &\quad + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle S^{n,h}(t), A^2_{\ell,k}(S^{n+1,h}(t))\big\rangle \dot{h}^2_{\ell,k}(t)\Big]dt. \end{aligned}$$

Let $\theta^n_t = d(S^{n,h}(t), S^{n+1,h}(t))$, then

$$d\theta^n_t = -\frac{1}{\sin\theta^n_t} d\big\langle S^{n,h}(t), S^{n+1,h}(t)\big\rangle.$$

Let

$$
\begin{aligned}
I_1(t) &= -\int_0^t \frac{1}{\sin\theta_s^n}\sum_{\ell=1}^{2^n}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big(\big\langle S^{n+1,h}(s),A^1_{\ell,k}(S^{n,h}(s))\big\rangle+\big\langle S^{n,h}(s),A^1_{\ell,k}(S^{n+1,h}(s))\big\rangle\big)\dot h^1_{\ell,k}(s)\\
&\quad+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big(\big\langle S^{n+1,h}(s),A^2_{\ell,k}(S^{n,h}(s))\big\rangle+\big\langle S^{n,h}(s),A^2_{\ell,k}(S^{n+1,h}(s))\big\rangle\big)\dot h^2_{\ell,k}(s)\Big]ds,\\
I_2(t) &= -\int_0^t \frac{1}{\sin\theta_s^n}\sum_{\ell=2^n+1}^{2^{n+1}}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle S^{n,h}(s),A^1_{\ell,k}(S^{n+1,h}(s))\big\rangle\dot h^1_{\ell,k}(s)\\
&\quad+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle S^{n,h}(s),A^2_{\ell,k}(S^{n+1,h}(s))\big\rangle\dot h^2_{\ell,k}(s)\Big]ds.
\end{aligned}
$$

We have,

$$
\begin{aligned}
I_1^2(t) &\le \Big(\int_0^t \frac{1}{\sin\theta_s^n}\Big(\sum_{\ell=1}^{2^n}\Big[\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\big(\big\langle S^{n+1,h}(s),A^1_{\ell,k}(S^{n,h}(s))\big\rangle+\big\langle S^{n,h}(s),A^1_{\ell,k}(S^{n+1,h}(s))\big\rangle\big)^2\\
&\quad+\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\big(\big\langle S^{n+1,h}(s),A^2_{\ell,k}(S^{n,h}(s))\big\rangle+\big\langle S^{n,h}(s),A^2_{\ell,k}(S^{n+1,h}(s))\big\rangle\big)^2\Big]\Big)^{\frac12}\\
&\quad\times\Big(\sum_{\ell=1}^{2^n}\Big[\sum_{k=1}^{D_{\ell,1}}|\dot h^1_{\ell,k}(s)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot h^2_{\ell,k}(s)|^2\Big]\Big)^{\frac12}ds\Big)^2.
\end{aligned}
$$

Using Proposition A.4 and Lemma 4.2, we see that,

$$
\begin{aligned}
I_1^2(t) &\le \Big(\int_0^t\Big(2\sum_{\ell=1}^{2^n}\{a_\ell[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\sin^2\theta_s^n\gamma_\ell'(\cos\theta_s^n)]+b_\ell[1-\gamma_\ell(\cos\theta_s^n)]\}\Big)^{\frac12}\\
&\quad\times\Big(\sum_{\ell=1}^{2^n}\Big[\sum_{k=1}^{D_{\ell,1}}|\dot h^1_{\ell,k}(s)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot h^2_{\ell,k}(s)|^2\Big]\Big)^{\frac12}ds\Big)^2\\
&\le 2\int_0^t\Big(\sum_{\ell=1}^{2^n}\{a_\ell[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\gamma_\ell'(\cos\theta_s^n)]+b_\ell[1-\gamma_\ell(\cos\theta_s^n)]\}\Big)ds\\
&\quad\times\int_0^t\sum_{\ell=1}^{2^n}\Big[\sum_{k=1}^{D_{\ell,1}}|\dot h^1_{\ell,k}(s)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot h^2_{\ell,k}(s)|^2\Big]ds\\
&\le 2\|h\|^2_{\mathcal H}\int_0^t\Big(\sum_{\ell=1}^{2^n}\{a_\ell[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\gamma_\ell'(\cos\theta_s^n)]+b_\ell[1-\gamma_\ell(\cos\theta_s^n)]\}\Big)ds.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned} I_2^2(t) &\le \Big(\int_0^t \frac{1}{\sin\theta_s^n}\Big(\sum_{\ell=2^n+1}^{2^{n+1}}\Big[\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\big\langle S^{n,h}(s), A^1_{\ell,k}(S^{n+1,h}(s))\big\rangle^2 \\ &\quad +\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\big\langle S^{n,h}(s), A^2_{\ell,k}(S^{n+1,h}(s))\big\rangle^2\Big]\Big)^{\frac12} \\ &\quad \times\Big(\sum_{\ell=2^n+1}^{2^{n+1}}\Big[\sum_{k=1}^{D_{\ell,1}}|\dot h^1_{\ell,k}(s)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot h^2_{\ell,k}(s)|^2\Big]\Big)^{\frac12}ds\Big)^2 \\ &\le 2\|h\|^2_{\mathcal H}\int_0^t\sum_{\ell=2^n}^{2^{n+1}}(a_\ell+b_\ell)ds. \end{aligned}$$

Therefore,

$$\begin{aligned} |\theta_t^n|^2 &\le 2I_1^2(t)+2I_2^2(t) \\ &\le 4\|h\|^2_{\mathcal H}\Big[\int_0^t\sum_{\ell=1}^{2^n}\{a_\ell[1-\cos\theta_s^n\gamma_\ell(\cos\theta_s^n)+\gamma_\ell'(\cos\theta_s^n)] \\ &\quad +b_\ell[1-\gamma_\ell(\cos\theta_s^n)]\}ds+\int_0^t\sum_{\ell=2^n+1}^{2^{n+1}}(a_\ell+b_\ell)ds\Big] \\ &\le 4\int_0^T\sigma_n^2(\theta_s^n)ds \\ &\le 4\int_0^t\Big((\theta_s^n)^2\log\frac{2\pi}{\theta_s^n}+2^{-n}\Big)ds. \end{aligned}$$

Using the similar arguments as that in [6], the above inequality implies that there exist constants C_1, C_2 such that

$$|\theta_t^n|^2\le C_1 2^{-ne^{-C_2t}}$$

and C_1, C_2 are independent of n, t. Hence,

$$|S^{n,h}(t)-S^{n+1,h}(t)|\le|\theta_t^n|\le C_1 2^{-ne^{-C_2t}}.$$

Thus, $S^{n,h}(t)$ uniformly converges to some function S^h in $C_x([0,T],S^d)$.

Next, we show that $\{S^h(t), t\ge 0\}$ satisfies the equation (3.3).

It suffices to show that for any $u\in S^d$,

$$\begin{aligned} d\langle u, S^h(t)\rangle &= \sum_{\ell=1}^\infty\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\langle u, A^1_{\ell,k}(S^h(t))\rangle\,\dot h^1_{\ell,k}(t)dt \\ &\quad +\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\langle u, A^2_{\ell,k}(S^h(t))\rangle\dot h^2_{\ell,k}(t)dt\Big\}. \end{aligned}$$

Set $\eta_t = \langle u, S^h(t)\rangle$, $\eta_t^n = \langle u, S^{n,h}(t)\rangle$, $\theta_t^n = d(u, S^{n,h}(t))$ and $\theta_t = d(u, S^h(t))$. Fix $N_0 > 0$, by Proposition A.4 and Lemma 4.2,

$$
\begin{aligned}
& |\sum_{\ell=N_0}^{2^n}\int_0^t\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(S^{n,h}(s))\big\rangle\,\dot h^1_{\ell,k}(s)ds + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(S^{n,h}(s))\big\rangle\dot h^2_{\ell,k}(s)\}ds\Big|^2 \\
\le\ & \int_0^t\sum_{\ell=N_0}^{2^n}(\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(S^{n,h}(s))\big\rangle^2 + \frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(S^{n,h}(s))\big\rangle^2)ds \\
& \times\int_0^t\sum_{\ell=N_0}^{2^n}(\sum_{k=1}^{D_{\ell,1}}|\dot h^1_{\ell,k}(s)|^2 + \sum_{k=1}^{D_{\ell,2}}|\dot h^2_{\ell,k}(s)|^2)ds \\
=\ & \int_0^t\sum_{\ell=N_0}^{2^n}(a_\ell + b_\ell)\sin^2\theta_s^n ds \times \int_0^t\sum_{\ell=N_0}^{2^n}(\sum_{k=1}^{D_{\ell,1}}|\dot h^1_{\ell,k}(s)|^2 + \sum_{k=1}^{D_{\ell,2}}|\dot h^2_{\ell,k}(s)|^2 ds \\
\le\ & t\|h\|^2_{\mathcal H}\sum_{N_0}^{2^n}(a_\ell + b_\ell) \le t\|h\|^2_{\mathcal H}\sum_{N_0}^{\infty}(a_\ell + b_\ell).
\end{aligned}
$$

Thus, for any $\varepsilon > 0$, there exists $N_0 > 0$,

$$
\begin{aligned}
& |\sum_{\ell=N_0}^{2^n}\int_0^t\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(S^{n,h}(s))\big\rangle\,\dot h^1_{\ell,k}(s)ds \\
& + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(S^{n,h}(s))\big\rangle\dot h^2_{\ell,k}(s)\Big\}ds| < \frac{\varepsilon}{2}.
\end{aligned}
$$

for $2^n > N_0$. By similar reasons, we also have

$$
|\sum_{\ell=N_0}^{\infty}\int_0^t\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(S^{h}(s))\big\rangle\,\dot h^1_{\ell,k}(s)ds + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(S^{h}(s))\big\rangle\dot h^2_{\ell,k}(s)\Big\}ds|
$$

$$
\le t\|h\|^2_{\mathcal H}\sum_{N_0}^{\infty}(a_\ell + b_\ell) < \frac{\varepsilon}{2}.
$$

On the other hand, because $S^{n,h} \Rightarrow S^h$ in $C_x([0,T], S^d)$, for any $\varepsilon > 0$, one can find $N_1 > 0$ such that for $n > N_1$,

$$
\begin{aligned}
& |\sum_{\ell=1}^{N_0-1}\int_0^t\Big\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}\big\langle u, A^1_{\ell,k}(S^{n,h}(s)) - A^1_{\ell,k}(S^h(s))\big\rangle\,\dot h^1_{\ell,k}(s)ds \\
& + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}\big\langle u, A^2_{\ell,k}(S^{n,h}(s)) - A^2_{\ell,k}(S^h(s))\big\rangle\dot h^2_{\ell,k}(s)\}ds\Big| < \frac{\varepsilon}{2}.
\end{aligned}
$$

Therefore, for any $\varepsilon > 0$, one can find $N_2 > 0$ such that for $n > N_2$,

$$
\begin{aligned}
& | \sum_{\ell=1}^{N_0-1} \int_0^t \{ \frac{da_\ell}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \langle u, A^1_{\ell,k}(S^{n,h}(s)) - A^1_{\ell,k}(S^h(s)) \rangle \dot{h}^1_{\ell,k}(s) ds \\
& + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} \langle u, A^2_{\ell,k}(S^{n,h}(s)) - A^2_{\ell,k}(S^h(s)) \rangle \dot{h}^2_{\ell,k}(s) \} ds | \\
& + | \sum_{\ell=N_0}^{2^n} \int_0^t \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} \langle u, A^1_{\ell,k}(S^{n,h}(s)) \rangle \, \dot{h}^1_{\ell,k}(s) ds \\
& + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} \langle u, A^2_{\ell,k}(S^{n,h}(s)) \rangle \dot{h}^2_{\ell,k}(s) \Big\} ds | \\
& + | \sum_{\ell=N_0}^{\infty} \int_0^t \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} \langle u, A^1_{\ell,k}(S^{h}(s)) \rangle \, \dot{h}^1_{\ell,k}(s) ds \\
& + \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} \langle u, A^2_{\ell,k}(S^{h}(s)) \rangle \dot{h}^2_{\ell,k}(s) \Big\} ds |. \\
& < \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we obtain that

$$
\begin{aligned}
d\langle u, S^h(t) \rangle &= \sum_{\ell=1}^{\infty} \Big\{ \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} \langle u, A^1_{\ell,k}(S^h(t)) \rangle \, \dot{h}^1_{\ell,k}(t) dt \\
&+ \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} \langle u, A^2_{\ell,k}(S^h(t)) \rangle \dot{h}^2_{\ell,k}(t) dt \Big\}.
\end{aligned}
$$

The uniqueness is deduced from similar estimates.

Lemma 4.3 *For any $N > 0$, the set $\{S^h : \|h\|_{\mathcal{H}} \le N\}$ is relatively compact in $C_x([0,T], S^d)$.*

Proof. By the Ascoli-Arzela lemma, we need to show that $\{S^h : \|h\|_{\mathcal{H}} \le N\}$ is uniformly bounded and equi-continuous. The first fact is obvious because $||S^h|| = 1$ for any $h \in \mathcal{H}$. Next we will show that $\{S^h : \|h\|_{\mathcal{H}} \le N\}$ is equi-continuous.

Let $\{u_i, i=1,\cdots,d+1\}$ be an orthonormal basis of R^{d+1}, by Proposition A.4 and Lemma 4.2,

$$\begin{aligned}
|<S^h(t)-S^h(s),u_i>|^2 &= |\int_s^t \sum_{\ell=1}^{\infty}[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}<A^1_{\ell,k}(S^h(u)),u_i>\ \dot{h}^1_{\ell,k}(u)du \\
&\quad +\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}<A^2_{\ell,k}(S^h(u)),u_i>\dot{h}^2_{\ell,k}(u)]du|^2 \\
&\le \int_s^t \sum_{\ell=1}^{\infty}(\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}\langle A^1_{\ell,k}(S^h(u)),u_i\rangle^2+\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}\langle A^2_{\ell,k}(S^h(u)),u_i\rangle^2)du \\
&\quad \times\int_s^t \sum_{\ell=1}^{\infty}(\sum_{k=1}^{D_{\ell,1}}|\dot{h}^1_{\ell,k}(u)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot{h}^2_{\ell,k}(u)|^2)du \\
&= \int_s^t \sum_{\ell=1}^{\infty}(a_\ell+b_\ell)\sin^2\theta_u du\times\int_s^t \sum_{\ell=1}^{\infty}(\sum_{k=1}^{D_{\ell,1}}|\dot{h}^1_{\ell,k}(u)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot{h}^2_{\ell,k}(u)|^2du \\
&\le \sum_{\ell=1}^{\infty}(a_\ell+b_\ell)\|h\|^2_{\mathcal{H}}|t-s|,
\end{aligned}$$

where $\theta_t = d(S^h(t), u_i)$. Thus,

$$|S^h(t)-S^h(s)|^2=\sum_{i=1}^{d+1}|<S^h(t)-S^h(s),u_i>|^2\le(d+1)\sum_{\ell=1}^{\infty}(a_\ell+b_\ell)\|h\|^2_{\mathcal{H}}|t-s|,$$

which finishes the proof.

Lemma 4.4 *The mapping $h\to S^h$ is continuous from $\{h,\|h\|_{\mathcal{H}}\le N\}$ with respect to the topology on Ω into $C_x([0,T],S^d)$.*

Proof. Let $h_n\in\mathcal{H}$ with $\|h_n\|_{\mathcal{H}}\le N$ and assume that h_n converges to h in Ω, then $h_n\to h$ weakly in $\mathcal{H}$. By lemma 4.2, $\{S^{h_n}, n\ge 1\}$ is relatively compact. Let $g\in C_x([0,T],S^d)$ be a limit of any convergent subsequence of $\{S^{h_{n_k}}, n_k\ge 1\}$. We shall finish the proof the lemma by showing that $g=S^h$. Now, for simplicity, we drop the subindex k.

$$S^{h_n}(t)=x+\int_0^t\sum_{\ell=1}^{\infty}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}(S^{h_n}(u))\,\dot{h}^{\,1}_{n,\ell,k}(u)+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}(S^{h_n}(u)),\dot{h}^{\,2}_{n\ell,k}(u)\Big]du,$$

$$S^h(t)=x+\int_0^t\sum_{\ell=1}^{\infty}\Big[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}(S^h(u))\,\dot{h}^1_{\ell,k}(u)\ +\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}(S^h(u))\dot{h}^2_{\ell,k}(u)\Big]du.$$

It is sufficient to show that $S^{h_n}\Rightarrow S^h$ in $C_x([0,T],S^d)$.

Write $S^{h_n}(t)-S^h(t)=I_3-I_4$ with I_3,I_4 being given by:

$$\begin{aligned}
I_3 &= \int_0^t \sum_{\ell=1}^{\infty}[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}(A^1_{\ell,k}(S^{h_n}(u))-A^1_{\ell,k}(S^h(u)))\dot{h}^1_{n\ell,k}(u)\\
&\quad+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}(A^2_{\ell,k}(S^{h_n}(u))-A^2_{\ell,k}(S^h(u)))\dot{h}^2_{n\ell,k}(u)]du,\\
I_4 &= \int_0^t \sum_{\ell=1}^{\infty}[\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}}A^1_{\ell,k}(S^h(u))\,(\dot{h}^1_{n\ell,k}(u)-\dot{h}^1_{\ell,k}(u))\\
&\quad+\sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}}A^2_{\ell,k}(S^h(u))(\dot{h}^2_{n\ell,k}(u)-\dot{h}^2_{\ell,k}(u))]du.
\end{aligned}$$

Let $\theta_t=d(S^h(t),S^{h_n}(t))$, by Proposition A.4 and Lemma 4.2,

$$\begin{aligned}
I_3 &\le \int_0^t [\sum_{\ell=1}^{\infty}(\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}|A^1_{\ell,k}(S^{h_n}(u))-A^1_{\ell,k}(S^h(u))|^2+\frac{db_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}|A^2_{\ell,k}(S^{h_n}(u))-A^2_{\ell,k}(S^h(u))|^2\big)]^{\frac12}\\
&\quad\times[\sum_{\ell=1}^{\infty}(\sum_{k=1}^{D_{\ell,1}}|\dot{h}^1_{n\ell,k}(u)|^2+\sum_{k=1}^{D_{\ell,2}}|\dot{h}^2_{n\ell,k}(u)|^2)]^{\frac12}du\\
&\le \|h_n\|_{\mathcal{H}}\Big(\int_0^t\sum_{\ell=1}^{\infty}[2da_\ell-2a_\ell(d-1+\cos\theta_u)\gamma_\ell(\cos\theta_u)-\cos\theta_u\sin^2\theta_u\gamma'_\ell(\cos\theta_u)\\
&\quad+2db_\ell-2b_\ell(d\cos\theta_u\gamma_\ell(\cos\theta_u)-\sin^2\theta_u\gamma'_\ell(\cos\theta_u))]du\Big)^{\frac12}\\
&\le \|h_n\|_{\mathcal{H}}\Big(\int_0^t 2[daG(0)-a((d-1+\cos^2\theta_u)G(\theta_u)+\cos\theta_u\sin\theta_uG'(\theta))\\
&\quad+dbG(0)-b(d\cos\theta_uG(\theta_u)+\sin\theta_uG'(\theta_u))]du\Big)^{\frac12}\\
&\le (\int_0^t C\theta_u^2\log\frac{2\pi}{\theta_u}du)^{\frac12}.
\end{aligned}$$

Let

$$\begin{aligned}
f^1_{\ell,k}(v) &= \int_0^v\sqrt{\frac{da_\ell}{D_{\ell,1}}}A^1_{\ell,k}(S^h(u))I_{[0,t]}(u)du,\\
f^2_{\ell,k}(v) &= \int_0^v\sqrt{\frac{db_\ell}{D_{\ell,2}}}A^2_{\ell,k}(S^h(u))I_{[0,t]}(u)du,
\end{aligned}$$

then $f=((f^1_{\ell,k}(v))_{\ell\ge1,1\le k\le D_{\ell,1}},(f^2_{\ell,k}(v))_{\ell\ge1,1\le k\le D_{\ell,2}})\in\mathcal{H}$, because of

$$\begin{aligned}
\|f\|_{\mathcal{H}} &= \sum_{\ell=1}^{\infty}[\int_0^t\frac{da_\ell}{D_{\ell,1}}\sum_{k=1}^{D_{\ell,1}}|A^1_{\ell,k}(S^h(u))|^2du+\int_0^t\frac{da_\ell}{D_{\ell,2}}\sum_{k=1}^{D_{\ell,2}}|A^2_{\ell,k}(S^h(u))|^2du]\\
&= \sum_{\ell=1}^{\infty}\int_0^t(a_\ell+b_\ell)du<\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned} I_4 =< f, h_n - h >_{\mathcal{H}} &= \int_0^t \sum_{\ell=1}^{\infty} \sqrt{\frac{da_\ell}{D_{\ell,1}}} \sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(S^h(u))(\dot{h^1_{n\ell,k}}(u) - \dot{h}^1_{\ell,k}(u))du \\ &+ \sqrt{\frac{db_\ell}{D_{\ell,2}}} \sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(S^h(u))(\dot{h^2_{n\ell,k}}(u) - \dot{h}^2_{\ell,k}(u))]du \\ &\longrightarrow 0 \quad as \quad n \to \infty. \end{aligned}$$

Combining above estimates,

$$\theta_t = d(S^h(t), S^{h_n}(t)) \le \frac{\pi}{2}[(\int_0^t C\theta_u^2 \log\frac{2\pi}{\theta_u}du)^{\frac{1}{2}} + | < f, h_n - h >_{\mathcal{H}} |].$$

Hence

$$\theta_t^2 \le 2C\int_0^t \theta_u^2 \log\frac{2\pi}{\theta_u}du + 2| < f, h_n - h >_{\mathcal{H}} |^2.$$

This implies

$$\theta_t \le C_4| < f, h_n - h >_{\mathcal{H}} |^{e^{-C_5 t}},$$

which yields

$$S^{h_n} \Rightarrow S^h \quad as \quad n \Rightarrow \infty.$$

Lemma 4.5 *I(f) is a good rate function.*

Proof. For any $a > 0$,

$$\{I(f) \le a\} = \{S^h, \frac{1}{2}\|h\|^2_{\mathcal{H}} \le a\} = S^h(\|h\|_{\mathcal{H}} \le \sqrt{2a}).$$

The subset $\{\|h\|_{\mathcal{H}} \le \sqrt{2a}\}$ is a compact set in Ω and $h \to S^h$ is a continuous map for any a. Therefore $\{I(f) \le a\}$ is a compact set for any a. So $I(f)$ is a good rate function.

5. The Proof Of Theorem 3.2

Let $x_t^{n,\varepsilon}$ be the solution to

(5.1)

$$dx_t^{n,\varepsilon} = \varepsilon\sum_{\ell=1}^{2^n}\{\sqrt{\frac{da_\ell}{D_{\ell,1}}}\sum_{k=1}^{D_{\ell,1}} A^1_{\ell,k}(x_t^{n,\varepsilon}) \circ dB^1_{\ell,k}(t) + \sqrt{\frac{db_\ell}{D_{\ell,2}}}\sum_{k=1}^{D_{\ell,2}} A^2_{\ell,k}(x_t^{n,\varepsilon}) \circ dB^2_{\ell,k}(t)\}, \quad x_0^{n,\varepsilon} = x.$$

We first have the following proposition.

Proposition 5.1 For any $\delta > 0$,

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2 \log P(\sup_{0\le t\le T}|x_t^\varepsilon - x_t^{n,\varepsilon}| > \delta) = -\infty. \tag{5.2}$$

Proof. Let $\theta_n^\varepsilon(t) = d(x_t^{n,\varepsilon}, x_t^\varepsilon)$. Using the similar estimates as that in [6], there exists a real-valued Brownian motion $W_n(t)$ such that

$$d\theta_n^\varepsilon(t) = -\varepsilon\sigma_n(t)dW_n(t) - \varepsilon^2 B_n(t)dt,$$

where $\sigma_n(t) = \sigma_n(\theta_n^\varepsilon(t))$, $B_n(t) = B_n(\theta_n^\varepsilon(t))$ are defined as in Lemma 4.2.

Let $\xi_n(t) = (\theta_n^\varepsilon)^2(t)$, we have

$$\begin{aligned} d\xi_n(t) &= 2\theta_n^\varepsilon(t)d\theta_n^\varepsilon(t) + d\theta_n^\varepsilon(t)d\theta_n^\varepsilon(t) \\ &= -2\varepsilon\theta_n^\varepsilon(t)\sigma_n(\theta_n^\varepsilon(t))dW_n(t) + \varepsilon^2(\sigma_n^2(\theta_n^\varepsilon(t)) - 2\theta_n^\varepsilon(t)B_n(\theta_n^\varepsilon(t)))dt, \end{aligned}$$

and

$$d\xi_n(t)d\xi_n(t) = 4\varepsilon^2\theta_n^\varepsilon(t)^2\sigma_n^2(\theta_n^\varepsilon(t))dt.$$

Introduce the function $\psi_\rho : (0, e^{-1}) \to R$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s\log\frac{2\pi}{s} + \rho}.$$

Then for any $0 < \xi < 1$,

$$\psi_\rho(\xi) \uparrow \psi_0(\xi) = \int_0^\xi \frac{ds}{s\log\frac{2\pi}{s}} = +\infty,$$

as $\rho \to 0$.
Define for $\lambda > 0$,

$$\Phi_{\rho,\lambda}(\xi) = e^{\lambda\psi_\rho(\xi)}.$$

We have

$$\Phi'_{\rho,\lambda}(\xi)(\xi\log\frac{2\pi}{\xi} + \rho) = \lambda\Phi_{\rho,\lambda}(\xi),$$

and

$$\begin{aligned} \Phi''_{\rho,\lambda}(\xi) &= \lambda^2\Phi_{\rho,\lambda}(\xi)\frac{1}{\xi\log\frac{2\pi}{\xi} + \rho} + \lambda\Phi_{\rho,\lambda}(\xi)\frac{1 - \log\frac{2\pi}{\xi}}{(\xi\log\frac{2\pi}{\xi} + \rho)^2} \\ &\le \lambda^2\Phi_{\rho,\lambda}(\xi)\frac{1}{(\xi\log\frac{2\pi}{\xi} + \rho)^2}, \qquad \text{if } \xi \le e^{-1}. \end{aligned}$$

Without loss of generality, we may assume $\delta < e^{-1}$. Define $\tau_n = \inf\{t \geq 0, \theta_n^\varepsilon(t) > \delta\}$. By Ito formula,

$$\begin{aligned}
\Phi_{\rho,\lambda}(\xi_n(t\wedge\tau_n)) &= 1 + \int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))d\xi_n(s) + \frac{1}{2}\int_0^{t\wedge\tau_n} \Phi''_{\rho,\lambda}(\xi_n(s))d\xi_n(s)d\xi_n(s) \\
&= 1 + 2\varepsilon\int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))(-\theta_n^\varepsilon(s)\sigma_n(\theta_n^\varepsilon(s)))dW_n(s) \\
&\quad +\varepsilon^2\int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))(\sigma_n^2(\theta_n^\varepsilon(s)))ds \\
&\quad -2\varepsilon^2\int_0^{t\wedge\tau_n} \Phi'_{\rho,\lambda}(\xi_n(s))(\theta_n^\varepsilon(s)B_n(\theta_n^\varepsilon(s)))ds \\
&\quad +2\varepsilon^2\int_0^{t\wedge\tau_n} \Phi''_{\rho,\lambda}(\xi_n(s))(\theta_n^\varepsilon(s)^2\sigma_n^2(\theta_n^\varepsilon(s)))ds. \qquad (5.3)
\end{aligned}$$

Using Lemma 4.2, $\exists N$, such that $n \geq N$,

$$\frac{\lambda\sigma_n^2(\theta_n^\varepsilon(s))}{\xi_n(s)\log\frac{2\pi}{\xi_n(s)}+\rho} \leq \frac{\lambda C(\theta_n^\varepsilon(s)^2\log\frac{2\pi}{\theta_n^\varepsilon(s)}+2^{-n})}{(\theta_n^\varepsilon(s))^2\log\frac{2\pi}{\theta_n^\varepsilon(s)^2}+\rho} \leq \lambda C_1,$$

$$\frac{\lambda\theta_n^\varepsilon(s)(-B_n(\theta_n^\varepsilon(s)))}{\xi_n(s)\log\frac{2\pi}{\xi_n(s)}+\rho} \leq \frac{\lambda C(\theta_n^\varepsilon(s)^2\log\frac{2\pi}{\theta_n^\varepsilon(s)}+2^{-n})}{(\theta_n^\varepsilon(s))^2\log\frac{2\pi}{\theta_n^\varepsilon(s)^2}+\rho} \leq \lambda C_2,$$

$$\frac{2\lambda^2\theta_n^\varepsilon(s)^2\sigma_n^2(\theta_n^\varepsilon(s))}{(\xi_n\log\frac{2\pi}{\xi_n}+\rho)^2} \leq \frac{2\lambda^2\theta_n^\varepsilon(s)^2(\theta_n^\varepsilon(s)^2\log\frac{2\pi}{\theta_n^\varepsilon(s)}+2^{-n})}{(\theta_n^\varepsilon(s)^2\log\frac{2\pi}{\theta_n^\varepsilon(s)^2}+\rho)^2} \leq 2\lambda^2 C_3.$$

Therefore, it follows from (5.3) that

$$E[\Phi_{\rho,\lambda}(\xi_n(t\wedge\tau_n))] \leq 1 + \varepsilon^2 C_4(\lambda^2+\lambda)E\int_0^t \Phi_{\rho,\lambda}(\xi_n(s\wedge\tau_n))ds,$$

which implies that

$$E[\Phi_{\rho,\lambda}(\xi_n(t\wedge\tau_n))] \leq E^{C_4(\lambda^2+\lambda)\varepsilon^2 t}.$$

Since

$$E[\Phi_{\rho,\lambda}(\xi_n(1\wedge\tau_n))] \geq E[\Phi_{\rho,\lambda}(\xi_n(1\wedge\tau_n)), \tau_n \leq 1] = e^{\lambda\psi_\rho(\delta^2)}P(\tau_n \leq 1),$$

we have

$$P(\sup_{0\leq t\leq 1}\theta_n^\varepsilon(t) > \delta) = P(\tau_n \leq 1) \leq e^{-\lambda\psi_\rho(\delta^2)}e^{C(\lambda^2+\lambda)\varepsilon^2}.$$

Taking $\lambda = \frac{1}{\varepsilon^2}$, we obtain that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\varepsilon^2\log P(\sup_{0\leq t\leq 1}\theta_n^\varepsilon(t) > \delta) \leq -\psi_\rho(\delta^2) + C.$$

Let $\rho \to 0$ to get (5.2). The proof is complete.

Define

$$I_n(f) = \inf\{\frac{1}{2}\|h^n\|_{\mathcal{H}}^2, S^{n,h}(t) = f\}, \qquad (5.4)$$

where

$$h^n = ((h^1_{\ell,k})_{1\le \ell\le 2^n, 1\le k\le D_{\ell,1}}, (h^2_{\ell,k})_{1\le \ell\le 2^n, 1\le k\le D_{\ell,2}}), \tag{5.5}$$

$$\|h^n\|^2_{\mathcal{H}} = \sum_{\ell=1}^{2^n}\Big\{\sum_{k=1}^{D_{\ell,1}}\int_0^T |\dot h^1_{\ell,k}(t)|^2dt + \sum_{k=1}^{D_{\ell,2}}\int_0^T |\dot h^2_{\ell,k}(t)|^2dt\Big\}. \tag{5.6}$$

It is obvious that

$$I_n(f) \ge I(f).$$

Proof of Theorem 3.2 For any closed subset $C \subset C_x([0,T], S^d)$ and $\delta > 0$,

$$\begin{aligned} P(x^\varepsilon \in C) &\le P(\|x^\varepsilon - x^{n,\varepsilon}\| \le \delta, x^\varepsilon \in C) + P(\|x^\varepsilon - x^{n,\varepsilon}\| \ge \delta, x^\varepsilon \in C) \\ &\le P(x^{n,\varepsilon} \in C_\delta) + P(\|x^\varepsilon - x^{n,\varepsilon}\| \ge \delta), \end{aligned}$$

where

$$C_\delta = \{f \in C_x([0,T], S^d), \inf_{g\in C} \|f - g\| \le \delta\}.$$

Therefore,

$$\begin{aligned} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(x^\varepsilon \in C) &\le \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(x^{n,\varepsilon} \in C_\delta) \vee \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(\|x^\varepsilon - x^{n,\varepsilon}\| > \delta) \\ &\le (-\inf_{f\in C_\delta} I_n(f) \vee \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(\|x^\varepsilon - x^{n,\varepsilon}\| > \delta)) \\ &\le (-\inf_{f\in C_\delta} I(f)) \vee \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(\|x^\varepsilon - x^{n,\varepsilon}\| > \delta). \end{aligned}$$

Let $n \to \infty$ to get

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log P(x^\varepsilon \in C) \le -\inf_{f\in C_\delta} I(f) \longrightarrow -\inf_{f\in C} I(f) \quad as \quad \delta \longrightarrow 0,$$

which gives the upper bound of Theorem 3.2(i).

Let $G \subset C_x([0,T], S^d)$ be an open subset. Take $f \in G$ with $I(f) < \infty$. Then there exists $h \in \mathcal{H}$ such that

$$f = S^h, \quad I(f) = \frac{1}{2}\|h\|_{\mathcal{H}}.$$

Let $f^n = S^{n,h^n}$, h^n be defined as (5.5). Then $f^n \Rightarrow f \quad as \quad n \to \infty$, and also $I_n(f^n) \le \frac{1}{2}\|h^n\|_{\mathcal{H}}$. Choose $\delta > 0$ such that $B_f(2\delta) = \{g \in C_x([0,T], S^d), \|f - g\| \le 2\delta\} \subset G$. Then, there exists $N > 0$ such that for $n > N$,

$$\|f^n - f\| < \delta, \quad B_{f^n}(\delta) \subset G.$$

Therefore,

$$\begin{aligned}P(x^{n,\varepsilon}\in B_{f^n}(\delta)) &\le P(\|x^{n,\varepsilon}-x^\varepsilon\|<\delta, x^{n,\varepsilon}\in B_{f^n}(\delta))+P(\|x^{n,\varepsilon}-x^\varepsilon\|\ge\delta, x^{n,\varepsilon}\in B_{f^n}(\delta))\\ &\le P(x^\varepsilon\in B_f(2\delta))+P(\|x^{n,\delta}-x^\varepsilon\|\ge\delta)\\ &\le P(x^\varepsilon\in G)+P(\|x^{n,\varepsilon}-x^\varepsilon\|\ge\delta).\end{aligned}$$

Thus,

$$\begin{aligned}-\frac{1}{2}\|h^n\|^2_{\mathcal{H}}\le -I_n(f^n) &\le \liminf_{\varepsilon\to0}\varepsilon^2\log P(x^{n,\varepsilon}\in B_{f^n}(\delta))\\ &\le \liminf_{\varepsilon\to0}\varepsilon^2\log P(x^\varepsilon\in G)\vee\limsup_{\varepsilon\to0}\varepsilon^2\log P(\|x^{n,\varepsilon}-x^\varepsilon\|\ge\delta).\end{aligned}$$

Let $n\to\infty$ to obtain

$$\liminf_{\varepsilon\to0}\varepsilon^2\log P(x^\varepsilon\in G)\ge -\frac{1}{2}\|h\|_{\mathcal{H}}=-I(f).$$

Because f is arbitrary,

$$\liminf_{\varepsilon\to0}\varepsilon^2\log P(x^\varepsilon\in G)\ge -\inf_{f\in G}I(f).$$

This completes the proof of theorem 3.2.

6. Appendix: Eigen-vector fields

In this section, we shall collect some notations and useful properties of the eigen vector fields of Δ. We refer readers to [6] [12] for proofs. Fix the point $P_o=(0,\cdots,0,1)\in S^d$. The group $\mathrm{SO}(d{+}1)$ acts transitively on S^d. The subgroup leaving P_o fixed is $\mathrm{SO}(d)$ so that $S^d=\mathrm{SO}(d{+}1)/\mathrm{SO}(d)$. Let χ_g be the action of $g\in\mathrm{SO}(d{+}1)$ on S^d, $\chi_g: x\to gx$. We have

$$d\chi_h(P_o): T_{P_o}S^d\to T_{P_o}S^d \quad\text{for}\quad h\in H$$

where T_PS^d denotes the tangent space at the point $P\in S^d$. Therefore $U: h\to d\chi_h(P_o)$ is a representation of $\mathrm{SO}(d)$; it is irreducible when $d\ge3$. Let $\{\varepsilon_1,\cdots,\varepsilon_d,\varepsilon_{d+1}\}$ be the canonical basis of $\mathbf{R}^{d+1}$ with $P_o=\varepsilon_{d+1}\in S^d$. For $1\le i\le d$, consider $\varphi_i(t)=\sin t\,\varepsilon_i+\cos t\,\varepsilon_{d+1}$. Then φ_i is a curve on S^d starting from P_o, having ε_i as the tangent vector at P_o. In this way, we shall identify $T_{P_o}S^d$ with $\mathbf{R}^d$

Let $\{T^\lambda;\ \lambda\in\Lambda\}$ be the family of equivalence class of unitary irreducible representations of $\mathrm{SO}(d{+}1)$.

Definition A.1 *We say that T^λ contains a copy of U if there exists a subspace W_λ of the base space V_λ of T^λ, which is invariant by all $\{T^\lambda(h);\ h \in H\}$ and such that the restriction of T^λ to W_λ is equivalent to U.*

Denote by Λ_o such sub-family of T^λ having this property. By theory of representation (see [12], [13]),

$$\{T^\lambda;\ \lambda \in \Lambda_o\} = \{T^{(d+1)\ell}, Q^{(d+1)\ell};\ \ell \geq 1\};$$

the base space of $T^{(d+1)\ell}$ is the space $\mathcal{H}_{d+1,\ell}$ of homogeneous harmonic polynomials on $\mathbf{R}^{d+1}$ of degree $\ell \geq 1$ and the base space of $Q^{(d+1)\ell}$ is the space of 2-differential forms $\mathcal{F}_{d+1,\ell}$ considered in [9]. Let's describe the subspace W_ℓ of $\mathcal{H}_{d+1,\ell}$ and $\hat{W}_\ell$ of $\mathcal{F}_{d+1,\ell}$, which are invariant by $h \in \mathrm{SO}(d)$. Let $\mathcal{R}_{d+1,\ell}$ be the space of homogeneous polynomials on $\mathbf{R}^{d+1}$ of degree ℓ, equipped with the inner product $< P, Q >= \int_{S^d} P(x)Q(x)\,dx$. Let H be the orthogonal projection from $\mathcal{R}^{d+1,\ell}$ onto $\mathcal{H}_{d+1,\ell}$. Then $W_\ell = H\big(x_{d+1}^{\ell-1}\cdot \mathcal{H}_{d,1}\big)$. Set $\Theta_i(x) = C_i\, H(x_{d+1}^{\ell-1}x_i)$ for $i = 1, \cdots, d$, where C_i are chosen so that we have an orthonormal basis of W_ℓ. The space $\hat{W}_\ell$ is the vector space spanned by $\{\hat{\Theta}_i = C_i\, H\big(dx_{d+1}\wedge d\Theta_i\big);\ i = 1, \cdots, d\}$. Completing $\{\Theta_i;\ i = 1, \cdots, d\}$ and $\{\hat{\Theta}_i;\ i = 1, \cdots, d\}$ into an orthonormal basis of $\mathcal{H}_{d+1,\ell}$ and $\mathcal{F}_{d+1,\ell}$, we denote by (T^ℓ_{ij}) and (Q^ℓ_{ij}) the associated matrices. For further discussions on this topic, we refer to the book [14]. The following results is taken from [13].

Proposition A.2([12][6]) *We have for $1 \leq i, j \leq d,\ g \in \mathrm{SO}(d+1)$,*

$$T^\ell_{ij}(g) = \gamma_\ell(t)\, g_{ij} + \gamma'_\ell(t)\, g_{i,d+1}g_{d+1,j}\,, \tag{A.1}$$

$$Q^\ell_{ij}(g) = \Big(t\gamma_\ell(t) - \frac{1-t^2}{d-1}\gamma'_\ell(t)\Big)\, g_{ij} + \Big(-\gamma_\ell(t) - \frac{t}{d-1}\gamma'_\ell(t)\Big)\, g_{i,d+1}g_{d+1,j} \tag{A.2}$$

where $t = g_{d+1,d+1}$ and

$$\gamma_\ell(\cos\theta) = \int_0^\pi (\cos\theta - \sqrt{-1}\sin\theta\cos\varphi)^{\ell-1}\,\sin^d\varphi\frac{d\varphi}{c_d} \tag{A.3}$$

with $c_d = \displaystyle\int_0^\pi \sin^d\varphi\, d\varphi$.

Remark that $\gamma_\ell(t)$ is real and $|\gamma_\ell(t)| \leq 1$. For $g = h \in \mathrm{SO}(d)$, $t = g_{d+1,d+1} = 1,\ g_{i,d+1} = 0$. By $(A.1)$, we see that $T^\ell_{ij}(h) = h_{ij}$. On the other hand, by the choice of basis,

$$T^\ell_{ki}(h) = 0 \quad \text{if} \quad k > d \qquad \textit{and} \quad 1 \leq i \leq d.$$

The same results hold for Q^ℓ. Now using the Peter-Weyl theorem, we get the spectral expansion for eigen vector fields $\{A^1_{\ell k}, A^2_{\ell k}\}$:

$$(A.4)\quad A^1_{\ell k}(gP_0) = \sqrt{\frac{D_{\ell,1}}{d}} \sum_{i=1}^{d} T^\ell_{ki}(g)\, d\chi_g(P_0)\varepsilon_i, \quad A^2_{\ell k}(gP_0) = \sqrt{\frac{D_{\ell,2}}{d}} \sum_{i=1}^{d} Q^\ell_{ki}(g)\, d\chi_g(P_0)\varepsilon_i,$$

where $D_{\ell,1} = \dim(\mathcal{H}_{d+1,\ell})$ and $D_{\ell,2} = \dim(\mathcal{F}_{d+1,\ell})$.

Proposition A.3([12][6]) *Let $x, y \in S^d$ and θ the angle between x, y. Then*

$$(A.5)\qquad \frac{d}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \big\langle A^1_{\ell k}(x), A^1_{\ell k}(y)\big\rangle_{\mathbf{R}^{d+1}} = (d-1+\cos^2\theta)\gamma_\ell(\cos\theta) - \cos\theta\sin^2\theta\gamma'_\ell(\cos\theta),$$

$$(A.6)\qquad \frac{d}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \big\langle A^2_{\ell k}(x), A^2_{\ell k}(y)\big\rangle_{\mathbf{R}^{d+1}} = d\cos\theta\gamma_\ell(\cos\theta) - \sin^2\theta\gamma'_\ell(\cos\theta)$$

Proposition A.4([12][6]) *Let $x, y \in S^d$ and θ the angle between them. Then*

$$(A.7)\qquad \frac{d}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \big\langle A^1_{\ell k}(x), y\big\rangle^2 = \frac{d}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \big\langle A^2_{\ell k}(x), y\big\rangle^2 = \sin^2\theta$$

$$(A.8)\quad \frac{d}{D_{\ell,1}} \sum_{k=1}^{D_{\ell,1}} \Big(\big\langle A^1_{\ell k}(x),\ y\big\rangle + \big\langle A^1_{\ell k}(y),\ x\big\rangle\Big)^2 = 2\sin^2\theta\,\Big[1 - \cos\theta\,\gamma_\ell(\cos\theta) + \sin^2(\theta)\,\gamma'_\ell(\cos\theta)\Big],$$

$$(A.10)\qquad \frac{d}{D_{\ell,2}} \sum_{k=1}^{D_{\ell,2}} \Big(\big\langle A^2_{\ell k}(x),\ y\big\rangle + \big\langle A^2_{\ell k}(y),\ x\big\rangle\Big)^2 = 2\sin^2\theta\,\Big[1 - \gamma_\ell(\cos\theta)\Big].$$

Acknowledgement: We thank Professor Shizan Fang for very useful discussions.

References

[1] J.M. Bismut: *Mécanique aléatoire*, Lect. notes in Math., Vol. 866 (1981), Springer-Verlag, Berlin, New-York.

[2] P. Dupuis, R.S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley, New York, 1997.

[3] K.D. Elworthy: *Stochastic differential equations on manifolds*, London Mathematical Society. Lect. Note Series 70 (1982), Cambridge University Press, Cambridge.

[4] S. Fang: Canonical Brownian motion on the diffeomorphism group of the circle, *J. Funct. Anal.*, **196** (2002), p.162-179.

[5] S. Fang and T. Zhang: A study of a class of stochastic differential equations with non-Lipschitzian coefficients, *Prob. Theory Rel. Fields 132(3),356-390.*

[6] S. Fang and T. Zhang: Isotropic stochastic flow of homeomorphisms on S^d for the critical Sobolev exponent. *J. Math. Pures et Appl.*85:4 (2006) 580-597.

[7] M.Freidlin and A. Wentzell: On small random pertubatons of dynamical system,*Russian Math. Surveys.*, **25** (1970), p. 1-55.

[8] H. Kunita: *Stochastic flows*, Cambridge Univ. Press, Cambridge, UK, 1988.

[9] Y. LeJan and O. Raimond: Integration of Brownian vector fields, *Ann. of Prob.*, **30** (2002), p. 826-873.

[10] P. Malliavin: The canonical diffusion above the difeomorphism group of the circle, *C. R. Acad. Sci.*, **329**, p. 325-329.

[11] P. Malliavin: *Stochastic Analysis*, Grunlehren des Math. **313**, Springer, Berlin, 1997.

[12] O. Raimond: Flots browniens isotropes sur la sphère, *Ann. Inst. H. Poincaré*, **35** (1999), p. 313-354.

[13] E. Stein: *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Study 63 (1970), Princeton.

[14] N.J. Vilenkin: *Fonctions spéciales et théorie de la représentation des groupes*, Dunod, Paris, 1969.