Perturbed Skorohod Equations and Perturbed Reflected Diffusion Processes

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Abstract

In this paper, we obtain the existence and uniqueness of the solution to the perturbed Skorohod equation $g(t) = f(t) + \alpha \max_{0\le s\le t} g(s) + h(t)$ for any real constant $\alpha < 1$ and any given continuous function f, where $g \ge 0$, and h is an increasing function which only increases at the time g is at zero. As an application, we establish the existence and uniqueness of some perturbed reflected diffusion processes.

Résumé: Dans ce papier, nous allons considérer l'équation de Skorohod perturbée $g(t) = f(t) + \alpha \max_{0\le s\le t} g(s) + h(t)$ pour tout nombre réel $\alpha < 1$ et tout fonction continue f; où $g \ge 0$ et h est une fonction croissante, strictement croissante aux points où g s'annule. L'existence et l'unicité seront obtenu. Comme une application, nous allons établir l'existence et l'unicité des processus de diffusion réfléchis perturbés.

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1 Introduction

There now exists a considerable body of literature devoted to the study of "perturbed" versions of familiar stochastic and deterministic equations; see e.g. [1]-[7], [9], [10], [12], and [14]. An example is the perturbed Skorokhod equation

$$X_t = x + B_t + \alpha \max_{0\le s\le t} X_s + L_t^X, \quad t \ge 0, \tag{1}$$

where B is a standard Brownian motion starting from 0, $\alpha < 1$ is a real constant, $x \geq 0$, and L^X denotes a local time at zero of X. Since for $\alpha = 0$ the solution is reflected Brownian motion any solution of (1) is referred to as a perturbed reflected Brownian motion, and the question of its existence and pathwise uniqueness has exercised several authors. As was pointed out in [10], if either $x > 0$ or $\alpha < 1/2$, an affirmative answer follows from the fact that the obvious deterministic version of (1) has a unique solution when B is replaced by an arbitrary continuous function, vanishing at zero. The remaining case was settled in [3], a slightly weaker result being published simultaneously in [6]. However neither of these results dealt with the deterministic version with $x = 0$ and $\alpha \in [1/2, 1)$, and our first main result, Theorem 3.1 fills this gap, by means of completely different arguments.
Our other main result seems to be the first to deal with the analogous question for a general diffusion. Specifically we study the equation

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \alpha \max_{0 \leq s \leq t} X_s + L_t^X, \quad t \geq 0, \tag{2}$$

where σ is a Lipschitz continuous function on R. Again the case $x > 0$ turns out to be relatively straight-forward, once we have established existence and pathwise uniqueness for perturbed diffusion processes in Theorem 2.1. For $x = 0$ we exploit our result on the Skorokhod equation together with Picard iteration to establish existence and pathwise uniqueness of a solution to (2), but unfortunately only in the case $\alpha < 1/2$.
Finally we consider a version of (2) in which the Itô integral is replaced by a Stratonovich integral. In this case we are able to establish existence of a unique solution for all values of $\alpha < 1$, under mild assumptions on σ.

2 Perturbed Diffusion Processes

Let $B_t, t \geq 0$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. $\mathcal{F}_t, t \geq 0$ denotes the filtration of the Brownian motion B. Let $\sigma(x)$, $b(x)$ be Lipschitz continuous functions on R, i.e.,there exists a constant C such that

$$|\sigma(x) - \sigma(y)| \leq C|x - y|, \tag{3}$$

$$|b(x) - b(y)| \leq C|x - y|. \tag{4}$$

For $\alpha < 1$, consider the following stochastic differential equation:

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)dB_s + \int_0^t b(Y_s)ds + \alpha \max_{0 \leq s \leq t} Y_s. \tag{5}$$

Theorem 2.1 . Assume that the random variable Y_0 is independent of B and $E[|Y_0|^2] < \infty$. There exists a unique, continuous, $\mathcal{F}_t$-adapted solution $Y_t, t \ge 0$ to equation (5) such that $E[\sup_{0\le s\le T}|Y_s|^2] < \infty$ for any $T > 0$.

Proof. We construct the solution by iteration . Let

$$Y_t^0 = \frac{Y_0}{1-\alpha}, \quad 0 \le t < \infty.$$

For $n \ge 0$ define Y_t^{n+1} to be the unique, continuous, adapted solution to the following equation:

$$Y_t^{n+1} = Y_0 + \int_0^t \sigma(Y_s^n)dB_s + \int_0^t b(Y_s^n)ds + \alpha \max_{0\le s\le t} Y_s^{n+1}. \quad (6)$$

Such a solution exists and can be expressed explicitly as

$$\begin{aligned} Y_t^{n+1} &= \frac{Y_0}{1-\alpha} + \int_0^t \sigma(Y_s^n)dB_s + \int_0^t b(Y_s^n)ds \\ &+ \frac{\alpha}{1-\alpha}\max_{0\le s\le t}\Big(\int_0^s \sigma(Y_u^n)dB_u + \int_0^s b(Y_u^n)du\Big). \end{aligned} \quad (7)$$

This is a consequence of the reflection principle. We will show that Y^n converges uniformly on compact intervals almost surely. It follows from (7) that

$$\begin{aligned} & |Y_s^{n+1} - Y_s^n| \\ \le\; & |\textstyle\int_0^s \sigma(Y_u^n)dB_u - \int_0^s \sigma(Y_u^{n-1})dB_u| + |\int_0^s b(Y_u^n)du - \int_0^s b(Y_u^{n-1})du| \\ & +\tfrac{|\alpha|}{1-\alpha}|\max_{0\le v\le s}(\textstyle\int_0^v \sigma(Y_u^n)dB_u + \int_0^v b(Y_u^n)du) \\ & -\max_{0\le v\le s}(\textstyle\int_0^v \sigma(Y_u^{n-1})dB_u + \int_0^v b(Y_u^{n-1})du)|) \\ \le\; & |\textstyle\int_0^s \sigma(Y_u^n)dB_u - \int_0^s \sigma(Y_u^{n-1})dB_u| + \int_0^s |b(Y_u^n)du - b(Y_u^{n-1})|du \\ & +\tfrac{|\alpha|}{1-\alpha}\max_{0\le v\le s}|\textstyle\int_0^v (\sigma(Y_u^n) - \sigma(Y_u^{n-1}))dB_u| \\ & +\tfrac{|\alpha|}{1-\alpha}\max_{0\le v\le s}|\textstyle\int_0^v (b(Y_u^n) - b(Y_u^{n-1}))du|, \end{aligned} \quad (8)$$

where we used the fact that $|\max_{0\le v\le s} f(v) - \max_{0\le v\le s} g(v)| \le \max_{0\le v\le s}|f(v) - g(v)|$ holds for any two continuous functions f and g. Thus,

$$\begin{aligned} \max_{0\le s\le t}|Y_s^{n+1} - Y_s^n| &\le (1+\frac{|\alpha|}{1-\alpha})\Big[\max_{0\le s\le t}|\int_0^s (\sigma(Y_u^n) - \sigma(Y_u^{n-1}))dB_u| \\ &+ \int_0^t |b(Y_u^n) - b(Y_u^{n-1})|du\Big]. \end{aligned} \quad (9)$$

In the rest of the proof, we will use C to denote a generic constant which may change from line to line. By Burkhölder's inequality,

$$\begin{aligned} E[\max_{0\le s\le t}|Y_s^{n+1} - Y_s^n|^2] &\le CE[\int_0^t (\sigma(Y_u^n) - \sigma(Y_u^{n-1}))^2 du] \\ &\quad +CE[\int_0^t (b(Y_u^n) - b(Y_u^{n-1}))^2 du] \\ &\le CE[\int_0^t |Y_u^n - Y_u^{n-1}|^2 du]. \end{aligned} \quad (10)$$

Iterating the inequality we get that for fixed $T > 0$,

$$E[\max_{0\le s\le T} |Y_s^{n+1} - Y_s^n|^2] \le C\frac{T^n}{n!}, \tag{11}$$

since $E[Y_0^2] < \infty$. Consequently,

$$P(\max_{0\le s\le T} |Y_s^{n+1} - Y_s^n| > \frac{1}{2^n}) \le C\frac{(4T)^n}{n!}. \tag{12}$$

By the Borel-Cantelli lemma, we see that Y^n converges uniformly to a continuous, adapted process Y on $[0,T]$ almost surely. Letting $n \to \infty$ in (6) it follows that Y is a solution to equation (5). As T is arbitrary, this proves the existence . Now suppose that $Y_t, t \ge 0$ and $Z_t, t \ge 0$ are two solutions to equation (5). Then we have, by the reflection principle,

$$\begin{aligned} Y_t &= \frac{Y_0}{1-\alpha} + \int_0^t \sigma(Y_s)dB_s \\ &\quad + \frac{\alpha}{1-\alpha}\max_{0\le s\le t}(\int_0^s \sigma(Y_u)dB_u + \int_0^s b(Y_u)du), \end{aligned} \tag{13}$$

$$\begin{aligned} Z_t &= \frac{Y_0}{1-\alpha} + \int_0^t \sigma(Z_s)dB_s \\ &\quad + \frac{\alpha}{1-\alpha}\max_{0\le s\le t}(\int_0^s \sigma(Z_u)dB_u + \int_0^s b(Z_u)du). \end{aligned} \tag{14}$$

Arguing as above, there is a constant C such that

$$|Y_t - Z_t| \le C\max_{0\le s\le t} |\int_0^s (\sigma(Y_u) - \sigma(Z_u))dB_u| + C\int_0^t |b(Y_u) - b(Z_u)|du. \tag{15}$$

Applying Burkhölder's inequality,

$$\begin{aligned} E[|Y_t - Z_t|^2] &\le CE[\int_0^t |\sigma(Y_u) - \sigma(Z_u)|^2 du] \qquad (16) \\ &\quad + CE[\int_0^t |b(Y_u) - b(Z_u)|^2 du] \\ &\le CE[\int_0^t |Y_u - Z_u|^2 du], \qquad (17) \end{aligned}$$

which implies $E[|Y_t - Z_t|^2] = 0$ by Gronwall's inequality. Hence, the solution is unique.

3 Perturbed Skorohod Equation

Set $W_0 = \{f \in C([0,\infty) \to R); f(0) = 0\}$ and $W^+ = \{f \in C([0,\infty) \to R); f(t) \ge 0 \quad \text{for all} \quad t \ge 0\}$. The aim of this section is to prove

Theorem 3.1. Given $f \in W_0$ and $0 \leq \alpha < 1$, there exist unique $g \in W^+$ and $h \in W^+$ such that

(i)
$$g(t) = f(t) + \alpha \max_{0 \leq s \leq t} g(s) + h(t), \tag{18}$$

(ii) $h(0) = 0$ and $t \to h(t)$ is non-decreasing,

(iii) $\int_0^t \chi_{\{g(s)=0\}} dh(s) = h(t)$.

(g, h) is called a solution to the perturbed Skorohod equation for the function f.

Remark. This gives a simpler proof of Theorem 1 of [3].

For clarity, we split the proof of the theorem into several lemmas. For a continuous function f and $0 \leq s \leq t$ define

$$\omega_{s,t}(f) = \max_{s \leq u,v \leq t} (f(u) - f(v)) = \max_{s \leq u,v \leq t} (|f(u) - f(v)|). \tag{19}$$

Lemma 3.2. Let (g, h) be a solution to the perturbed Skorohod equation for the function f. There exists a constant C independent of g and f such that

$$\omega_{s,t}(g) \leq C\omega_{s,t}(f) \quad \text{for all} \quad 0 \leq s \leq t. \tag{20}$$

Proof. Fix $0 \leq s < t$. We observe that if $g(v) > 0$ for all $v \in (s, t)$ then

$$g(v) - g(s) = f(v) - f(s) + \alpha(\max_{0 \leq u \leq v} g(u) - \max_{0 \leq u \leq s} g(u)) \quad \text{for all} \quad v \in [s, t], \tag{21}$$

and if $\max_{0 \leq u \leq t} g(u) = \max_{0 \leq u \leq s} g(u)$

$$g(v) - g(s) = f(v) - f(s) + h(v) - h(s) \quad \text{for all} \quad v \in [s, t]. \tag{22}$$

Without loss of generality we assume $\omega_{s,t}(g) > 0$ and $\omega_{s,t}(g) = g(v_0) - g(u_0)$ for some $u_0, v_0 \in [s, t]$.

Case 1. $u_0 < v_0$.

Let $l_0 = v_0 \wedge \inf\{v \geq u_0; g(v) = \max_{0 \leq u \leq v} g(u)\}$. As $\max_{0 \leq u \leq l_0} g(u) = \max_{0 \leq u \leq u_0} g(u)$, by (22) ,

$$g(v) - g(u_0) = f(v) - f(u_0) + h(v) - h(u_0) \quad \text{for all} \quad v \in [u_0, l_0]. \tag{23}$$

It follows by the reflection principle that for $v \in [u_0, l_0]$,

$$\begin{aligned} g(v) - g(u_0) &= f(v) - f(u_0) - \inf_{u_0 \leq u \leq v} \{(g(u_0) + f(u) - f(u_0)) \wedge 0\} \\ &\leq f(v) - f(u_0) - \inf_{u_0 \leq u \leq v} \{(f(u) - f(u_0)) \wedge 0\} \\ &\leq |f(v) - f(u_0)| + \max_{u_0 \leq u \leq v} \{|f(u) - f(u_0)|\} \\ &\leq 2 \max_{u_0 \leq u \leq v} \{|f(u) - f(u_0)|\}. \end{aligned} \tag{24}$$

In particular,

$$g(l_0) - g(u_0) \leq 2 \max_{u_0 \leq u \leq l_0} \{|f(u) - f(u_0)|\}. \tag{25}$$

On the other hand, we can assume $g(u) > 0$ for all $u \in [l_0, v_0]$. Otherwise, replace u_0 by $\bar{u}_0 = \sup\{u; u \leq v_0, g(u) = 0\}$. Thus, by (21)

$$g(v_0) - g(l_0) = f(v_0) - f(l_0) + \alpha(\max_{0 \leq u \leq v_0} g(u) - \max_{0 \leq u \leq l_0} g(u)). \tag{26}$$

If $l_0 = v_0$, then (20) follows already from (25). Assume now $l_0 < v_0$. In this case it is easy to see that $g(l_0) = \max_{0 \leq u \leq l_0} g(u)$ and $g(v_0) = \max_{s \leq u \leq t} g(u)$. Therefore, we also have $g(v_0) = \max_{0 \leq u \leq v_0} g(u)$. Consequently,

$$g(v_0) - g(l_0) = f(v_0) - f(l_0) + \alpha(g(v_0) - g(l_0)). \tag{27}$$

Hence,

$$g(v_0) - g(l_0) = \frac{1}{1-\alpha}(f(v_0) - f(l_0)). \tag{28}$$

Combining (25) and (28) yields

$$\begin{aligned} g(v_0) - g(u_0) &= g(v_0) - g(l_0) + g(l_0) - g(u_0) \\ &\leq \frac{1}{1-\alpha}(f(v_0) - f(l_0)) + 2 \max_{u_0 \leq u \leq l_0} \{|f(u) - f(u_0)|\} \\ &\leq (2 + \frac{1}{1-\alpha}) \max_{s \leq u, v \leq t} \{|f(u) - f(v)|\}. \end{aligned} \tag{29}$$

Case 2. $v_0 < u_0$.

We can assume $g(u) > 0$ for all $u \in [v_0, u_0]$. Otherwise , we can replace u_0 by $\bar{u}_0 = \inf\{u > v_0; g(u) = 0\}$. Thus, by (21),

$$g(u_0) - g(v_0) = f(u_0) - f(v_0) + \alpha(\max_{0 \leq u \leq u_0} g(u) - \max_{0 \leq u \leq v_0} g(u)). \tag{30}$$

Since $g(v_0) = \max_{v_0 \leq u \leq u_0} g(u)$, we have

$$g(u_0) - g(v_0) = f(u_0) - f(v_0) \leq \omega_{s,t}(f). \tag{31}$$

Combining case 1 and case 2 completes the proof.

Lemma 3.3. Let (g_n, h_n) be a solution to the perturbed Skorohod equation for a function f_n. If $f_n \to f$ uniformly on compact intervals as $n \to \infty$, then a solution (g, h) to the perturbed Skorohod equation for the function f exists.

Proof. Without loss of generality, we can restrict ourself to the finite interval $[0, 1]$. Since $f_n \to f$ as $n \to \infty$, $\{f_n, n \geq 1\}$ is a bounded set of equicontinuous functions. By Lemma 3.2, we see that $K = \{g_n, n \geq 1\}$ is also a bounded set of equicontinuous functions. By the Arzela-Ascoli theorem, there exists a subsequence $g_{n_k}, k \geq 1$, that converges uniformly to a continuous function g on $[0,1]$. It is seen from (18) that $h_{n_k}, k \geq 1$ also converges uniformly to a non-decreasing continuous function h on $[0,1]$ and

$$g(t) = f(t) + \alpha \max_{0 \leq s \leq t} g(s) + h(t). \tag{32}$$

To show that (g,h) is a solution to the perturbed Skorohod equation for the function f, it remains to prove that $\int_0^t \chi_{\{g(s)=0\}} dh(s) = h(t)$ for $t \geq 0$. To see this it is enough to show that for any continuous function $l(t)$ on $(0,\infty)$ with compact support, $\int_0^t l(g(s))dh(s) = 0$. This in fact follows from

$$\int_0^t l(g(s))dh(s) = \lim_{k \to \infty} \int_0^t l(g_{n_k}(s))dh_{n_k}(s) = 0. \tag{33}$$

Proof of Theorem 3.1 .

Existence.

Again we restrict ourself to the interval $[0,1]$. For the existence, by Lemma 3.3 we can assume $f \in H_2^1[0,1]$, where

$$H_2^1[0,1] := \{f; f : [0,1] \to R \text{ is absolutely continuous and } \int_0^1 f'(s)^2 ds < \infty\}, \tag{34}$$

where $f'(s)$ stands for the derivative of f. We will use the so called penalized method. Define

$$\phi(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$

Consider the following functional equation

$$g_\varepsilon(t) = f(t) + \alpha \max_{0 \leq s \leq t} g_\varepsilon(s) - \frac{1}{\varepsilon} \int_0^t \phi'(g_\varepsilon(s))ds, \tag{35}$$

where $\phi'(x)$ stands for the derivative of ϕ.
Note that there is no "local time" term in equation (35). The existence of the solution of the equation (35) can be obtained in exactly the same way as Theorem 2.1. In fact, let

$$g_\varepsilon^0(t) = 0, \quad 0 \leq t < \infty,$$

and define $g_\varepsilon^{n+1}(t)$ to be the unique continuous solution of the equation:

$$g_\varepsilon^{n+1}(t) = f(t) - \frac{1}{\varepsilon} \int_0^t \phi'(g_\varepsilon^n(s))ds + \alpha \max_{0 \leq s \leq t} g_\varepsilon^{n+1}(s).$$

By the reflection principle,

$$g_\varepsilon^{n+1}(t) = f(t) - \frac{1}{\varepsilon} \int_0^t \phi'(g_\varepsilon^n(s))ds$$

$$+\frac{\alpha}{1-\alpha}\max_{0\le s\le t}\Big(f(s)-\frac{1}{\varepsilon}\int_0^s\phi'(g_\varepsilon^n(u))du\Big).$$

Following the proof of Theorem 2.1 sentence by sentence, we arrive at

$$\max_{0\le s\le t}|g_\varepsilon^{n+1}(t)-g_\varepsilon^n(t)|$$

$$\le(1+\frac{|\alpha|}{1-\alpha})\frac{1}{\varepsilon}\int_0^t|\phi'(g_\varepsilon^n(s))-\phi'(g_\varepsilon^{n-1}(s))|ds$$

$$\le(1+\frac{|\alpha|}{1-\alpha})C_\varepsilon\int_0^t|g_\varepsilon^n(s)-g_\varepsilon^{n-1}(s)|ds.$$

Iterating the above inequality as in the proof of Theorem 2.1, it is seen that g_ε^n converges uniformly on any finite interval $[0,T]$ to a continuous function g_ε, which clearly is a solution of equation (35).
Put $A^\varepsilon(t)=\max_{0\le s\le t}g_\varepsilon(s)$, then

$$\phi(g_\varepsilon(t))+\frac{1}{\varepsilon}\int_0^t\phi'(g_\varepsilon(s))^2ds=\int_0^t\phi'(g_\varepsilon(s))f'(s)ds+\alpha\int_0^t\phi'(g_\varepsilon(s))dA^\varepsilon(s). \tag{36}$$

Since $\phi'(x)\le 0$ and $A^\varepsilon(t)$ is increasing in t, it follows from (36) that

$$\frac{1}{\varepsilon}\int_0^t\phi'(g_\varepsilon(s))^2ds\le\int_0^t\phi'(g_\varepsilon(s))f'(s)ds$$

$$\le(\int_0^t\phi'(g_\varepsilon(s))^2ds)^{\frac{1}{2}}(\int_0^t f'(s)^2ds)^{\frac{1}{2}} \tag{37}$$

Consequently,

$$\frac{1}{\varepsilon^2}\int_0^t\phi'(g_\varepsilon(s))^2ds\le\int_0^t f'(s)^2ds, \tag{38}$$

and futhermore, by the Hölder inequality and (38),

$$\phi(g_\varepsilon(t))\le\int_0^t\phi'(g_\varepsilon(s))f'(s)ds$$

$$\le(\int_0^t\phi'(g_\varepsilon(s))^2ds)^{\frac{1}{2}}(\int_0^t f'(s)^2ds)^{\frac{1}{2}}\le\varepsilon\int_0^t f'(s)^2ds. \tag{39}$$

Set $h^\varepsilon(t)=-\frac{1}{\varepsilon}\int_0^t\phi'(g_\varepsilon(s))ds$. Then by (38) there exists a sequence $\varepsilon_n\to 0$ such that $h^{\varepsilon_n}(t)$ converges unformly to $h(t)\in H_2^1[0,1]$. On the other hand, it follows from the equation (35) that

$$|g_{\varepsilon_n}(t)-g_{\varepsilon_m}(t)|\le\alpha|A^{\varepsilon_n}(t)-A^{\varepsilon_m}(t)|+|h^{\varepsilon_n}(t)-h^{\varepsilon_m}(t)|$$

$$\le\alpha\max_{0\le s\le t}|g_{\varepsilon_n}(s)-g_{\varepsilon_m}(s)|+|h^{\varepsilon_n}(t)-h^{\varepsilon_m}(t)|. \tag{40}$$

Hence,

$$\max_{0\le s\le 1}|g_{\varepsilon_n}(s)-g_{\varepsilon_m}(s)|\le\frac{1}{1-\alpha}\max_{0\le s\le 1}|h_{\varepsilon_n}(s)-h_{\varepsilon_m}(s)|. \tag{41}$$

Therefore, $\{g_{\varepsilon_m}(t), m \geq 1\}$ is a Cauchy sequence in $C([0,1])$. Let g denote the limit. We will show that (g,h) is a solution to the perturbed Skorohod equation for the function f. First of all, it is clear from the construction that

$$g(t) = f(t) + \alpha \max_{0\leq s\leq t} g(s) + h(t), \tag{42}$$

and $h(t)$ is a non-negative increasing function (because h_{ε_n} is). Secondly, it follows from (39) that $\phi(g(t)) = 0$, which implies that $g(t) \geq 0$ for all $t \geq 0$. Now for any continuous function $l(t)$ on $(0,\infty)$ with compact support, say that the support is contained in $[\delta_1, \delta_2]$ for some $\delta_1 > 0$, we have

$$\begin{aligned} |\int_0^t l(g_{\varepsilon_n}(s)) dh^{\varepsilon_n}(s)| &\leq \int_0^t |l(g_{\varepsilon_n}(s))| dh^{\varepsilon_n}(s) \\ &= \frac{1}{\varepsilon_n}\int_0^t |l(g_{\varepsilon_n}(s))||\phi'(g_{\varepsilon_n}(s))| ds \\ &\leq C\int_0^t \chi_{[\delta_1,\delta_2]}(g_{\varepsilon_n}(s))|\phi'(g_{\varepsilon_n}(s))| ds = 0, \end{aligned}$$

since $\phi'(x) = 0$ for $x > 0$.
The identity $\int_0^t \chi_{\{g(s)=0\}} dh(s) = h(t)$ for $t \geq 0$ can now be proved similarly as in the proof of Lemma 3.3. Thus, we complete the proof of existence.

Uniqueness:

For any $\delta > 0$, let $g_\delta(t)$ be the unique solution to equation

$$g_\delta(t) = (1-\alpha)\delta + f(t) + \alpha \max_{0\leq s\leq t} g_\delta(s) + h_\delta(t). \tag{43}$$

Such an equation was studied in [10] and [3]. Let (g,h) be a solution to equation (18). A " round trip" of g is defined to be a section of the path g lying between two maxima and containing a visit to zero. It follows from the proof of Lemma 3 in [3] that

$$\sup_{0\leq s\leq t} |g_\delta(s) - g(s)| \leq \delta(\alpha^* + 1)^{2n_t+1}, \tag{44}$$

where $\alpha^* = \frac{\alpha}{1-\alpha}$, n_t is the number of round trips completed by g by time t. This implies that $g(t) = \lim_{\delta\to 0} g_\delta(t)$, which gives the uniqueness.

4 Perturbed Reflected Diffusions

Let σ be as in section 2 . For $x \geq 0$, consider the stochastic differential equation:

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \alpha \max_{0\leq s\leq t} X_s + L_t. \tag{45}$$

Definition 4.1. We say that $(X_t, L_t, t \geq 0)$ is a solution to equation (45) if

(i) $X_0 = x, X_t \geq 0$ for $t \geq 0$,

(ii) X_t, L_t are continuous and adapted to the filtration of B,

(iii) L_t is non-decreasing with $L_0 = 0$ and

$$\int_0^t \chi_{\{X_s=0\}} dL_s = L_t,$$

(iv) $(X_t, L_t, t \geq 0)$ satisfies the equation (45) almost surely for every $t > 0$.

The cases $x = 0$ and $x > 0$ are quite different. We will treat them separately .

Theorem 4.2 Assume $\alpha < 1$ and σ is Lipschitz. If $x > 0$, there exists a unique solution $(X_t, L_t, t \geq 0)$ to equation (45).

Proof. We construct the solution iteratively in a similar way to [10]. Define Y_t^0 to be the unique solution to the equation:

$$Y_t^0 = x + \int_0^t \sigma(Y_s^0) dB_s + \alpha \max_{0 \leq s \leq t} Y_s^0. \tag{46}$$

It is known from section 2 that such a solution exists. Set $T_1 = \inf\{t \geq 0; Y_t^0 = 0\}$. Then $T_1 > 0$ a.s. as $x > 0$. Define

$$X_t = Y_t^0, \qquad L_t = 0 \quad \text{for} \quad 0 \leq t \leq T_1. \tag{47}$$

Put $B_t^1 = B_{t+T_1} - B_{T_1}$ for $t \geq 0$. It is well known that $B_t^1, t \geq 0$ is a standard Brownian motion independent of $\mathcal{F}_{T_1}$. Consider the stochastic differential equation with refecting boundary:

$$\begin{gathered} Z_t^1 = \int_0^t \sigma(Z_s^1) dB_s^1 + L_t^1, \\ Z_t^1 \geq 0, Z_0^1 = 0, \\ L_0^1 = 0, \int_0^t \chi_{\{Z_s^1=0\}} dL_s^1 = L_t^1. \end{gathered} \tag{48}$$

The definition of a solution to this equation is the same as Definition 4.1 with $x = 0$ and $\alpha = 0$. It is known that a unique solution (Z_t^1, L_t^1) to the equation (48) exists, see e.g. [11]. In general, suppose that (X_t, L_t) has been defined for $0 \leq t \leq T_{2n-1}$. We can construct (X_t, L_t) for $T_{2n-1} \leq t \leq T_{2n+1}$ as follows. Let Z_t^{2n-1} be the solution to the equation:

$$\begin{gathered} Z_t^{2n-1} = \int_0^t \sigma(Z_s^{2n-1}) dB_s^{2n-1} + L_t^{2n-1}, \\ Z_t^{2n-1} \geq 0, Z_0^{2n-1} = 0, \\ L_0^{2n-1} = 0, \int_0^t \chi_{\{Z_s^{2n-1}=0\}} dL_s^{2n-1} = L_t^{2n-1}, \end{gathered} \tag{49}$$

where $B_t^{2n-1} = B_{t+T_{2n-1}} - B_{T_{2n-1}}$. Put $T_{2n} = \inf\{t > T_{2n-1}; Z_{t-T_{2n-1}}^{2n-1} = \max_{0 \leq s \leq T_{2n-1}} X_s\}$ and set

$$X_t = Z_{t-T_{2n-1}}^{2n-1}, \qquad L_t = L_{T_{2n-1}} + L_{t-T_{2n-1}}^{2n-1} \quad \text{for} \quad T_{2n-1} \leq t \leq T_{2n}. \tag{50}$$

Let Y_t^{2n} denote the solution to equation:

$$Y_t^{2n} = (1-\alpha)X_{T_{2n}} + \int_0^t \sigma(Y_s^{2n})dB_s^{2n} + \alpha \max_{0\le s\le t} Y_s^{2n}, \tag{51}$$

where $B_t^{2n} = B_{t+T_{2n}} - B_{T_{2n}}$. Set $T_{2n+1} = \inf\{t > T_{2n}, Y_{t-T_{2n}} = 0\}$ and

$$X_t = Y_{t-T_{2n}}^{2n}, \qquad L_t = L_{T_{2n}} \quad \text{for} \quad T_{2n} \le t \le T_{2n+1}. \tag{52}$$

By this procedure, we obtain a sequence of increasing stopping times $T_n, n \ge 0$. Set $T = \lim_{n\to\infty} T_n$. Then T is again a stopping time, and (X_t, L_t) is a well defined continuous process for all $0 \le t < T$. We will show that $(X_t, L_t, t < T)$ satisfies the equation (45) in the sense of Definition 4.1. To achieve this, it is sufficient to prove that (X_t, L_t) satisfies the equation (45) for $T_{2n} \le t \le T_{2n+1}$ and $n = 0, 1...,$. We will do this by induction. It is obvious that (X_t, L_t) is a solution to equation (45) for $0 \le t \le T_1$. If $T_1 \le t \le T_2$, it follows that

$$\begin{aligned}
X_t &= Z_{t-T_1}^1 \\
&= \textstyle\int_0^{t-T_1} \sigma(Z_s^1)dB_s^1 + L_{t-T_1}^1 \\
&= \textstyle\int_0^{t-T_1} \sigma(Z_s^1)dB_{s+T_1} + L_t \\
&= \textstyle\int_{T_1}^t \sigma(X_u)dB_u + L_t \\
&= X_{T_1} + \textstyle\int_{T_1}^t \sigma(X_u)dB_u + L_t \\
&= x + \textstyle\int_0^{T_1} \sigma(X_s)dB_s + \alpha \max_{0\le s\le T_1} X_s + \int_{T_1}^t \sigma(X_u)dB_u + L_t \\
&= x + \textstyle\int_0^t \sigma(X_s)dB_s + \alpha \max_{0\le s\le t} X_s + L_t,
\end{aligned} \tag{53}$$

since $\max_{0\le s\le T_1} X_s = \max_{0\le s\le t} X_s$ for $T_1 \le t \le T_2$, and $X_{T_1} = 0$. Furthermore if $T_1 \le t \le T_2$, we see that

$$\begin{aligned}
\int_0^t \chi_{\{X_s=0\}} dL_s &= \int_{T_1}^t \chi_{\{X_s=0\}} dL_{s-T_1}^1 \\
&= \int_0^{t-T_1} \chi_{\{Z_s^1=0\}} dL_s^1 = L_{t-T_1}^1 = L_t.
\end{aligned} \tag{54}$$

Thus we have showed that (X_t, L_t) is a solution to equation (45) for $0 \le t \le T_2$. Suppose that (X_t, L_t) satisfies the equation (45) for $0 \le t \le T_{2n}$. If $T_{2n} \le t \le T_{2n+1}$,it follows that

$$\begin{aligned}
X_t &= Y_{t-T_{2n}}^{2n} \\
&= (1-\alpha)X_{T_{2n}} + \int_0^{t-T_{2n}} \sigma(Y_s^{2n})dB_s^{2n} + \alpha \max_{0\le s\le t-T_{2n}} Y_s^{2n} \\
&= x + \int_0^{T_{2n}} \sigma(X_s)dB_s + \alpha \max_{0\le s\le T_{2n}} X_s + L_{T_{2n}} - \alpha X_{T_{2n}} \\
&\quad + \int_0^{t-T_{2n}} \sigma(Y_s^{2n})dB_{s+T_{2n}} + \alpha \max_{0\le s\le t-T_{2n}} Y_s^{2n} \\
&= x + \int_0^t \sigma(X_s)dB_s + \alpha \max_{T_{2n}\le s\le t} X_s + L_t \\
&= x + \int_0^t \sigma(X_s)dB_s + \alpha \max_{0\le s\le t} X_s + L_t,
\end{aligned} \tag{55}$$

where we have used the fact that $X_{T_{2n}} = \max_{0\le s\le T_{2n}} X_s$ and $Y_0^{2n} = X_{T_{2n}}$ from their definitions. Since $X_t \neq 0$ for $T_{2n} \le t < T_{2n+1}$, we also have

$$\int_0^t \chi_{\{X_s=0\}} dL_s = \int_0^{T_{2n}} \chi_{\{X_s=0\}} dL_s = L_{T_{2n}} = L_t.$$

So (X_t, L_t) satisfies the equation (45) also for $T_{2n} \le t \le T_{2n+1}$. Repeating similar arguments as for (53), we also can show that (X_t, L_t) satisfies the equation (45) for $T_{2n+1} \le t \le T_{2n+2}$.

Finally we show that $T = \infty$ a.s. By the construction of X, we have that

$$\begin{aligned} 0 = X_{T_{2n+1}} &= \max_{0\le s\le T_{2n}} X_s + \int_{T_{2n}}^{T_{2n+1}} \sigma(X_s)dB_s \\ &+\alpha(\max_{0\le s\le T_{2n+1}} X_s - \max_{0\le s\le T_{2n}} X_s) + L_{T_{2n+1}} - L_{T_{2n}} \end{aligned} \quad (56)$$

Suppose $T < \infty$ with positive probability. Letting $n \to \infty$ in (56), we get $0 = \max_{0\le s\le T} X_s$ which contradicts the fact that $X_0 = (1-\alpha)^{-1}x > 0$. The proof of existence is complete.

On the other hand, it is easily seen that the solution is unique since it is unique on each interval $[T_n, T_{n+1}]$.

Theorem 4.3. Assume $x = 0$. If $0 \le \alpha < \frac{1}{2}$, then there exists a unique solution $(X_t, L_t, t \ge 0)$ to equation (45).

Proof. We will use the Picard iteration method. Define $X_t^0 \equiv 0$ and (X_t^{n+1}, L_t^{n+1}) to be the unique solution to the equation:

$$X_t^{n+1} = \int_0^t \sigma(X_s^n)dB_s + \alpha \max_{0\le s\le t} X_s^{n+1} + L_t^{n+1}. \quad (57)$$

The existence and uniqueness of this solution follow from section 3. Observe that by the reflection principle,

$$L_t^{n+1} = -\inf_{s\le t}\{(\int_0^s \sigma(X_u^n)dB_u + \alpha \max_{0\le u\le s} X_u^{n+1}) \wedge 0\}. \quad (58)$$

Now (57) and (58) imply that

$$\begin{aligned} |X_t^{n+1} - X_t^n| &\le |\int_0^t (\sigma(X_s^n) - \sigma(X_s^{n-1}))dB_s| \\ &\quad + \sup_{s\le t} |\int_0^s (\sigma(X_u^n) - \sigma(X_u^{n-1}))dB_u| \\ &\quad +2\alpha \sup_{s\le t} |X_s^{n+1} - X_s^n|. \end{aligned} \quad (59)$$

Consequently,

$$\sup_{s\le t} |X_s^{n+1} - X_s^n| \le \frac{2}{1-2\alpha} \sup_{s\le t} |\int_0^s (\sigma(X_u^n) - \sigma(X_u^{n-1}))dB_u|. \quad (60)$$

By Burkhölder's inequality,

$$\begin{aligned} E[\sup_{s\le t} |X_s^{n+1} - X_s^n|^2] &\le C_\alpha E[\int_0^t (\sigma(X_s^n) - \sigma(X_s^{n-1}))^2 du] \\ &\le C_\alpha E[\int_0^t (X_s^n - X_s^{n-1})^2 du]. \end{aligned} \quad (61)$$

Thus, we deduce that for any fixed $T > 0$,

$$E[\sup_{s\leq T}|X_s^{n+1} - X_s^n|^2] \leq \frac{(C_{\alpha,T})^n}{n!}, \tag{62}$$

which yields

$$P(\sup_{s\leq T}|X_s^{n+1} - X_s^n| > \frac{1}{4^n}) \leq \frac{(4C_{\alpha,T})^n}{n!}. \tag{63}$$

Using the Borel-Cantelli lemma, it follows that X_t^n converges uniformly to a continuous, adapted process X on $[0,T]$ almost surely. Again by the Borel-Cantelli argument, it is also seen that $M_n(t) := \int_0^t \sigma(X_s^n)dB_s$ converges uniformly on $[0,T]$ to $M(t) = \int_0^t \sigma(X_s)dB_s$ almost surely. Thus, by (57), we see that L_t^n converges uniformly to a continuous non-decreasing process L on $[0,T]$ almost surely. Letting $n \to \infty$ in (57) gives

$$X_t = \int_0^t \sigma(X_s)dB_s + \alpha \max_{0\leq s\leq t} X_s + L_t. \tag{64}$$

To show that (X_t, L_t) is a solution to (45), we need to prove

$$\int_0^t \chi_{\{X_s=0\}}dL_s = L_t.$$

This will follow if we can show that for any $f \in C_0(0,\infty)$

$$\int_0^t f(X_s)dL_s = 0. \tag{65}$$

Indeed,

$$\int_0^t f(X_s)dL_s = \lim_{n\to\infty}\int_0^t f(X_s^n)dL_s^n = 0.$$

Next we show the uniqueness. Let (X_t^1, L_t^1),(X_t^2, L_t^2) be two solutions to the equation (45). Using similar arguments as above, it can be shown that

$$E[\sup_{s\leq t}|X_s^1 - X_s^2|^2] \leq C_\alpha E[\int_0^t (X_s^1 - X_s^2)^2 du]. \tag{66}$$

By Gronwall's inequality, it follows that $X^1 = X^2$, and hence $L^1 = L^2$.

Remark 4.4. Under the conditions in Theorem 4.3 the existence can also be obtained from Theorem 4.2 by letting $x \to 0$ and showing the corresponding solutions form a Cauchy sequence in the $L^2(\Omega)$-space.

Now, consider the stochastic differential equation:

$$X_t = x + \int_0^t \sigma(X_s) \circ dB_s + \alpha \max_{0\leq s\leq t} X_s + L_t, \tag{67}$$

where $\circ$ means the Stratonovich integration. A solution to this equation is defined as in Definition 4.1 with Ito integration replaced by the Stratonovich integration.

Theorem 4.5. Assume that there exists a constant $c > 0$ such that $\sigma(x) \geq c$, and $\frac{d\sigma(x)}{dx}$ is continuous. Then for any $0 \leq \alpha < 1$ and $x \geq 0$ there exists a unique solution $(X_t, L_t, t \geq 0)$ to equation (67).

Proof. We prove the theorem for $x = 0$. Other cases are similar. Introduce two functions $F_1(x)$, $F_2(x)$ on R defined by

$$F_1(x) = \int_0^x \frac{1}{\sigma(u)} du, \qquad F_2(x) = \int_0^x \sigma(F_2(u)) du.$$

Note that $F_1, F_2 \in C^2(R)$, $F_1 = F_2^{-1}$, and

$$F_1(x) = 0 \Leftrightarrow x = 0, \qquad F_1(x) \geq 0 \Leftrightarrow x \geq 0, \tag{68}$$

$$F_2(x) = 0 \Leftrightarrow x = 0, \qquad F_2(x) \geq 0 \Leftrightarrow x \geq 0. \tag{69}$$

It follows from section 3 (also [3]) that there is a unique solution, denoted by (X_t^0, L_t^0), to the equation

$$X_t^0 = B_t + \alpha \max_{0 \leq s \leq t} X_s^0 + L_t^0. \tag{70}$$

Define $X_t = F_2(X_t^0)$ and $L_t = \int_0^t \sigma(X_s) dL_s^0$. Put $m_t^0 = \max_{0 \leq s \leq t} X_s^0$. By It$\hat{o}'s$ formula, we have that

$$\begin{aligned} X_t &= \int_0^t F_2'(X_s^0) \circ dX_s^0 \\ &= \int_0^t \sigma(X_s) \circ dB_s + \alpha \int_0^t \sigma(X_s) dm_s^0 + L_t. \end{aligned} \tag{71}$$

Since F_2 is a monotone increasing function, it follows that

$$\max_{0 \leq s \leq t} X_s = \max_{0 \leq s \leq t} F_2(X_s^0) = F_2(m_t^0). \tag{72}$$

On the other hand,

$$\begin{aligned} F_2(m_t^0) &= \int_0^t F_2'(m_s^0) dm_s^0 = \int_0^t \sigma(F_2(m_s^0)) dm_s^0 \\ &= \int_0^t \sigma(F_2(X_s^0)) dm_s^0 = \int_0^t \sigma(X_s) dm_s^0, \end{aligned} \tag{73}$$

where we have used the fact that the measure dm^0 does not charge the set $\{s; m_s^0 > x_s^0\}$. Thus (71) becomes

$$X_t = \int_0^t \sigma(X_s) \circ dB_s + \alpha \max_{0 \leq s \leq t} X_s + L_t. \tag{74}$$

It is clear that $L_t, t \geq 0$ is a continuous , non-decreasing process. Moreover,

$$\int_0^t \chi_{\{0\}}(X_s) dL_s = \int_0^t \chi_{\{0\}}(X_s^0) dL_s = \int_0^t \chi_{\{0\}}(X_s^0) \sigma(X_s) dL_s^0 = L_t.$$

This shows that $(X_t, L_t, t \geq 0)$ is a solution to equation (67).
On the other hand, similar arguments also show that for any solution $(X_t, L_t, t \geq 0)$ to equation (67), $(X_t^0 = F_1(X_t), L_t^0 = \int_0^t \frac{1}{\sigma(X_s)} dL_s)$ is a solution to equation (70). Thus, the uniqueness of the solution to equation (67) is a consequence of the uniqueness of the solution to equation (70). The proof is complete.

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