Perturbation of Symmetric Markov Processes

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In Memory of Professor Martin L. Silverstein

Abstract

We present a path-space integral representation of the semigroup associated with the quadratic form obtained by a lower order perturbation of the L^2-infinitesimal generator $\mathcal{L}$ of a general symmetric Markov process. An illuminating concrete example for $\mathcal{L}$ is $\Delta_D - (-\Delta)^s_D$, where D is a bounded Euclidean domain in $\mathbb{R}^d$, $s \in]0,1[$, Δ_D is the Laplacian operator in D with zero Dirichlet boundary condition and $-(-\Delta)^s_D$ is the fractional Laplacian operator in D with zero exterior condition. The strong Markov process corresponding to $\mathcal{L}$ is a Lévy process that is the sum of Brownian motion in $\mathbb{R}^d$ and an independent symmetric $(2s)$-stable process in $\mathbb{R}^d$ killed upon exiting domain D. This probabilistic representation is a combination of Feynman-Kac and Girsanov formulas. Crucial to the development is to use the extension of Nakao's stochastic integral for zero-energy additive functionals and the associated Itô formula, both of which were recently developed in [3].

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1 Introduction

Let $A(x) := (a_{ij}(x))_{1\le i,j\le d}$ be a symmetric matrix-valued function on $\mathbb{R}^n$ that is uniformly elliptic and bounded. It is well-known that there is a conservative symmetric diffusion $\{\Omega, X, \mathbf{P}_x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$ with infinitesimal generator $\mathcal{L} := \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$. Moreover X has the following Fukushima's decomposition (see [10])

$$X_t = X_0 + M_t + N_t, \qquad t \ge 0, \tag{1.1}$$

where $M = (M^1, \cdots, M^d)$ is a square-integrable martingale additive functional of X with quadratic covariant $\langle M^i, M^j\rangle_t = \int_0^t a_{ij}(X_s)ds$ and $N = (N^1, \cdots, N^d)$ is a continuous additive functional of X locally of zero energy. Let b and $\widehat{b}$ be two $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$, and c a measurable function on $\mathbb{R}^d$ such that $|b| + |\widehat{b}| \in L^{p_1}(\mathbb{R}^d)$ for some $p_1 > d$ (resp. $p_1 \ge 2$) and $c \in L^{p_1}(\mathbb{R}^d)$ for some $p_2 > d/2$ (resp. $p_2 \ge 1$) under $d \ge 2$ (resp. $d = 1$). In [14], Lunt, Lyons and Zhang showed that the semigroup $\{T_t, t \ge 0\}$ of the following operator

$$\widetilde{\mathcal{L}}\varphi(x) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\varphi(x)}{\partial x_j}\right) + b(x)\cdot\nabla\varphi(x) - \operatorname{div}\left(\widehat{b}(x)\varphi(x)\right) + c(x)\varphi(x)$$

is given by

$$T_t f(x) = \mathbf{E}_x\left[Z_t f(X_t)\right], \tag{1.2}$$

where

$$\begin{aligned} Z_t \quad = \quad \exp\Bigg(&\int_0^t (A^{-1}b)(X_s)\cdot dM_s + \left(\int_0^t (A^{-1}\widehat{b})(X_s)\cdot dM_s\right)\circ r_t \\ &-\frac{1}{2}\int_0^t \left((b-\widehat{b})\cdot A^{-1}(b-\widehat{b})\right)(X_s)\,ds + \int_0^t c(X_s)\,ds\Bigg). \end{aligned} \tag{1.3}$$

Here r_t is the time-reversal operator on Ω from time $t > 0$, that is, given a path $\omega \in \Omega$,

$$r_t(\omega)(s) := \begin{cases} \omega(t-s), & \text{if } 0 \le s \le t, \\ \omega(0), & \text{if } s \ge t, \end{cases}$$

Recently, Fitzsimmons and Kuwae [9] extended the above result from symmetric diffusions X on $\mathbb{R}^d$ associated with bounded uniformly elliptic divergence form operators to general symmetric diffusions with no killings inside the state space. The purpose of this paper is to establish similar results for general symmetric Markov processes which may have discontinuous sample paths and killings inside the state space. An illuminating concrete example to keep in mind while reading this paper is that X being a discontinuous symmetric Lévy process killed upon exiting a domain, such as X is the sum of a Brownian motion on $\mathbb{R}^d$ with an independent symmetric α-stable process on $\mathbb{R}^d$ that is killed upon leaving an open ball. When X is a discontinuous symmetric Markov process, its martingale additive functional may be discontinuous. This discontinuity causes a lot of challenges when studying the transformation of X of the form Z_t given in analogous to (1.3).

One of the challenges is to define stochastic integral for zero-energy additive functionals of X and to establish the associated Itô formula. Nakao [17] has defined such kind of stochastic integral for a class of integrand but it is too restrictive for our investigation. In our recent paper [3], we established the needed stochastic integration theory for zero-energy additive functionals of X as well as the corresponding Itô formula via time-reversal technique. The main result of the current paper extends not only the results in [14] and [9] but also Feynman-Kac transforms by continuous additive functionals of zero energy studied in Chen and Zhang [8] and the pure-jump Girsanov transforms and discontinuous Feynman-Kac transforms in Chen [1] and in Chen and Song [5]-[6]. The following is a more detailed description of this paper.

Throughout this paper, $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ is an m-symmetric right Markov process on a Lusin space E, where m is a positive σ-finite measure with full topological support on E. Here $\Omega := D([0,\infty[\to E_\partial)$ is the totality of right-continuous, left-limited (*rcll*, for short) sample paths from $[0,\infty[$ to E_∂. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \partial\}$ be the life time of X. As usual, $\mathcal{F}_\infty$ and $\mathcal{F}_t$ are the minimal augmented σ-algebras obtained from $\mathcal{F}^0_\infty := \sigma\{X_s \mid 0 \leq s < \infty\}$, $\mathcal{F}^0_t := \sigma\{X_s \mid 0 \leq s \leq t\}$ under $\mathbf{P}_x$; see Section 3 below for more details. We sometimes use a filtration denoted by $(\mathcal{M}_t)$ on $(\Omega, \mathcal{M})$ in order to represent several filtrations, for example, $(\mathcal{F}^0_t)$, $(\mathcal{F}^0_{t+})$ on $(\Omega, \mathcal{F}^0_\infty)$, $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}_\infty)$ and others introduced later. We set $X_t(\omega) := \partial$ for $t \geq \zeta(\omega)$ and use θ_t to denote the shift operator defined by $\theta_t(\omega)(s) := \omega(t+s)$, $t, s \geq 0$. Let ω_∂ be the path starting from ∂. Then $\omega_\partial(s) \equiv \partial$ for all $s \in [0,\infty[$. Note that $\theta_{\zeta(\omega)}(\omega) = \omega_\partial$ for all $\omega \in \Omega$, $\{\omega_\partial\} \in \mathcal{F}^0_0 \subset \mathcal{F}^0_t$ for all $t > 0$ and $\mathbf{P}_x(\{\omega_\partial\}) \leq \mathbf{P}_x(X_0 = \partial) = 0$ for $x \in E$. For a Borel subset B of E, $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ (the *exit time* of B) is an $(\mathcal{F}_t)$-stopping time. If B is closed, then τ_B is an $(\mathcal{F}^0_{t+})$-stopping time. Also, ζ is an $(\mathcal{F}^0_t)$-stopping time because $\{\zeta \leq t\} = \{X_t = \partial\} \in \mathcal{F}^0_t$, $t \geq 0$.

The transition semigroup of X, $\{P_t, t \geq 0\}$, is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta], \qquad t \geq 0.$$

Each P_t may be viewed as an operator on $L^2(E;m)$; collectively these operators form a strongly continuous semigroup of self-adjoint contractions. The Dirichlet form associated with X is the bilinear form

$$\mathcal{E}(u,v) := \lim_{t \downarrow 0} \frac{1}{t}(u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \left\{ u \in L^2(E;m) \,\middle|\, \sup_{t>0}\, t^{-1}(u - P_t u, u)_m < \infty \right\}.$$

Here we use the notation $(f,g)_m := \int_E f(x)g(x)\, m(dx)$. Since $(\mathcal{E}, \mathcal{E})$ is a quasi-regular Dirichlet form, we know from [4] that $(\mathcal{E}, \mathcal{F})$ is quasi-homeomorphic to a regular Dirichlet form on a locally compact metric space. Thus without loss of generality, we may and do assume that X is an m-symmetric Hunt process on a locally compact metric space E, whose associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular in $L^2(E;m)$ and m is a positive Radon measure with full topological support on

E. For notions such as quasi-continuous, quasi-everywhere (abbreviated as q.e. or $\mathcal{E}$-q.e.), $\mathcal{E}$-nest, martingale additive functional, continuous additive functionals, $\mathcal{F}_{\rm loc}$, etc. we refer the reader to [10] and [15]. In particular, we recall that an increasing sequence of closed sets $\{F_n\}$ is called an $\mathcal{E}$-nest if $\cup_{n=1}^\infty \mathcal{F}_{F_n}$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$, where $\mathcal{F}_{F_n} := \{u \in \mathcal{F} : u = 0 \ m\text{-a.e. on } E \setminus F_n\}$. A function f is said to be locally in $\mathcal{F}$ (denoted as $f \in \mathcal{F}_{\rm loc}$) if there is an increasing sequence of finely open Borel sets $\{D_k, k \geq 1\}$ with $\cup_{k=1}^\infty D_k = E$ q.e. and for every $k \geq 1$, there is $f_k \in \mathcal{F}$ such that $f = f_k$ m-a.e. on D_k.

The main purpose of this paper is to establish a probabilistic representation (via a combination of Girsanov and Feynman-Kac transformations) of certain lower-order perturbations of the Dirichlet form $\mathcal{E}$. To discuss these perturbations we need to establish some notation.

A positive continuous additive functional (PCAF in abbreviation) of X (call it A) determines a measure $\mu = \mu_A$ on the Borel subsets of E via the formula

$$\int_E f(x)\mu(dx) =\uparrow \lim_{t\to 0} \frac{1}{t}\mathbf{E}_m \left[\int_0^t f(X_s)\, dA_s\right], \tag{1.4}$$

in which $f : E \to [0, \infty]$ is Borel measurable. The measure μ is necessarily *smooth*, in the sense that μ charges no exceptional set of X and there is an $\mathcal{E}$-nest $\{F_n\}$ of closed subsets of E such that $\mu(F_n) < \infty$ for each $n \in \mathbb{N}$. Conversely, given a smooth measure μ, there is a unique PCAF A^μ such that (1.4) holds with $A = A^\mu$. In the sequel we refer to this bijection between smooth measures and PCAFs as the *Revuz correspondence*, and to μ as the Revuz measure of A^μ.

A smooth measure ν is said to be of the *Hardy class* if there are constants $\delta > 0$ and $\gamma \geq 0$ such that

$$\int_E \tilde{u}^2\, d\nu \leq \delta \cdot \mathcal{E}(u, u) + \gamma \cdot (u, u)_m \qquad \text{for every } u \in \mathcal{F}.$$

It is well known that for every $u \in \mathcal{F}$, u has a quasi-continuous m-version $\tilde{u}$. As a rule we take u to be represented by its quasi-continuous version (when such exists), and drop the tilde from the notation. Let $\overset{\circ}{\mathcal{M}}$ and $\mathcal{N}$ denote, respectively, the space of MAFs of finite energy and the space of continuous additive functionals of zero energy. For $u \in \mathcal{F}$, the following Fukushima decomposition holds:

$$u(X_t) - u(X_0) = M_t^u + N_t^u \qquad \text{for } t \in [0, \infty[, \tag{1.5}$$

$\mathbf{P}_x$-a.s. for $\mathcal{E}$-q.e. $x \in E$, where $M^u \in \overset{\circ}{\mathcal{M}}$ and $N^u \in \mathcal{N}$.

If M is a locally square-integrable martingale additive functional (MAF) on $[\![0, \zeta[\![$ of X, then the process $\langle M\rangle$ (the dual predictable projection of $[M]$) is a PCAF, and the associated Revuz measure (as in (1.4)) is denoted by $\mu_{\langle M\rangle}$ (see [3]). More generally, if M^u is the martingale part in the Fukushima decomposition (1.5) of $u \in \mathcal{F}$, then $\langle M^u, M\rangle$ is a CAF locally of bounded variation, and we have the associated Revuz measure $\mu_{\langle M^u, M\rangle}$, which is locally the difference of smooth (positive) measures.

Now let M and $\widehat{M}$ be two locally square-integrable MAFs on $I(\zeta)$ such that $\mu_{\langle M\rangle}$ and $\mu_{\langle \widehat{M}\rangle}$ are of the Hardy class, and let A^μ be a CAF locally of bounded variation whose Revuz measure μ has total variation $|\mu|$ of the Hardy class. Here $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$, where ζ_i is the totally inaccessible part of ζ (see the comment before Definition 2.1). As the main result of this paper (Theorem 3.1), we show that under a suitable condition on the δ coefficients in the Hardy inequality for $\mu_{\langle M\rangle}$, $\mu_{\langle \widehat{M}\rangle}$, and $|\mu|$, the form perturbation $(\mathcal{Q},\mathcal{F})$ of $(\mathcal{E},\mathcal{F})$ defined by

$$\begin{aligned}\mathcal{Q}(f,g) &= \mathcal{E}(f,g)-\int_E f(x)\,\mu_{\langle M^g,\widehat{M}\rangle}(dx)-\int_E g(x)\,\mu_{\langle M^f,M\rangle}(dx)-\int_E f(x)g(x)\,\mu(dx)\\ &\qquad -\int_{E\times E} f(y)g(x)\varphi(x,y)\psi(y,x)N(x,dy)\mu_H(dx).\end{aligned}$$

determines a strongly continuous semigroup $\{T_t, t\geq 0\}$ of operators on $L^2(E;m)$, where φ and ψ are Borel functions bounded below and away from -1, coming from the jump part of M and $\widehat{M}$ respectively, and $\{T_t, t\geq 0\}$ admits the representation

$$T_t f(x) := \mathbf{E}_x\left[Z_t f(X_t)\right]$$

where

$$Z_t=\operatorname{Exp}(M_t+A^\mu_t+\langle M^c,\widehat{M}^c\rangle_t)\cdot\operatorname{Exp}(\widehat{M}_t)\circ r_t(1+\psi(X_t,X_{t-})),\quad \text{for } t<\zeta. \tag{1.6}$$

Here r_t is the time-reversal operator defined on the path space Ω of X as follows: Given a path $\omega\in\{t<\zeta\}$,

$$r_t(\omega)(s)=\begin{cases}\omega((t-s)_-), & \text{if } 0\leq s\leq t,\\ \omega(0), & \text{if } s\geq t,\end{cases}$$

in which, for $r>0$, $\omega(r_-):=\lim_{s\uparrow r}\omega(s)$. (The restriction of the measure $\mathbf{P}_m$ to $\mathcal{F}_t$ is invariant under r_t on $\Omega\cap\{\zeta>t\}$.) Also in (1.6), the symbol Exp denotes the familiar Doléans-Dade stochastic exponential: if Y is a semimartingale with $Y_0=0$, then $Z=\operatorname{Exp}(Y)$ is the unique solution of the SDE

$$Z_t=1+\int_0^t Z_{s-}\,dY_s,$$

and is given explicitly by the formula

$$\operatorname{Exp}(Y_t)=\exp\left(Y_t-\frac{1}{2}\langle Y^c,Y^c\rangle_t\right)\prod_{s\in]0,t]}(1+\Delta Y_s)e^{-\Delta Y_s}.$$

As mentioned previously, this result was first obtained by Lunt, Lyons and Zhang [14] when X is a conservative symmetric diffusion on $\mathbb{R}^d$ whose L^2-infinitesimal generator is a bounded uniformly elliptic divergence form operator, and then by Fitzsimmons and Kuwae [9] in case X is a diffusion process on a Lusin space E with no killing inside E. The jumps of X, as allowed in the context of the present paper, complicate the study. It calls for first to develop certain aspects of the stochastic calculus of symmetric Markov processes (in particular a general enough version of Itô's formula) to deal with these complications, which have been addressed very recently in our separate paper [3]. See section 2 below for a quick review

A special case ($\widehat{M}=0$ and M purely discontinuous) of the above result was obtained by Chen and Song [5] (see also [1] and [6]) in the broader context of "nearly symmetric" right Markov processes, under somewhat more stringent conditions on $\mu_{\langle M\rangle}$ and μ (in [5] $\mu_{\langle M\rangle}$ is assumed to be in the Kato class while μ is only assumed to satisfy the condition $\|G\mu^+\|_\infty < 1$). Recall that the Revuz measure ν of a PCAF A^ν is said to be of the Kato class provided

$$\lim_{t\to 0} \|\mathbf{E}.[A^\nu_t]\|_\infty = 0,$$

and that the Kato class is a subclass of the Hardy class. We write $\mathbf{K}(X)$ for the Kato class and define $\mathbf{K}_0(X) := \{\nu \in \mathbf{K}(X) \mid \nu(E) < \infty\}$.

The remainder of the paper is organized as follows. A quick review of the needed stochastic integration with respect to a continuous additive functional of zero energy is given in Section 2. Section 3 contains the statement of our main result (Theorem 3.1) as well as some auxiliary lemmas needed for its proof. The proof of Theorem 3.1 is completed in section 4. In section 5 we show that the Feynman-Kac formula for zero-energy CAF perturbations (Theorem 2.1 of Chen and Song [6]) can be deduced from Theorem 3.1 of the present paper.

2 Stochastic integral for Dirichlet processes

In this section, we give a quick review of Nakao's [17] definition of stochastic integral with respect to a additive functional of zero energy and our time-reversal approach of stochastic integral for Dirichlet processes developed in [3].

Let $(N(x,dy), H_t)$ be a Lévy system for X; that is, $N(x,dy)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ and H_t is a PCAF with bounded 1-potential such that for any nonnegative Borel function ϕ on $E_\partial \times E_\partial$ vanishing on the diagonal and any $x \in E_\partial$,

$$\mathbf{E}_x\left(\sum_{s\le t}\phi(X_{s-}, X_s)\right) = \mathbf{E}_x\left(\int_0^t \int_{E_\partial} \phi(X_s, y) N(X_s, dy) dH_s\right).$$

To simplify notation, we will write

$$N\phi(x) := \int_{E_\partial} \phi(x,y) N(x,dy)$$

and

$$(N\phi * H)_t := \int_0^t N\phi(X_s) dH_s.$$

Let μ_H be the Revuz measure of the PCAF H. Then the jumping measure J and the killing measure κ of X are given by

$$J(dx,dy) = \frac{1}{2} N(x,dy)\mu_H(dx), \quad \text{and} \quad \kappa(dx) = N(x,\{\partial\})\mu_H(dx).$$

These measure feature in the Beurling-Deny decomposition of $\mathcal{E}$: for $f, g \in \mathcal{F}$,

$$\mathcal{E}(f,g) = \mathcal{E}^{(c)}(f,g) + \int_{E\times E} (f(x)-f(y))(g(x)-g(y)) J(dx,dy) + \int_E f(x)g(x)\kappa(dx),$$

where $\mathcal{E}^{(c)}$ is the strongly local part of $\mathcal{E}$.

For $u \in \mathcal{F}$, the martingale part M_t^u in (1.5) can be decomposed as

$$M_t^u = M_t^{u,c} + M_t^{u,j} + M_t^{u,\kappa}, \ {}^\forall t \in [0, \infty[, \mathbf{P}_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E,$$

where $M_t^{u,c}$ is the continuous part of martingale M^u, and

$$\begin{aligned} M_t^{u,j} &= \lim_{\varepsilon \downarrow 0} \left\{ \sum_{0<s\le t} (u(X_s) - u(X_{s-})) 1_{\{|u(X_s)-u(X_{s-})|>\varepsilon\}} 1_{\{s<\zeta\}} \right. \\ &\qquad \left. - \int_0^t \left(\int_{\{y\in E:\, |u(y)-u(X_s)|>\varepsilon\}} (u(y) - u(X_s)) N(X_s, dy) \right) dH_s \right\}, \\ M_t^{u,\kappa} &= \int_0^t u(X_s) N(X_s, \{\partial\}) dH_s - u(X_{\zeta-}) 1_{\{t \ge \zeta_i\}}, \end{aligned}$$

are the jump and killing parts of M^u, respectively. See Theorem A.3.9 of [10]. The limit in the expression for $M^{u,j}$ is in the sense of convergence in the norm of $\overset{\circ}{\mathcal{M}}$ and of convergence in probability under $\mathbf{P}_x$ for $\mathcal{E}$-q.e. $x \in E$ (see [10]). The Revuz measure $\mu_{\langle M^u \rangle}$ of $\langle M^u \rangle$ will usually be denoted by $\mu_{\langle u \rangle}$.

Let $\mathcal{N}^* \subset \mathcal{N}$ denote the class of continuous additive functionals of the form $N^u + \int_0^\cdot g(X_s) ds$ for some $u \in \mathcal{F}$ and $g \in L^2(E; m)$. Nakao [17] constructed a certain linear map Γ from $\overset{\circ}{\mathcal{M}}$ into $\mathcal{N}^*$ in the following way. It is shown in [17] that, for every $Z \in \overset{\circ}{\mathcal{M}}$, there is a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}_1(w, f) = \frac{1}{2} \mu_{\langle M^f + M^{f,\kappa},\, Z \rangle}(E) \quad \text{for every } f \in \mathcal{F}. \tag{2.1}$$

This unique w is denoted by $\gamma(Z)$. The operator Γ is defined by

$$\Gamma(Z)_t = N_t^{\gamma(Z)} - \int_0^t \gamma(Z)(X_s) ds \qquad \text{for } Z \in \overset{\circ}{\mathcal{M}}. \tag{2.2}$$

It is shown in Nakao [17] that $\Gamma(Z)$ can be characterized by the following equation

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_{g \cdot m} [\Gamma(Z)_t] = -\frac{1}{2} \mu_{\langle M^g + M^{g,\kappa},\, Z \rangle}(E) \qquad \text{for every } g \in \mathcal{F}_b. \tag{2.3}$$

So in particular we have $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Nakao [17] used the operator Γ to define a stochastic integral

$$\int_0^t f(X_s) dN_s^u := \Gamma(f * M^u)_t - \frac{1}{2} \langle M^{f,c} + M^{f,j},\, M^{u,c} + M^{u,j} \rangle_t, \tag{2.4}$$

where $u \in \mathcal{F}$, $f \in \mathcal{F} \cap L^2(E; \mu_{\langle u \rangle})$ and $(f * M^u)_t := \int_0^t f(X_{s-}) dM_s^u$. If we define

$$\widetilde{\mathcal{N}} := \{ N \in \mathcal{N} \mid N = N^u + A^\mu \text{ for some } u \in \mathcal{F} \text{ and some signed smooth measure } \mu \},$$

then we see by (2.2) that $\int_0^\cdot f(X_s) dN_s^u \in \widetilde{\mathcal{N}}$ if $u \in \mathcal{F}$ and $f \in \mathcal{F} \cap L^2(E; \mu_{\langle u \rangle})$. However Nakao's definition of stochastic integral places restrictions on the integrand $f(X_t)$ and on the integrator

N^u that are too stringent for our study of the perturbation theory of general symmetric Markov processes. We now recall the definition of the stochastic integral introduced in our recent paper [3] using time-reversal.

For T an $(\mathcal{F}_t)$-stopping time, we will use T_p and T_i to denote, respectively, the predictable and totally inaccessible parts of the given $(\mathcal{F}_t)$-stopping time T of X, that is, $T_p := T_{\Lambda_p}$ and $T_i := T_{\Lambda_i}$, where $\Lambda_p := \{T < \infty, X_{T-} = X_T\}$, $\Lambda_i := \{T < \infty, X_{T-} \in E, X_{T-} \neq X_T\}$ (see Theorem 44.5 in M. Sharpe [18]). It is shown in [18] that T_p and T_i are $(\mathcal{F}_t)$-stopping times if T is an $(\mathcal{F}_t)$-stopping time. We set $I(T) := [\![0, T[\![\cup [\![T_i]\!]$.

For a locally square-integrable MAF M_t on $I(\zeta)$, it is shown in [3] (cf. [2, Lemma 3.2]) that there is a Borel function φ on $E_\partial \times E_\partial$ with $\varphi(x,x) = \varphi(\partial, x) = 0$ for all $x \in E_\partial$ so that

$$M_t - M_{t-} = \varphi(X_{t-}, X_t) \qquad \text{for every } t \in]0, \zeta_p[, \ \mathbf{P}_m\text{-a.s.}$$

Such a function φ is unique up to a measure J^*-null set on $E_\partial \times E_\partial$, where J^* denotes the measure $\frac{1}{2}N(x,dy)\mu_H(dx)$ on $E_\partial \times E_\partial$. We will call φ the jump function of M.

Definition 2.1 Let M be a locally square-integrable MAF on $I(\zeta)$ with jump function φ. Assume

$$\int_0^t \int_E \left(\widehat{\varphi}^2 1_{\{|\widehat{\varphi}| \le 1\}} + |\widehat{\varphi}| 1_{\{|\widehat{\varphi}| > 1\}}\right)(X_s, y) N(X_s, dy) dH_s < \infty, \ ^\forall t < \zeta, \quad \mathbf{P}_x\text{-a.s.} \tag{2.5}$$

for $\mathcal{E}$-q.e. $x \in E$, where $\widehat{\varphi}(x,y) := \varphi(x,y) + \varphi(y,x)$. Define, $\mathbf{P}_m$-a.s. on $[0, \zeta[$,

$$\Lambda(M)_t := -\frac{1}{2}\left(M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t\right) \qquad \text{for } t \in [0, \ \zeta[, \tag{2.6}$$

where K_t is the purely discontinuous local MAF on $I(\zeta)$ with

$$K_t - K_{t-} = -\widehat{\varphi}(X_{t-}, X_t), \quad t < \zeta, \quad \mathbf{P}_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E. \tag{2.7}$$

It is shown in [3, Theorem 3.5] that $\Lambda(M) = \Gamma(M)$ when M is a MAF of X having finite energy. In other words, the above Λ operator extends Nakao's Γ operator.

Note that for $f \in \mathcal{F}_{\rm loc}$, $M^{f,c}$ is well defined as a continuous MAF on $[0, \zeta[$ of locally finite energy. Moreover, for $f \in \mathcal{F}_{\rm loc}$ and a locally square-integrable MAF M on $I(\zeta)$,

$$t \mapsto (f * M)_t := \int_0^t f(X_{s-}) dM_s$$

is a locally square-integrable MAF on $I(\zeta)$. For a locally square-integrable MAF M on $I(\zeta)$, denote by M^c its continuous part, which is also a locally square-integrable MAF on $I(\zeta)$ (see Theorem 8.23 in [11]). The following definition of stochastic integral is introduced in [3].

Definition 2.2 (Stochastic integral) Suppose that M is a locally square-integrable MAF on $I(\zeta)$ and $f \in \mathcal{F}_{\rm loc}$. Let $\varphi : E_\partial \times E_\partial \to \mathbb{R}$ be a jump function for M, and assume that φ satisfies condition (2.5). Define $\mathbf{P}_m$-a.s. on $[0, \zeta[$ by,

$$\begin{aligned}
&\int_0^t f(X_{s-})\, d\Lambda(M)_s \\
&:= \quad \Lambda(f * M)_t - \frac{1}{2}\langle M^{f,c},\, M^c\rangle_t + \frac{1}{2}\int_0^t \int_E (f(y) - f(X_s))\varphi(y, X_s) N(X_s, dy) dH_s, \qquad (2.8)
\end{aligned}$$

whenever $\Lambda(f * M)$ is well defined and the third term in the right hand side of (2.8) is absolutely convergent.

It is shown in [3, Remark 3.8(ii) and Theorem 4.6] that the above defined stochastic integral extends Nakao's definition of stochastic integral (2.4) and enjoys a generalized Itô's formula.

3 Perturbation

Recall that a smooth measure μ is in the *Hardy class* (write $\mu \in \mathbf{H}(X)$) if there are constants $\delta \in]0, \infty[$ and $\gamma \in [0, \infty[$ such that

$$\int_E u^2 d\mu \le \delta \mathcal{E}(u, u) + \gamma \int_E u^2 dm, \quad {}^\forall u \in \mathcal{F}. \qquad (3.1)$$

A well-known sufficient condition for $\mu \in \mathbf{H}(X)$ is that for some $\delta > 0$ and $\beta \ge 0$ the β-potential $U^\beta \mu$ is bounded above $\mathcal{E}$-q.e. by δ, in which case $\gamma = \delta\beta$ does the job in (3.1).

Let M, $\widehat{M}$ be two locally square-integrable MAFs on $I(\zeta)$. Let M^c and $\widehat{M}^c$ denote the continuous parts of M and $\widehat{M}$ respectively, and let φ and ψ be jump functions for M and $\widehat{M}$ respectively; thus φ and ψ are Borel functions on $E_\partial \times E_\partial$, vanishing on the diagonal, such that

$$M_t - M_{t-} = \varphi(X_{t-}, X_t) \quad \text{and} \quad \widehat{M}_t - \widehat{M}_{t-} = \psi(X_{t-}, X_t), \quad {}^\forall t \in]0, \zeta_p[\quad \mathbf{P}_m\text{-a.s.}$$

We assume $\varphi > -1$ and $\psi > -1$ on $E_\partial \times E_\partial$. Let $\langle M\rangle$ and $\langle \widehat{M}\rangle$ denote the dual predictable projections of $[M]$ and $[\widehat{M}]$ respectively. Note that

$$\langle M\rangle_t = \langle M^c\rangle_t + \int_0^t \int_{E_\partial} \varphi(X_s, y)^2 N(X_s, dy) dH_s, \qquad t < \zeta,$$

and

$$\langle \widehat{M}\rangle_t = \langle \widehat{M}^c\rangle_t + \int_0^t \int_{E_\partial} \psi(X_s, y)^2 N(X_s, dy) dH_s, \qquad t < \zeta.$$

Let μ be a signed smooth measure; thus μ uniquely determines a continuous additive functional A^μ of bounded variation on each compact time interval. Let $\mu_{\langle M\rangle}$ and $\mu_{\langle \widehat{M}\rangle}$ be the smooth measures associated with the PCAFs $\langle M\rangle_t$ and $\langle \widehat{M}\rangle_t$. Then

$$\mu_{\langle M\rangle} = \mu_{\langle M^c\rangle} + N(\varphi^2)\mu_H \qquad \text{and} \qquad \mu_{\langle \widehat{M}\rangle} = \mu_{\langle \widehat{M}^c\rangle} + N(\psi^2)\mu_H.$$

We assume $\mu_{\langle M\rangle}$, $\mu_{\langle\widehat{M}\rangle}$ and $|\mu|$ are in $\mathbf{H}(X)$. Let $\delta(\langle M\rangle)$, $\delta(\langle\widehat{M}\rangle)$, $\delta(A^{\mu^+})$, $\delta(\varphi^2)$ and $\delta(\psi^2)$ denote the coefficient of $\mathcal{E}(u)$ and $\gamma(\langle M\rangle)$, $\gamma(\langle\widehat{M}\rangle)$, $\gamma(A^{\mu^+})$, $\gamma(\varphi^2)$ and $\gamma(\psi^2)$ the coefficient of $\|u\|_2^2$ in the estimate (3.1) for $\mu_{\langle M\rangle}$, $\mu_{\langle\widehat{M}\rangle}$, μ^+, $N(1_{E\times E}\cdot\varphi^2)\mu_H$ and $N(1_{E\times E}\cdot\psi^2)\mu_H$, respectively. We assume that

$$\delta_0 := \sqrt{2\delta(\langle M\rangle)} + \sqrt{2\delta(\langle\widehat{M}\rangle)} + \delta(A^{\mu^+}) + \sqrt{\delta(\varphi^2)\delta(\psi^2)} < 1. \tag{3.2}$$

Given these elements, we define a quadratic form $\mathcal{Q}$ on $\mathcal{F}$: For $f, g \in \mathcal{F}$,

$$\begin{aligned}\mathcal{Q}(f,g) \quad := \quad & \mathcal{E}(f,g) - \int_E g d\mu_{\langle M^f, M\rangle} - \int_E f d\mu_{\langle M^g, \widehat{M}\rangle} - \int_E fg d\mu \\ & - \int_{E\times E} f(y)g(x)\varphi(x,y)\psi(y,x)N(x,dy)\mu_H(dx).\end{aligned} \tag{3.3}$$

It is easy to check that there is a constant $C > 0$ that

$$|\mathcal{Q}(f,g)| \le C\mathcal{E}_1(f,f)^{1/2}\mathcal{E}_1(g,g)^{1/2}, \quad f,g \in \mathcal{F}. \tag{3.4}$$

Moreover,

$$\mathcal{Q}_\alpha(f,f) \ge (1-\delta_0)\mathcal{E}(f,f) + (\alpha - \alpha_0)\|f\|_2^2, \quad f \in \mathcal{F}, \tag{3.5}$$

where

$$\begin{aligned}\alpha_0 \quad := \quad & \gamma(\langle M\rangle)\sqrt{2/\delta(\langle M\rangle)} + \gamma(\langle\widehat{M}\rangle)\sqrt{2/\delta(\langle\widehat{M}\rangle)} \\ & + \gamma(A^{\mu^+}) + \sqrt{\delta(\varphi^2)\delta(\psi^2)}\left\{\frac{\gamma(\varphi^2)}{\delta(\varphi^2)} \vee \frac{\gamma(\psi^2)}{\delta(\psi^2)}\right\}.\end{aligned}$$

The quadratic form $(\mathcal{Q}, \mathcal{F})$ is closed as a form on $L^2(E;m)$. Standard resolvent theory now yields the existence of a strongly continuous semigroup $(Q_t)_{t\ge 0}$ of operators on $L^2(E;m)$ with $\|Q_t\|_{2\to 2} \le e^{\alpha_0 t}$ for all $t \ge 0$.

Define a multiplicative functional $Z = (Z_t)$ by

$$Z_t := \mathrm{Exp}(\widehat{M}_t) \circ r_t\, \mathrm{Exp}\left(M_t + A_t^\mu + \langle M^c, \widehat{M}^c\rangle_t\right)\,(1 + \psi(X_t, X_{t-})). \tag{3.6}$$

and an operator

$$T_t f(x) := \mathbf{E}_x\left[Z_t\, f(X_t)\right]. \tag{3.7}$$

The main result of this paper is the following.

Theorem 3.1 *Assume that $\mu_{\langle M\rangle}, \mu_{\langle\widehat{M}\rangle}$ and $|\mu|$ are all in the Hardy class $\mathbf{H}(X)$ and that δ_0 defined in (3.2) is less than 1. Then $\{T_t, t \ge 0\}$ defined by (3.7) coincides with the strongly continuous semigroup $\{Q_t, t \ge 0\}$ on $L^2(E;m)$ associated with $(\mathcal{Q}, \mathcal{F})$.*

The rest of this section is devoted to the statement and proof of two lemmas needed for the proof of Theorem 3.1.

Lemma 3.2 (i) *If $\mu_{\langle M^c\rangle}, \mu_{\langle \widehat{M}^c\rangle}$, $|\mu|$, $N(|\varphi|)\mu_H$ and $N(|\psi|)\mu_H$ are measures in $\mathbf{K}(X)$, then the semigroup $\{T_t, t\geq 0\}$ defined by (3.7) is a bounded linear operator in $L^2(E;m)$.*

(ii) *Let F be a closed set and G its fine interior under X. If*

$$1_F(\mu_{\langle M^c\rangle}+\mu_{\langle \widehat{M}^c\rangle}+|\mu|+N(|\varphi|)\mu_H+N(|\psi|)\mu_H)\in\mathbf{K}(X),$$

and if $\Lambda(\widehat{M}^c)_t=N_t^{\rho}-\int_0^t\rho(X_s)ds$ $\mathbf{P}_m$-a.s. on $\{t<\tau_G\}$ for some $\rho\in\mathcal{F}$ bounded on G such that $1_F\mu_{\langle\rho\rangle}\in\mathbf{K}(X)$, then there exists a constant $k>0$ such that for non-negative $f,g\in L^2(G;m)$

$$\mathbf{E}_m\left[f(X_t)g(X_0)\sup_{s\in[0,\,t\wedge\tau_G[}Z_s\right]\leq k\,e^{k\,t}\|f\|_2\|g\|_2\qquad \textit{for } t\geq 0.$$

Proof. (i): Since $\log(1+t)\leq t^+(:=t\vee 0)$, for $t<\zeta$

$$\begin{aligned}
\mathrm{Exp}\,(M_t) &= \exp\left(M_t^c-\frac{1}{2}\langle M^c\rangle_t+M_t^d+\sum_{0<s\leq t}(\log(1+\varphi(X_{s-},X_s))-\varphi(X_{s-},X_s))\right)\\
&= \exp\left(M_t^c-\frac{1}{2}\langle M^c\rangle_t-\int_0^t N(\varphi)(X_s)dH_s+\sum_{0<s\leq t}\log(1+\varphi(X_{s-},X_s))\right)\\
&\leq \exp\left(M_t^c-\frac{1}{2}\langle M^c\rangle_t+\int_0^t N(\varphi^-)(X_s)dH_s+\sum_{0<s\leq t}\varphi^+(X_{s-},X_s)\right). \qquad (3.8)
\end{aligned}$$

where we use the fact that $m_t:=\sum_{0<s\leq t}\varphi(X_{s-},X_s)-\int_0^t N(\varphi)(X_s)dH_s$ is a purely discontinuous martingale and coincides with M_t^d because $N(|\varphi|)\mu_H\in\mathbf{K}(X)$. Similarly

$$\begin{aligned}
&\mathrm{Exp}\left(\widehat{M}_t\right)\circ r_t\cdot(1+\overline{\psi}(X_{t-},X_t))\\
&= \exp\left(\widehat{M}_t^c\circ r_t-\frac{1}{2}\langle\widehat{M}^c\rangle_t-\int_0^t N(\psi)(X_s)dH_s+\sum_{0<s\leq t}\log(1+\overline{\psi}(X_{s-},X_s))\right)\\
&\leq \exp\left(\widehat{M}_t^c\circ r_t-\frac{1}{2}\langle\widehat{M}^c\rangle_t+\int_0^t N(\psi^-)(X_s)dH_s+\sum_{0<s\leq t}\overline{\psi}^+(X_{s-},X_s)\right), \qquad (3.9)
\end{aligned}$$

where $\overline{\psi}(x,y):=\psi(y,x)$, $\overline{\psi}^+(x,y):=\psi^+(y,x)$. Decompose the CAF A^μ as the difference $A^{\mu^+}-A^{\mu^-}$ of PCAFs with mutually singular Revuz measures μ^+ and μ^-, respectively. Then $\mu^+\leq|\mu|$, and because $\mu_{\langle M^c\rangle}+\mu_{\langle\widehat{M}^c\rangle}+|\mu|+N(|\varphi|)\mu_H+N(|\psi|)\mu_H\in\mathbf{K}(X)$, so also

$$\eta:=\frac{9}{2}[\mu_{\langle M^c\rangle}+\mu_{\langle\widehat{M}^c\rangle}]+3\mu^++3N(\varphi^-)\mu_H+3N(\psi^-)\mu_H\in\mathbf{K}(X). \qquad (3.10)$$

Let f and g be non-negative elements of $L^2(E;m)$. Then by Hölder's inequality and the expression (3.6) for Z_t,

$$\begin{aligned}
\mathbf{E}_m[g(X_0)Z_tf(X_t)] &\leq \mathbf{E}_m[g(X_0)^2e^{3M_t^c-(9/2)\langle M^c\rangle_t}]^{1/3}\\
&\quad\times\mathbf{E}_m[f(X_t)^2e^{3\widehat{M}_t^c\circ r_t-(9/2)\langle\widehat{M}^c\rangle_t}]^{1/3}\mathbf{E}_m[g(X_0)e^{B_t+D_t}f(X_t)]^{1/3},
\end{aligned} \qquad (3.11)$$

where $D_t := 3\sum_{0<s\le t}\overline{\psi}^+(X_{s-},X_s)$ and

$$B_t := \frac{3}{2}\langle M^c\rangle_t + \frac{3}{2}\langle \widehat{M}^c\rangle_t + 3A_t^{\mu^+} + 3\int_0^t N(\varphi^- + \psi^-)(X_s)dH_s + 3\sum_{0<s\le t}\varphi^+(X_{s-},X_s)$$

is the sum of the PCAF associated with the Revuz measure η and the discontinuous increasing AF $3\sum_{0<s\le t}\varphi^+(X_{s-},X_s)$. Note that $\widehat{D}_t := 3\sum_{0<s\le t}\psi^+(X_{s-},X_s) = D_t\circ r_t$ $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$.

Now $e^{3M_t^c-(9/2)\langle M^c\rangle_t}$ is a positive supermartingale, so $\mathbf{E}_x[e^{3M_t^c-(9/2)\langle M^c\rangle_t}]\le 1$ for $\mathcal{E}$-q.e. $x\in E$. Thus the first factor on the right side of (3.11) is no bigger than $\|g\|_2^{2/3}$. Because $\langle\widehat{M}^c\rangle$ is even, the middle factor on the right side of (3.11) is equal to

$$\mathbf{E}_m[f(X_t)^2 e^{3\widehat{M}_t^c\circ r_t-(9/2)\langle\widehat{M}^c\rangle_t\circ r_t}]^{1/3} = \mathbf{E}_m[f(X_0)^2 e^{3\widehat{M}_t^c-(9/2)\langle\widehat{M}^c\rangle_t}]^{1/3}\le\|f\|_2^{2/3}, \qquad (3.12)$$

because $e^{3\widehat{M}_t^c-(9/2)\langle\widehat{M}^c\rangle_t}$ is also a positive supermartingale. Finally, by Proposition 2.3 in Chen and Song [5], the cube of the last factor in (3.11) is estimated by

$$\begin{aligned}
\mathbf{E}_{gm}[e^{2B_t}f(X_t)]^{1/2}\mathbf{E}_{gm}[e^{2D_t}f(X_t)]^{1/2} &= \mathbf{E}_m[e^{2B_t}f(X_t)g(X_0)]^{1/2}\mathbf{E}_m[e^{2D_t\circ r_t}f(X_0)g(X_t)]^{1/2}\\
&\le \|\mathbf{E}_\cdot[e^{4B_t}]\|_\infty^{1/4}\,\|\mathbf{E}_\cdot[e^{4\widehat{D}_t}]\|_\infty^{1/4}\,\|f\|_2\,\|g\|_2\\
&\le k_0e^{k_0t}\,\|f\|_2\,\|g\|_2
\end{aligned}$$

for some $k_0>0$. Hence, (3.10) implies that (3.11) is no bigger than $k_0^{1/3}e^{k_0t/3}\|f\|_2^{1/3}\|g\|_2^{1/3}$. Combining these estimates we find that

$$\mathbf{E}_m[f(X_t)Z_tg(X_0)]\le k_0^{1/3}\cdot e^{k_0t/3}\|f\|_2\|g\|_2, \qquad (3.13)$$

which proves the assertion.

(ii): By (3.8) and (3.9), we have that $\mathbf{P}_m$-a.s. on $\{t<\tau_G\}$

$$\begin{aligned}
Z_t &\le \exp\left(-2\Lambda(\widehat{M}^c)_t\right)\operatorname{Exp}\left(M_t^c-\widehat{M}_t^c\right)\\
&\quad\times\exp\left(\sum_{0<s\le t}(\varphi^+ + \overline{\psi}^+)(X_{s-},X_s)\right)\exp\left(A_t^\mu+\int_0^t N(\varphi^-+\psi^-)(X_s)dH_s\right)\\
&= \exp\left(2\rho(X_0)-2\rho(X_t)+2\int_0^t\rho(X_s)ds+2M_t^\rho\right)\operatorname{Exp}\left(M_t^c-\widehat{M}_t^c\right)\\
&\quad\times\exp\left(\sum_{0<s\le t}(\varphi^+ + \overline{\psi}^+)(X_{s-},X_s)\right)\exp\left(A_t^\mu+\int_0^t N(\varphi^-+\psi^-)(X_s)dH_s\right).
\end{aligned}$$

Then, $\mathbf{P}_m$-a.s. on $\{t < \tau_G\}$

$$\sup_{0\le s<t} Z_s \le \exp\left[(4+2t)\|\rho\|_{G,\infty}\right] \sup_{0\le s<t} \exp\left[2(1_F * M^\rho)_s\right] \tag{3.14}$$

$$\times \sup_{0\le s<t} \exp\left((1_F * K^c)_s - \frac{1}{2}\langle 1_F * K^c\rangle_s\right) \tag{3.15}$$

$$\times \exp\left(\sum_{0<s\le t} 1_F(X_{s-})\overline{\psi}^+(X_{s-},X_s)\right) \tag{3.16}$$

$$\times \exp\left((1_F A^{\mu^+})_t + \int_0^t 1_F(X_s)N(\varphi^- + \psi^-)(X_s)dH_s + \sum_{0<s\le t} 1_F(X_{s-})\varphi^+(X_{s-},X_s)\right), \tag{3.17}$$

where $K^c := M^c - \widehat{M}^c$ and $1_F * K^c_t := \int_0^t 1_F(X_{s-})dK^c_s$ and $(1_F A^{\mu^+})_t := \int_0^t 1_F(X_s)dA^{\mu^+}_s$. Applying Doob's inequality to the submartingale $\exp(1_F * M^\rho)_t$ together with Lemma 4.1(i) in Chen and Zhang [8] we see that, because $1_F\mu_{\langle\rho\rangle} \in \mathbf{K}(X)$, the expectation of the eighth power of the second factor of (3.14) is estimated by

$$\begin{aligned}\left\|\mathbf{E}_\cdot\left[\sup_{0\le s<t} \exp\left(16(1_F * M^\rho)_s\right)\right]\right\|_\infty &\le \left(\frac{16}{15}\right)^{16} \left\|\mathbf{E}_\cdot\left[\exp\left(16(1_F * M^\rho)_t\right)\right]\right\|_\infty \\ &\le k_1 e^{k_1 t}\end{aligned}$$

for some $k_1 > 0$. Since $1_F\mu_{\langle K^c\rangle} \in \mathbf{K}(X)$, $\exp\left((1_F * K^c) - \frac{1}{2}\langle 1_F * K^c\rangle\right)$ is a martingale. Applying Doob's inequality to $\exp\left((1_F * K^c) - \frac{1}{2}\langle 1_F * K^c\rangle\right)$, the expectation of the eighth power of (3.15) is estimated by

$$\begin{aligned}&\mathbf{E}_x\left[\left|\sup_{0\le s\le t} \exp((1_F * K^c)_s - \tfrac{1}{2}\langle 1_F * K^c\rangle_s)\right|^8\right] \\ \le\ & \left(\frac{8}{7}\right)^8 \mathbf{E}_x\left[\exp(8(1_F * K^c)_t - 4\langle 1_F * K^c\rangle_t)\right] \\ \le\ & \left(\frac{8}{7}\right)^8 \mathbf{E}_x[\exp(16(1_F * K^c)_t - 128\langle 1_F * K^c\rangle_t)]^{1/2}\, \mathbf{E}_x[\exp(120\langle 1_F * K^c\rangle_t)]^{1/2} \\ \le\ & \left(\frac{8}{7}\right)^8 \sqrt{k_2} e^{(k_2/2)t}\end{aligned} \tag{3.18}$$

for some $k_2 > 0$, because $\exp(16(1_F * K^c)_t - 128\langle 1_F * K^c\rangle_t)$ is a martingale. Noting $1_F N(|\varphi| + |\psi|)\mu_H \in \mathbf{K}(X)$, by using Proposition 2.3 in [8] again, the expectations of the eighth power of (3.16) (after time reversion with respect to the part process on G) and (3.17) are estimated by $k_3 e^{k_3 t}$ for some $k_3 > 0$. Denote by $C_t^{(1)}$, $C_t^{(2)}$, $C_t^{(3)}$ and $C_t^{(4)}$, the second factor of (3.14), (3.15), (3.16) and (3.17) respectively. Then

$$\mathbf{E}_{1_G m}\left[f(X_t)g(X_0) \sup_{0\le s<t} Z_s : t < \tau_G\right] \le e^{(4+2t)\|\rho\|_{G,\infty}} \prod_{i=1}^4 \mathbf{E}_{1_G m}[|C_t^{(i)}|^4 f(X_t)g(X_0) : t < \tau_G]^{1/4}.$$

For $i = 1, 2, 4$,

$$\mathbf{E}_{1_G m}\left[|C_t^{(i)}|^4 f(X_t)g(X_0) : t < \tau_G\right] \le \|f\|_2\|g\|_2\|\mathbf{E}.[|C_t^{(i)}|^8 : t < \tau_G]\|_{G,\infty},$$

while for $i = 3$,

$$\mathbf{E}_{1_G m}\left[|C_t^{(3)}|^4 f(X_t)g(X_0) : t < \tau_G\right] \le \|f\|_2\|g\|_2\|\mathbf{E}.[|C_t^{(3)} \circ r_t|^8 : t < \tau_G]\|_{G,\infty}.$$

Here r_t is the time reverse operator under the part process on G. Therefore we have the desired estimate. □

Under the assumptions of Theorem 3.1, it is easy to show that the bilinear form $(\mathcal{Q}, \mathcal{F})$ is a closed, lower-bounded quadratic form. Therefore there exists a strongly continuous semigroup, $\{Q_t, t \ge 0\}$, associated with $(\mathcal{Q}, \mathcal{F})$. Let $(L^{\mathcal{Q}}, \mathcal{D}(L^{\mathcal{Q}}))$ be its corresponding L^2-generator. On the other hand, it can be shown that the operators $\{T_t, t \ge 0\}$ defined in (3.7) forms a strongly continuous semigroup on $L^2(E; m)$. Denote the L^2-generator of $\{T_t, t \ge 0\}$ by $(L, \mathcal{D}(L))$.

Lemma 3.3 *Suppose* $\mu_{\langle M\rangle} + \mu_{\langle \widehat{M}\rangle} + |\mu| + N(|\varphi|)\mu_H \in \mathbf{K}(X)$, $\mu_{\langle \widehat{M}\rangle}(E) + \int_E N(|\psi|)d\mu_H < \infty$ *and* $-1 < c_1 \le \varphi, \psi \le c_2 < \infty$ *for some constants* c_1, c_2*. Then, for* $f \in \mathcal{D}(L^{\mathcal{Q}})$

$$\begin{aligned} Z_t f(X_t) &= f(X_0) + \int_0^t Z_{s-} d(M_s^f + U_s^f) + \int_0^t Z_{s-} f(X_{s-}) d(M_s^c - \widehat{M}_s^c + W_s) \\ &\quad + \int_0^t Z_s L^{\mathcal{Q}} f(X_s) ds, \end{aligned} \tag{3.19}$$

where W *and* U^f *are purely discontinuous local MAFs on* $I(\zeta)$ *with*

$$W_t - W_{t-} = \varphi(X_{t-}, X_t) + \psi(X_t, X_{t-}) + \varphi(X_{t-}, X_t)\psi(X_t, X_{t-}), \quad t < \zeta \tag{3.20}$$

and

$$U_t^f - U_{t-}^f = (f(X_t) - f(X_{t-}))(W_t - W_{t-}), \quad t < \zeta. \tag{3.21}$$

Proof. Putting $\overline{\psi}(x, y) := \psi(y, x)$, we see $\int_E N(1_{E\times E}|\overline{\psi}|^2)d\mu_H = \int_E N(1_{E\times E}|\psi|^2)d\mu_H < \infty$ and $\int_E N(1_{E\times E}|\overline{\psi}|)d\mu_H = \int_E N(1_{E\times E}|\psi|)d\mu_H < \infty$. In view of Theorem 5.1.3 in [10], we have $\mathbf{E}_x[\int_0^t N(1_{E\times E}(|\overline{\psi}| + |\overline{\psi}|^2))(X_s)dH_s] < \infty$ for m-a.e. $x \in E$ for each $t > 0$. On the other hand, $N(|\varphi|)\mu_H \in \mathbf{K}(X)$ implies $\mathbf{E}_x[\int_0^t N(|\varphi|)(X_s)dH_s] < \infty$ for m-a.e. $x \in E$ for each $t > 0$. Thus, we have the purely discontinuous local MAF W (resp. U^f) on $I(\zeta)$ with the property (3.20) (resp. (3.21)). Since $\mu_{\langle \widehat{M}\rangle}(E) < \infty$, $\widehat{M}$ is a MAF having finite energy and so by (2.2) and [3, Theorem 3.5] there is some $\rho \in \mathcal{F}$ such that

$$\Lambda(\widehat{M})_t = N_t^\rho - \int_0^t \rho(X_s)ds = \rho(X_t) - \rho(X_0) - M_t^\rho - \int_0^t \rho(X_s)ds \quad \mathbf{P}_m\text{-a.s. on } \{t < \zeta\}.$$

By Definition 2.1,

$$\widehat{M}_t \circ r_t = -2\Lambda(\widehat{M})_t - \widehat{M}_t - \psi(X_t, X_{t-}) - \widehat{K}_t \quad \mathbf{P}_m\text{-a.s. on } \{t < \zeta\},$$

where $\widehat{K}$ is a purely discontinuous local MAF on $I(\zeta)$ with

$$\widehat{K}_t - \widehat{K}_{t-} = -\psi(X_{t-}, X_t) - \psi(X_t, X_{t-}),\ t < \zeta.$$

The condition $\int_E N(|\psi|)d\mu_H = \int_E N(|\overline{\psi}|)d\mu_H \in \mathbf{K}(X)$
implies the integrability of $\widehat{K}$, hence, $\widehat{K}$ is a martingale. Therefore on $\{t < \zeta\}$,

$$\begin{aligned}
& \mathrm{Exp}(\widehat{M_t}) \circ r_t\,(1 + \psi(X_t, X_{t-})) \\
= & \left(\exp\left(\widehat{M}_t - \frac{1}{2}\langle \widehat{M^c}\rangle_t\right) \prod_{0<s\le t} (1 + \psi(X_{s-}, X_s))e^{-\psi(X_{s-},X_s)}\right) \circ r_t\,(1 + \psi(X_t, X_{t-})) \\
= & \exp\left(\widehat{M}_t \circ r_t - \frac{1}{2}\langle \widehat{M^c}\rangle_t\right)\left(\prod_{0<s<t} (1 + \psi(X_s, X_{s-}))e^{-\psi(X_s,X_{s-})}\right)(1 + \psi(X_t, X_{t-})) \\
= & \exp\left(-2\Lambda(\widehat{M})_t - \widehat{M}_t - \widehat{K}_t - \frac{1}{2}\langle \widehat{M^c}\rangle_t\right) \prod_{0<s\le t} (1 + \psi(X_s, X_{s-}))e^{-\psi(X_s,X_{s-})} \\
= & \exp\left(-2\Lambda(\widehat{M})_t\right) \mathrm{Exp}\left(-\widehat{M}^c_t\right) \mathrm{Exp}\left(-\widehat{M}^d_t - \widehat{K}_t\right) \\
= & \exp\left(-2\Lambda(\widehat{M})_t\right) \mathrm{Exp}\left(-\widehat{M}_t - \widehat{K}_t\right).
\end{aligned}$$

We see

$$W_t = M^d_t - \widehat{M}^d_t - \widehat{K}_t + \sum_{0<s\le t} (\varphi\cdot\overline{\psi})(X_{s-}, X_s) - \int_0^t N(\varphi\cdot\overline{\psi})(X_s)dH_s,\quad t < \zeta.$$

Thus

$$\begin{aligned}
Z_t &= \exp\left(-2\Lambda(\widehat{M})_t\right) \mathrm{Exp}\left(-\widehat{M}_t - \widehat{K}_t\right) \mathrm{Exp}\left(M_t + A^\mu_t + \langle M^c, \widehat{M^c}\rangle_t\right) \\
&= \exp\left(-2\Lambda(\widehat{M})_t\right) \mathrm{Exp}\left(M_t - \widehat{M}_t - \widehat{K}_t + A^\mu_t - [M^d,\, \widehat{M}^d + \widehat{K}]_t\right) \\
&= \exp\left(-2\Lambda(\widehat{M})_t\right) \mathrm{Exp}\left(M_t - \widehat{M}_t - \widehat{K}_t + A^\mu_t + \sum_{0<s\le t} (\varphi\cdot\overline{\psi})(X_{s-}, X_s)\right) \\
&= \exp\left(-2\Lambda(\widehat{M})_t\right) \mathrm{Exp}\left(M^c_t - \widehat{M}^c_t + W_t + A^\mu_t + \int_0^t N(\varphi\cdot\overline{\psi})(X_s)dH_s\right) \qquad (3.22) \\
&:= \exp(-2\rho(X_t))\, Z^1_t \qquad (3.23)
\end{aligned}$$

with

$$Z^1_t := \exp\left(2\rho(X_0) + 2M^\rho_t + 2\int_0^t \rho(X_s)ds\right) \mathrm{Exp}\left(M^c_t - \widehat{M}^c_t + W_t + A^\mu_t + \int_0^t N(\varphi\cdot\overline{\psi})(X_s)dH_s\right). \qquad (3.24)$$

Note that

$$1 + W_t - W_{t-} = (1 + \varphi(X_{t-}, X_t))(1 + \psi(X_t, X_{t-})) > 0.$$

Let $f \in \mathcal{D}(L^{\mathcal{Q}}) \subset \mathcal{F}$. Then

$$\begin{aligned}\mathcal{E}(f,\, g) &= \int_E (-L^{\mathcal{Q}} f)(x) g(x) m(dx) + \int_E f(x) \mu_{\langle M^g, \widehat{M}\rangle}(dx) + \int_E g(x) \mu_{\langle M^f, M\rangle}(dx) \\ &\quad + \int_E f(x) g(x) \mu(dx) + \int_E g(x) \left(\int_E f(y) \varphi(x,y) \psi(y,x) N(x,dy) \right) \mu_H(dx).\end{aligned}$$

Applying (2.3) to the MAF $f * \widehat{M}$ having finite energy, we have

$$\lim_{t\to 0} \frac{1}{t} \mathbf{E}_{g\, m} \left[2\Gamma(f * \widehat{M})_t \right] = - \int_E f(x) \mu_{\langle M^g, \widehat{M}\rangle}(dx) \qquad \text{for every } g \in \mathcal{F}_b, \tag{3.25}$$

where $(f * \widehat{M})_t = \int_0^t f(X_{s-}) d\widehat{M}_s$. By (3.24)-(3.25) above and Theorem 5.2.4 in [10],

$$\begin{aligned}N_t^f &= \int_0^t L^{\mathcal{Q}} f(X_s) ds + 2\Gamma(f * \widehat{M})_t - \langle M^f, M\rangle_t - \int_0^t f(X_s) dA_s^\mu \\ &\quad - \int_0^t \left(\int_E f(y) \varphi(X_s, y) \psi(y, X_s) N(X_s, dy) \right) dH_s.\end{aligned} \tag{3.26}$$

Note that it was shown in [3, Remark 3.8(ii)] that $\Gamma(f * \widehat{M}) = \Lambda(f * \widehat{M})$ and that by Definition 1.2,

$$\Lambda(f * \widehat{M})_t = \int_0^t f(X_{s-})\, d\Lambda(\widehat{M})_s + \frac{1}{2} \langle M^f, \widehat{M}^c\rangle_t - \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s)) \psi(y, X_s) N(X_s, dy) dH_s. \tag{3.27}$$

It follows from (3.26)-(3.27)

$$\begin{aligned}N_t^f &= \int_0^t L^{\mathcal{Q}} f(X_s) ds + 2 \int_0^t f(X_{s-}) dN_s^\rho - 2 \int_0^t f(X_s) \rho(X_s) ds + \langle M^f, \widehat{M}^c - M\rangle_t - \int_0^t f(X_s) dA_s^\mu \\ &\quad - \int_0^t \left(\int_E \Big(f(y) - f(X_s)) \psi(y, X_s) + f(y) \varphi(X_s, y) \psi(y, X_s) \Big) N(X_s, dy) \right) dH_s.\end{aligned}$$

By [3, Theorem 4.2],

$$\begin{aligned}&\int_0^t e^{-2\rho(X_{s-})} dN_s^f - 2 \int_0^t e^{-2\rho(X_{s-})} f(X_{s-}) dN_s^\rho \\ = &\int_0^t e^{-2\rho(X_s)} L^{\mathcal{Q}} f(X_s) ds - 2 \int_0^t e^{-2\rho(X_s)} f(X_s) \rho(X_s) ds \\ &+ \int_0^t e^{-2\rho(X_s)} d\langle M^f, \widehat{M}^c - M\rangle_s - \int_0^t e^{-2\rho(X_s)} f(X_s) dA_s^\mu \\ &- \int_0^t e^{-2\rho(X_s)} \left(\int_E \Big(f(y) - f(X_s)) \psi(y, X_s) + f(y) \varphi(X_s, y) \psi(y, X_s) \Big) N(X_s, dy) \right) dH_s.\end{aligned}$$

Now by the Itô's formula in [3, Theorem 4.7] and the above identity, we have

$$
\begin{aligned}
& e^{-2\rho(X_t)}f(X_t) - e^{-2\rho(X_0)}f(X_0) \\
= & \int_0^t e^{-2\rho(X_{s-})} d(M_s^f + N_s^f) - 2\int_0^t e^{-2\rho(X_{s-})} f(X_{s-}) d(M_s^\rho + N_s^\rho) \\
& -2\int_0^t e^{-2\rho(X_{s-})} d\langle M^{\rho,c}, M^{f,c}\rangle_s + 2\int_0^t e^{-2\rho(X_{s-})} f(X_{s-}) d\langle M^{\rho,c}, M^{\rho,c}\rangle_s \\
& + \sum_{0<s\le t} [\Delta(e^{-2\rho(X_s)}f(X_s)) + 2e^{-2\rho(X_{s-})} f(X_{s-})\Delta(\rho(X_s)) - e^{-2\rho(X_{s-})}\Delta(f(X_s))] \\
= & \int_0^t e^{-2\rho(X_{s-})} dM_s^f - 2\int_0^t e^{-2\rho(X_{s-})} f(X_{s-}) dM_s^\rho + \int_0^t e^{-2\rho(X_s)} L^{\mathcal{Q}} f(X_s) ds \\
& -2\int_0^t e^{-2\rho(X_s)} f(X_s)\rho(X_s) ds + \int_0^t e^{-2\rho(X_s)} d\langle M^f, \widehat{M}^c - M\rangle_s - \int_0^t e^{-2\rho(X_s)} f(X_s) dA_s^\mu \\
& - \int_0^t e^{-2\rho(X_s)} \left(\int_E (f(y) - f(X_s))\psi(y, X_s) + f(y)\varphi(X_s, y)\psi(y, X_s))\, N(X_s, dy) \right) dH_s \\
& -2\int_0^t e^{-2\rho(X_{s-})} d\langle M^{\rho,c}, M^{f,c}\rangle_s + 2\int_0^t e^{-2\rho(X_{s-})} f(X_{s-}) d\langle M^{\rho,c}, M^{\rho,c}\rangle_s \\
& + \sum_{0<s\le t} [\Delta(e^{-2\rho(X_s)}f(X_s)) + 2e^{-2\rho(X_{s-})} f(X_{s-})\Delta(\rho(X_s)) - e^{-2\rho(X_{s-})}\Delta(f(X_s))], \qquad (3.28)
\end{aligned}
$$

which is a semimartingale. Note that Z_t^1 can be rewritten as

$$
Z_t^1 = \operatorname{Exp}\left(M_t^c - \widehat{M}_t^c + W_t + A_t^\mu + \int_0^t N(\varphi \cdot \overline{\psi})(X_s) dH_s + 2\int_0^t \rho(X_s) ds \right) \exp\left(2\rho(X_0) + M_t^{2\rho} \right).
$$

Now by Itô's formula for semimartingales,

$$
\begin{aligned}
& Z_t^1 - Z_0^1 \\
= & \int_0^t Z_{s-}^1 d\Big(M_s^c - \widehat{M}_s^c + W_s + 2M_s^\rho + A_s^\mu + \int_0^s N(\varphi \cdot \overline{\psi})(X_u) dH_u + 2\int_0^s \rho(X_u) du \Big) \\
& +2\int_0^t Z_{s-}^1 d\langle M^{\rho,c} + M^c - \widehat{M}^c,\, M^{\rho,c}\rangle_s + \sum_{0<s\le t} Z_{s-}^1 \Big(e^{2\Delta(\rho(X_s))} - 1 - 2\Delta(\rho(X_s)) \Big) \qquad (3.29) \\
& + \sum_{0<s\le t} Z_{s-}^1 \Big(\varphi(X_{s-}, X_s) + \psi(X_s, X_{s-}) + \varphi(X_{s-}, X_s)\psi(X_s, X_{s-}) \Big) \Big(e^{2\Delta(\rho(X_s))} - 1 \Big).
\end{aligned}
$$

Applying Itô's formula to

$$
Z_t f(X_t) = Z_t^1 \cdot (e^{-2\rho(X_t)} f(X_t))
$$

and using (3.28)-(3.29), we get (3.19) after many terms cancel out. This calculation is tedious and must be done with care. It is fairly easy to calculate out the martingale part, the quadratic variation part and continuous additive part while applying Itô's formula. However the calculation of the jump part of $Z_t^1 \cdot (e^{-2\rho(X_t)} f(X_t))$ using Itô's formula can be frustrating. The best way to calculate $Z_t f(X_t) - Z_{t-} f(X_{t-})$ directly is perhaps the following. It follows from (3.22) and that the Doléans-Dade's exponential solve an SDE, we see that

$$Z_t - Z_{t-} = Z_{t-}(W_t - W_{t-})$$

and so

$$\begin{aligned} Z_t f(X_t) - Z_{t-} f(X_{t-}) &= Z_{t-}\Big((1 + W_t - W_{t-}) f(X_t) - f(X_{t-})\Big) \\ &= Z_{t-}\Big((1 + W_t - W_{t-})\Delta(f(X_t)) + f(X_{t-})(W_t - W_{t-})\Big). \end{aligned}$$

□

4 Proof of Theorem 3.1.

Lemma 4.1 *Under the conditions of Theorem 3.1, the quadratic form $(\mathcal{Q}, \mathcal{F})$ defined in (3.3) possesses the positivity preserving property in the sense of [16].*

Proof. By Proposition 1.3(i) in [16], it suffices to show $\mathcal{Q}(f^+, f^-) \leq 0$ for $f \in \mathcal{F}$. Let $\mathcal{Q}^{(c)+(k)}$ be the sum of continuous part and killing part of $\mathcal{Q}$:

$$\begin{aligned} \mathcal{Q}^{(c)+(k)}(f,g) \;:=\; & \mathcal{E}^{(c)}(f,g) + \mathcal{E}^{(k)}(f,g) \\ & - \int_E f(x)\mu_{\langle M^{g,c}, \widehat{M}^c\rangle}(dx) - \int_E g(x)\mu_{\langle M^{f,c}, M^c\rangle}(dx) - \int_E f(x)g(x)\mu(dx) \\ & + \int_E f(x)g(x)\psi(x,\partial)N(x,\{\partial\})\mu_H(dx) + \int_E f(x)g(x)\varphi(x,\partial)N(x,\{\partial\})\mu_H(dx). \end{aligned}$$

Then we see $\mathcal{Q}^{(c)+(k)}(f^+, f^-) = 0$ because $\mu^c_{\langle f\rangle}(f = 0) = 0$ and $\mu^c_{\langle f^+\rangle}(f < 0) = \mu^c_{\langle f^-\rangle}(f > 0) = 0$, where $\mu^c_{\langle u\rangle} := \mu_{\langle M^{u,c}\rangle}$ for $u \in \mathcal{F}$. If we let $\mathcal{Q}^{(j)}(f,g) := \mathcal{Q}(f,g) - \mathcal{Q}^{(c)+(k)}(f,g) = \mathcal{E}^{(j)}(f,g) + 2\int_{E\times E\setminus d}(f(x) - f(y))g(x)\varphi(x,y)J(dx,dy) + 2\int_{E\times E\setminus d}(g(x) - g(y))f(x)\psi(x,y)J(dx,dy)$, then $Q^{(j)}(f^+, f^-) = -2\int_{E\times E} f^+(y)f^-(x)(1 + \varphi(x,y))(1 + \psi(y,x))J(dx,dy) \leq 0$. □

Lemma 4.2 *Let G be a finely open (nearly) Borel subset of E and consider the part space $(\mathcal{E}_G, \mathcal{F}_G)$ of $(\mathcal{E}, \mathcal{F})$ on $L^2(G;m)$. Then $\mathcal{Q}$ on $\mathcal{F}_G$ has the following expression: For $f, g \in \mathcal{F}_G$,*

$$\begin{aligned} \mathcal{Q}(f,g) \;=\; & \mathcal{E}(f,g) - \int_G f(x)\mu^G_{\langle M^g, \widehat{M}\rangle}(dx) - \int_G g(x)\mu^G_{\langle M^f, M\rangle}(dx) - \int_G f(x)g(x)\mu^G(dx) \\ & - \int_{G\times G} f(y)g(x)\varphi(x,y)\psi(y,x)N(x,dy)\mu_H(dx), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \mu^G_{\langle M^g, \widehat{M}\rangle}(dx) &:= \mu_{\langle M^{g,c}, \widehat{M}^c\rangle}(dx) + \Big(\int_{G_\partial}(g(y) - g(x))\psi(x,y)N(x,dy)\Big)\mu_H(dx), \\ \mu^G_{\langle M^f, M\rangle}(dx) &:= \mu_{\langle M^{f,c}, M^c\rangle}(dx) + \Big(\int_{G_\partial}(f(y) - f(x))\varphi(x,y)N(x,dy)\Big)\mu_H(dx), \\ \mu^G(dx) &:= \mu(dx) - \Big(\int_{E\setminus G}(1 + \varphi(x,y) + \psi(x,y))N(x,dy)\Big)\mu_H(dx). \end{aligned}$$

Proof. The proof is an easy calculation. We also note that for $f, g \in \mathcal{F}_G$, $\mathcal{E}(f,g) = \mathcal{E}^c(f,g) + \int_{G\times G}(f(x)-f(y))(g(x)-g(y))J(dx,dy) + \int_G f(x)g(x)\kappa^G(dx)$, where $\kappa^G = \kappa + \frac{1}{2}N(x, E\setminus G)\mu_H$. □

Proof of Theorem 3.1. As in the analogous argument in [9], we can construct a common $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbb{N}}$ (of compact sets) such that (i) $1_{F_n}(\mu_{\langle M\rangle}+\mu_{\langle\widehat{M}\rangle}+|\mu|) \in \mathbf{K}_0(X)$ for each $n\in\mathbb{N}$ and (ii) there exists $\rho_n \in \mathcal{F}$ such that $\Lambda(\widehat{M}^c)_t = N_t^{\rho_n} - \int_0^t \rho_n(X_s)ds$ for $0 \le t < \tau_{F_n}$ $\mathbf{P}_m$-a.s., $\rho_n|_{F_n} \in C(F_n)$ and $1_{F_n}\mu_{\langle\rho_n\rangle} \in \mathbf{K}_0(X)$ for each $n \in \mathbb{N}$. Here we use the fact $\Lambda(1_{F_n} * \widehat{M}^c)_t = \Lambda(\widehat{M}^c)_t$ for $0 \le t < \tau_{F_n}$ $\mathbf{P}_m$-a.s. and $\mu_{\langle\widehat{M}^c\rangle}(F_n) < \infty$. The latter implies that $1_{F_n} * \widehat{M}^c$ is a MAF of finite energy under X and there exists a $\rho_n \in \mathcal{F}$ such that $\Lambda(1_{F_n} * \widehat{M}^c)_t = \Gamma(1_{F_n} * \widehat{M}^c)_t = N_t^{\rho_n} - \int_0^t \rho_n(X_s)ds$.

Let E_n denote the fine interior of F_n and define $\mathcal{F}^{(n)} := \{u \in \mathcal{F} \mid u = 0 \;\; \mathcal{E}\text{-q.e. on } E_n^c\}$, and let $\mathcal{Q}^{(n)}$ denote the restriction of $\mathcal{Q}$ to $\mathcal{F}^{(n)}$. Clearly $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$ is a quasi-regular positivity preserving coercive closed form on $L^2(E_n; m)$ satisfying the same hypothesis as $(\mathcal{Q}, \mathcal{F})$. In fact, $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$ is related to the restriction of $\mathcal{E}$ to $\mathcal{F}^{(n)}$ (which is the Dirichlet form of the part process X^{E_n}) in exactly the same way that $(\mathcal{Q}, \mathcal{F})$ is related to $(\mathcal{E}, \mathcal{F})$.

(1) First assume that $|\varphi|$ and $|\psi|$ are bounded above and below away from 0. Note that $1_{F_n}N(|\varphi|+|\psi|)\mu_H \in \mathbf{K}_0(X)$ because of the boundedness of $|\varphi|$ and $|\psi|$ away from 0. Then the conditions of Lemmas 3.2 and 3.3 are satisfied by X^{E_n} and $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$. Let $(Q_t^{(n)})_{t>0}$ and $(V_\alpha^{(n)})_{\alpha>\alpha_0}$ be the semigroup and resolvent on $L^2(E_n; m)$ associated with $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$. Let $(L^{\mathcal{Q},(n)}, \mathcal{D}(L^{\mathcal{Q},(n)}))$ denote the infinitesimal generator of $(Q_t^{(n)})$.

Consider a bounded $f \in \mathcal{D}(L^{\mathcal{Q},(n)})$. Let τ_n be the first exit time of X from E_n. Then by Lemma 3.3 (applied to X^{E_n} and $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$),

$$\begin{aligned} f(X_{t\wedge\tau_n})Z_{t\wedge\tau_n} = f(X_0) + \int_0^{t\wedge\tau_n} Z_{s-}\, d(M_s^{f,(n)} + U_s^{f,(n)}) \\ + \int_0^{t\wedge\tau_n} Z_{s-}f(X_{s-})\, d(M_s^{c,(n)} - \widehat{M}_s^{c,(n)} + W_s^{(n)}) + \int_0^{t\wedge\tau_n} Z_s L^{\mathcal{Q},(n)} f(X_s)\, ds, \end{aligned} \tag{4.2}$$

because $f(X_{\tau_n}) = 0$, $\mathbf{P}_m$-a.s. Here was used the fact that $Z^{(n)}$ (resp. $M^{c,(n)}$, $W^{(n)}$, $U^{f,(n)}$), the analog of Z (resp. M^c, W, U^f) with respect to X^{E_n} and $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$, coincides with Z (resp. M^c, W, U^f) on $[0, \tau_n[$. Let $\{T_k\}$ be an increasing sequence of $(\mathcal{F}_t)$-stopping times with $T_k \uparrow \zeta$ as $k \to \infty$ reducing the local martingale terms on the right hand side of (3.19). Replacing t by $t \wedge T_k$ in (4.2) and taking expectations we obtain

$$\mathbf{E}_x[f(X_{t\wedge T_k\wedge\tau_n})Z_{t\wedge T_k\wedge\tau_n}] = f(x) + \mathbf{E}_x\left[\int_0^{t\wedge T_k\wedge\tau_n} Z_s(L^{\mathcal{Q},(n)}f)(X_s)\, ds\right], \tag{4.3}$$

for m-a.e. $x \in E_n$. Hence, for non-negative $g \in L^2(E_n; m)$ we have

$$\mathbf{E}_{gm}[f(X_{t\wedge T_k\wedge\tau_n})Z_{t\wedge T_k\wedge\tau_n}] = (f,g)_{L^2(E_n;m)} + \mathbf{E}_{gm}\left[\int_0^{t\wedge T_k\wedge\tau_n} Z_s(L^{\mathcal{Q},(n)}f)(X_s)\, ds\right] \tag{4.4}$$

Because f is bounded, Lemma 3.2(ii) for $1_{E_n}, g \in L^2(E_n; m)$ permits us to conclude that, as $k \to \infty$, the left side of (4.4) converges to $\mathbf{E}_{gm}[f(X_{t\wedge\tau_n})Z_{t\wedge\tau_n}]$. On the other hand, because

$$\left|\mathbf{E}_x\left[\int_0^{t\wedge\tau_n} Z_s(L^{\mathcal{Q},(n)}f)(X_s)\,ds\right]\right| \le \int_0^t T_s^{(n)}|L^{\mathcal{Q},(n)}f|(x)\,ds, \tag{4.5}$$

the left hand side of (4.5) is in $L^2(E_n;m)$ from Lemma 3.2(i), where

$$T_s^{(n)}f(x) := \mathbf{E}_x[f(X_s)Z_s : s < \tau_n]. \tag{4.6}$$

Passing to the limit in (4.4) as $k \to \infty$, we obtain

$$\mathbf{E}_{gm}[f(X_{t\wedge\tau_n})Z_{t\wedge\tau_n}] = (f,g)_{L^2(E_n;m)} + \mathbf{E}_{gm}\left[\int_0^{t\wedge\tau_n} Z_s(L^{\mathcal{Q},(n)}f)(X_s)\,ds\right] \tag{4.7}$$

first for non-negative, and then for all $g \in L^2(E_n;m)$. Then

$$\mathbf{E}_x[f(X_{t\wedge\tau_n})Z_{t\wedge\tau_n}] = f(x) + \mathbf{E}_x\left[\int_0^{t\wedge\tau_n} Z_s(L^{\mathcal{Q},(n)}f)(X_s)\,ds\right], \tag{4.8}$$

for m-a.e. $x \in E_n$, provided $f \in \mathcal{D}(L^{\mathcal{Q},(n)})$ is bounded. For $f \in \mathcal{D}(L^{\mathcal{Q},(n)})$ of the form $V_\alpha^{(n)}g$ $(0 \le g \in L^2(E_n;m))$, by the construction of the nest $\{F_n\}$, there is a sequence $\{g_k\}$ of non-negative elements of $L^2(E;m)$ such that $f_k := V_\alpha^{(n)}g_k$ is in $L^\infty(E_n;m)$, f_k converges in $L^2(E_n;m)$ to f and $L^{\mathcal{Q},(n)}f_k = \alpha f_k - g_k$ converges to $L^{\mathcal{Q},(n)}f$ in $L^2(E_n;m)$. [The boundedness of f_k comes from the inequality $V_\alpha^{(n)}f \le 1_{F_n}\cdot V_\alpha f$ for all non-negative $f \in L^2(E;m)$, which is a consequence of the fact that these resolvents are associated with quasi-regular positivity preserving forms (see the argument in [9]). Substituting f_k for f in (4.8) and then passing the limit as $k \to \infty$, we see that (4.8) is valid for all $f \in \mathcal{D}(L^{\mathcal{Q},(n)})$, since any such f can be written as $V_\alpha^{(n)}g_1 - V_\alpha^{(n)}g_2$ for non-negative $g_1, g_2 \in L^2(E_n;m)$. That is, we have,

$$T_t^{(n)}f(x) = f(x) + \int_0^t T_s^{(n)}(L^{\mathcal{Q},(n)}f)(x)\,ds, \qquad m\text{-a.e. } x \in E_n, \tag{4.9}$$

for all $f \in \mathcal{D}(L^{\mathcal{Q},(n)})$. This implies the strong continuity of $T_t^{(n)}$ on $\mathcal{D}(L^{\mathcal{Q},(n)})$, hence on $L^2(E_n;m)$. Note that $T_t^{(n)}$ maps $L^2(E_n;m)$ into itself by Lemma 3.2(i), because

$$1_{E_n}(\mu_{\langle M\rangle} + \mu_{\langle\widehat{M}\rangle} + N(|\varphi|)\mu_H + N(|\psi|)\mu_H + |\mu|)$$

is a Kato class smooth measure with respect to X^{E_n}. Hence

$$\lim_{t\downarrow 0}\frac{T_t^{(n)}f - f}{t} = L^{\mathcal{Q},(n)}f. \tag{4.10}$$

Thus, using $(L^{(n)}, \mathcal{D}(L^{(n)}))$ to denote the infinitesimal generator of $(T_t^{(n)})_{t>0}$,

$$\mathcal{D}(L^{\mathcal{Q},(n)}) \subset \mathcal{D}(L^{(n)}) \text{ and } L^{\mathcal{Q},(n)} = L^{(n)} \text{ on } \mathcal{D}(L^{\mathcal{Q},(n)}). \tag{4.11}$$

Let $G_\alpha^{(n)} := \int_0^\infty e^{-\alpha t} T_t^{(n)}\,dt$ be the resolvent operator associated with $L^{(n)}$. Fix $u \in L^2(E_n;m)$ and define, for $\alpha > \alpha_0$, $v := V_\alpha^{(n)} u$. Then $v \in \mathcal{D}(L^{\mathcal{Q},(n)}) \subset \mathcal{D}(L^{(n)})$ and $L^{(n)} v = L^{\mathcal{Q},(n)} v = \alpha v - u$. But $\mathcal{D}(L^{(n)})$ coincides with $G_\alpha^{(n)}(L^2(E_n;m))$, so there exists $g \in L^2(E_n;m)$ such that $v = G_\alpha^{(n)} g$, in which case $L^{(n)} v = \alpha v - g$. It follows that $u = g$, and then that $G_\alpha^{(n)} u = V_\alpha^{(n)} u$, for all $u \in L^2(E_n;m)$. This identity of resolvents implies that the associated semigroups $(T_t^{(n)})$ and $(Q_t^{(n)})$ coincide under the boundedness of $|\varphi|$, $|\psi|$ away from 0.

(2) Secondly, we assume only that ψ is bounded below away from 0. For general $\varphi > -1$, define

$$\varphi_\ell(x,y) := (1_{\{\frac{1}{\ell} < |\varphi| < \ell\}} \varphi)(x,y).$$

Clearly φ_ℓ satisfies the condition for φ in step (1). Let $M^{d,\ell}$ be a purely discontinuous MAF on $I(\zeta)$ such that $\Delta M_t^{d,\ell} = \varphi_\ell(X_{t-}, X_t)$, $t \in]0,\zeta[$, and set $M^\ell := M^c + M^{d,\ell}$. Then we see $\mu_{\langle M^\ell - M\rangle} = N((\varphi_\ell - \varphi)^2)\mu_H \le N(\varphi^2)\mu_H$. Hence we see that M^ℓ converges uniformly to M on any compact subinterval of $[0, \tau_{F_n}[$ $\mathbf{P}_m$-a.s., because of the convergence of energy $\mathbf{e}(1_{F_n} * (M^\ell - M)) \to 0$. By replacing M with M^ℓ, we consider $\mathcal{Q}^\ell$, Q_t^ℓ, Z_t^ℓ, T_t^ℓ, G_α^ℓ instead of $\mathcal{Q}$, Q_t, Z_t, T_t, G_α, respectively and also consider $Q_t^{\ell,(n)}$, $T_t^{\ell,(n)}$, $G_\alpha^{\ell,(n)}$ instead of $Q_t^{(n)}$, $T_t^{(n)}$, $G_\alpha^{(n)}$ respectively. From (1), we already know that $Q_t^{\ell,(n)}$ coincides with $T_t^{\ell,(n)}$. To show the coincidence, we first prove that $T_t^{(n)} f \in L^2(E_n;m)$ and $T_t^{\ell,(n)} f$ weakly converges to $T_t^{(n)} f$ for any Borel function $f \in L^2(E_n;m)$.

In order to prove this weak convergence, we will follow the approach in Chen and Zhang [8] by showing that there exists a constant $\breve{\alpha}_0$ independent of ℓ, ℓ_0 with $\ell \ge \ell_0$ such that for any nonnegative Borel $f, g \in L^2(E_n;m)$

$$\sup_{\ell \ge \ell_0} \mathbf{E}_m \left[f(X_t) g(X_0) Z_t^{\ell_0} \left(Y_t^\ell\right)^2 : t < \tau_n \right] \le e^{\breve{\alpha}_0 t} \|f\|_2 \|g\|_2, \tag{4.12}$$

where $Y_t^\ell := Z_t^\ell \left(Z_t^{\ell_0}\right)^{-1} = \mathrm{Exp}\left(M_t^{d,\ell}\right) \mathrm{Exp}\left(M_t^{d,\ell_0}\right)^{-1} = \mathrm{Exp}\left(M_t^{d,\ell} - M_t^{d,\ell_0}\right)$ for $\ell \ge \ell_0 \ge 1$. Here we use $[M^{d,\ell_0}, M^{d,\ell} - M^{d,\ell_0}]_t = 0$ for $\ell \ge \ell_0 \ge 1$. From (4.12) we see the uniform integrability of $\{Y_t^\ell, \ell \ge \ell_0\}$ under the law $1_{\{t<\tau_n\}} f(X_t) g(X_0) Z_t^{\ell_0} \mathbf{P}_m$, which implies the desired weak convergence. Indeed, from (1), we can conclude $\|T_t^{\ell,(n)}\|_{2\to 2} \le e^{\alpha_0} t$, hence $\|G_\alpha^{\ell,(n)}\|_{2\to 2} \le 1/(\alpha - \alpha_0)$. By Fatou's lemma we have for Borel $f \in L^2(E_n;m)$

$$\int_{E_n} |T_t^{(n)} f(x)|^2 m(dx) \le \varliminf_{l\to\infty} \int_{E_n} |T_t^{\ell,(n)} f(x)|^2 m(dx) \le e^{\alpha_0 t} \|f\|_2^2.$$

Hence $\|T_t^{(n)}\|_{2\to 2} \le e^{\alpha_0} t$ and $\|G_\alpha^{(n)}\|_{2\to 2} \le 1/(\alpha - \alpha_0)$. By (4.12), for each $f \in L^2(E_n;m)$, $T_t^{\ell,(n)} f$ converges to $T_t^{(n)} f$ weakly on $L^2(E_n;m)$ and consequently $G_\alpha^{\ell,(n)} f$ converges to $G_\alpha^{(n)} f$ weakly on $L^2(E_n;m)$ as $\ell \to \infty$.

We now prove (4.12). Since

$$\begin{aligned}
\left(Y_t^\ell\right)^2 &= \left(\mathrm{Exp}\left(M_t^{d,\ell}-M_t^{d,\ell_0}\right)\right)^2 \\
&= \exp\left(2M_t^{d,\ell}-2M_t^{d,\ell_0}\right)\prod_{0<s\le t}(1+\Delta(M^{d,\ell}-M^{d,\ell_0})_s)^2\exp(-2\Delta(M^{d,\ell}-M^{d,\ell_0})_s) \\
&= \exp\left(2M_t^{d,\ell}-2M_t^{d,\ell_0}\right)\prod_{0<s\le t}\left(1+(1+\Delta(M^{d,\ell}-M^{d,\ell_0})_s)^2-1\right) \\
&\qquad\times\exp\left(1-(1+\Delta(M^{d,\ell}-M^{d,\ell_0})_s)^2\right)\exp\left((\Delta(M^{d,\ell}-M^{d,\ell_0})_s)^2\right) \\
&= \mathrm{Exp}\left(K_t^\ell\right)\exp\left((N(1_{\{1/\ell<|\varphi|\le 1/\ell_0,\ell_0\le|\varphi|<\ell\}}\varphi^2)*H)_t\right),
\end{aligned}$$

where K_t^ℓ is the purely discontinuous MAF on $I(\zeta)$ with

$$\begin{aligned}
\Delta K_s^\ell &= (1+\Delta(M^{d,\ell}-M^{d,\ell_0})_s)^2-1 \\
&= 2\Delta(M^{d,\ell}-M^{d,\ell_0})_s+(\Delta(M^{d,\ell}-M^{d,\ell_0})_s)^2 \\
&= 1_{\{1/\ell<|\varphi|\le 1/\ell_0,\ell_0\le|\varphi|<\ell\}}(2\varphi+\varphi^2)(X_{s-},X_s),\quad s\in]0,\zeta[.
\end{aligned}$$

Thus $\check{Z}_t^\ell := Z_t^\ell Y_t^\ell = Z_t^{\ell_0}\left(Y_t^\ell\right)^2$ is of the same form as Z_t^ℓ. Indeed,

$$\begin{aligned}
\check{Z}_t^\ell &= \mathrm{Exp}\left(\widehat{M}_t\right)\circ r_t\mathrm{Exp}\left(M_t^{\ell_0}+K_t^\ell+A_t^\mu+(N(1_{\{1/\ell<|\varphi|\le 1/\ell_0\}}\varphi^2)*H)_t+\langle M^c,\widehat{M}^c\rangle_t\right) \\
&\quad\times(1+\psi(X_t,X_{t-}))
\end{aligned}$$

and the corresponding form $\check{\mathcal{Q}}^\ell$ on $\mathcal{F}^{(n)}$ is given by

$$\begin{aligned}
\check{\mathcal{Q}}^\ell(f,g) &= \mathcal{E}(f,g)-\int_{E_n}f(x)\mu^{E_n}_{\langle M^g,\widehat{M}\rangle}(dx)-\int_{E_n}g(x)\mu^{E_n}_{\langle M^f,M^{\ell_0}+K_t^\ell\rangle}(dx) \\
&\quad-\int_{E_n}f(x)g(x)\left(\mu^{E_n}+N(1_{E_n\times E_n}1_{\{1/\ell<|\varphi|\le 1/\ell_0,\ell_0\le|\varphi|<\ell\}}\varphi^2)\mu_H\right)(dx) \qquad (4.13)\\
&\quad-\int_{E_n\times E_n}f(y)g(x)1_{\{1/\ell<|\varphi|\le 1/\ell_0,\ell_0\le|\varphi|<\ell\}}\varphi(x,y)\psi(y,x)N(x,dy)\mu_H(dx)
\end{aligned}$$

for $f,g\in\mathcal{F}^{(n)}$. Then the constant $\check{\delta}_0^\ell$ corresponding to $\check{\mathcal{Q}}^\ell$ on $\mathcal{F}^{(n)}$ is given by

$$\begin{aligned}
\check{\delta}_0^\ell &:= \sqrt{2\delta(\langle M^{\ell_0}+K_t^\ell\rangle)}+\sqrt{2\delta(\langle\widehat{M}\rangle)}+\delta(A^{\mu^+}) \\
&\quad+\delta(1_{\{1/\ell<|\varphi|\le 1/\ell_0,\ell_0\le|\varphi|<\ell\}}\varphi^2)+\sqrt{\delta(1_{\{1/\ell<|\varphi|\le 1/\ell_0,\ell_0\le|\varphi|<\ell\}}\varphi^2)}\sqrt{\delta(\psi^2)}
\end{aligned}$$

and it is estimated by

$$\check{\delta}_0 := \sqrt{2\delta(10\langle M\rangle)}+\sqrt{2\delta(\langle\widehat{M}\rangle)}+\delta(A^{\mu^+})+\delta(\varphi^2)+\sqrt{\delta(\varphi^2)}\sqrt{\delta(\psi^2)}.$$

Here we use $[M^{\ell_0},K^\ell]=0$, $\langle M^{\ell_0}\rangle\le\langle M\rangle$ and $\langle K^\ell\rangle\le\langle 3M\rangle$. Note that $\check{\delta}_0$ can be taken to be less than 1 because $1_{E_n}(\mu_{\langle M\rangle}+\mu_{\langle\widehat{M}\rangle}+|\mu|)\in\mathbf{K}(X)$. Therefore by (1) we have (4.12). Here $\check{\alpha}_0$ is given by

$$\breve{\alpha}_0 := \gamma(10\langle M\rangle)\sqrt{2/\delta(10\langle M\rangle)} + \gamma(\widehat{M})\sqrt{2/\delta(\langle \widehat{M}\rangle)} \\ +\gamma(A^{\mu^+}) + \gamma(\varphi^2) + \big(\delta(\varphi^2)\delta(\psi^2)\big)^{1/2}\left\{\frac{\gamma(\varphi^2)}{\delta(\varphi^2)} \vee \frac{\gamma(\psi^2)}{\delta(\psi^2)}\right\}.$$

Next we show that $\{G_\alpha^{(n)}, \alpha > \alpha_0\}$ is the resolvent associated with $(\mathcal{Q}^{(n)}, \mathcal{F}^{(n)})$. Fix a bounded Borel $f \in L^2(E_n; m)$. We easily see

$$\mathcal{E}_1(G_\alpha^{\ell,(n)}f, G_\alpha^{\ell,(n)}f) \le M\mathcal{Q}_\alpha^{\ell,(n)}(G_\alpha^{\ell,(n)}f, G_\alpha^{\ell,(n)}f) = M(f, G_\alpha^{\ell,(n)}f) \le \frac{M}{\alpha-\alpha_0}\|f\|_2^2.$$

So $\{G_\alpha^{\ell,(n)}f, n \in \mathbb{N}\}$ is $\mathcal{E}_1^{1/2}$-bounded. Taking a subsequence if necessary, $G_\alpha^{\ell,(n)}f$ converges weakly to some $f_0 \in \mathcal{F}^{(n)}$ and its Cesàro mean strongly converges to f_0. Hence $f_0 = G_\alpha^{(n)}f$.

Since

$$\begin{aligned}&\mathcal{Q}^{\ell,(n)}(u,v) - \mathcal{Q}^{(n)}(u,v)\\ &= -\int_{E_n} v\, d\mu^{E_n}_{\langle M^u, M^{\ell,d}-M^d\rangle} - \int_{E_n\times E_n} u(y)v(x)((\varphi_\ell - \varphi)\cdot\overline{\psi})(x,y)N(x,dy)\mu_H(dx),\end{aligned}$$

we have

$$\begin{aligned}&|\mathcal{Q}^{\ell,(n)}(u,v) - \mathcal{Q}^{(n)}(u,v)|\\ &\le \left(\int_{E_n} v^2 N(\varphi^2)d\mu_H\right)^{1/2}\left(\int_{E_n\times E_n}(u(x)-u(y))^2 1_{\{|\varphi|\le 1/\ell, \ell\le|\varphi|\}}(x,y)N(x,dy)\mu_H(dx)\right)^{1/2}\\ &\quad + \left(\int_{E_n} v^2 N(\varphi^2)d\mu_H\right)^{1/2}\left(\int_{E_n} u^2 N(1_{\{|\overline{\varphi}|\le 1/\ell, \ell\le|\overline{\varphi}|\}}\psi^2)d\mu_H\right)^{1/2}\end{aligned}$$

Taking $u := G_\alpha^{(n)}f$, $v := g_\ell := G_\alpha^{\ell,(n)}f - G_\alpha^{(n)}f$ and noting $\sup_{\ell\in\mathbb{N}}\mathcal{E}_1(g_\ell, g_\ell) < \infty$, we have

$$\begin{aligned}&\mathcal{E}_1(G_\alpha^{\ell,(n)}f - G_\alpha^{(n)}f, G_\alpha^{\ell,(n)}f - G_\alpha^{(n)}f)\\ &\quad\le M\mathcal{Q}_\alpha^{\ell,(n)}(G_\alpha^{\ell,(n)}f - G_\alpha^{(n)}f, G_\alpha^{\ell,(n)}f - G_\alpha^{(n)}f)\\ &\quad= M\left(\mathcal{Q}_\alpha^{(n)}(G_\alpha^{(n)}f, g_\ell) - \mathcal{Q}_\alpha^{\ell,(n)}(G_\alpha^{(n)}f, g_\ell)\right)\\ &\quad= M\left|\mathcal{Q}^{\ell,(n)}(G_\alpha^{(n)}f, g_\ell) - \mathcal{Q}^{(n)}(G_\alpha^{(n)}f, g_\ell)\right|\\ &\quad\to 0 \quad \text{as } \ell\to\infty.\end{aligned}$$

We also see

$$\begin{aligned}\mathcal{Q}_\alpha^{(n)}(G_\alpha^{(n)}f, v) &= \lim_{\ell\to\infty}\mathcal{Q}_\alpha^{(n)}(G_\alpha^{\ell,(n)}f, v)\\ &= \lim_{\ell\to\infty}\left(\mathcal{Q}_\alpha^{(n)}(G_\alpha^{\ell,(n)}f, v) - \mathcal{Q}_\alpha^{\ell,(n)}(G_\alpha^{\ell,(n)}f, v)\right) + (f,v)_{L^2(E_n;m)}\\ &= (f,v)_{L^2(E_n;m)}.\end{aligned}$$

Now $G_\alpha^{(n)} f = V_\alpha^{(n)} f$ holds for any Borel $f \in L^2(E_n;m)$. Therefore, we have the desired result for general $\varphi > -1$ and the lower boundedness of ψ away from 0.

(3) Finally, we show the coincidence for general $\varphi, \psi > -1$. By duality, it suffices to prove the coincidence $\hat{G}_\alpha^{(n)} f = \hat{V}_\alpha^{(n)} f$ for Borel $f \in L^2(E_n;m)$, where $\hat{G}_\alpha f = \int_0^\infty e^{-\alpha t}\hat{T}_t f dt$ and $\hat{T}_t f(x) := \mathbf{E}_x[Z_t \circ r_t f(X_t) : t < \tau_n]$. Considering the approximation $\psi_\ell := 1_{\{|\psi|>1/\ell\}}\psi$ for ψ, we can apply the result in (2) and the proof is quite similar with (2) [The boundedness of ψ away from 0 in (2) is only used for the application of (1)]. Therefore, we have the coincidence $\hat{G}_\alpha^{(n)} f = \hat{V}_\alpha^{(n)} f$, hence $G_\alpha^{(n)} f = V_\alpha^{(n)} f$ for any Borel $f \in L^2(E_n;m)$.

As in [9], by using Lemma 3.6 in [12], we have that G_α coincides with V_α, consequently T_t coincides with Q_t by use of the positivity preserving property of $(\mathcal{Q},\mathcal{F})$ (see Lemma 4.1). □

5 Feynman-Kac type formula for $\Lambda(M)$

In this section, we show that Theorem 3.1 yields an extension of Feynman-Kac formula for zero energy CAF N^u, studied by Chen and Zhang in [8], where u is a function in $\mathcal{F}$ having Kato class energy measure $\mu_{\langle M^u\rangle}$.

Let M be a locally square-integrable MAF on $I(\zeta)$. Let φ be the jumping function for M, that is, φ vanishes on the diagonal of $E_\partial \times E_\partial$ and $\Delta M_t = \varphi(X_{t-}, X_t)$ for ${}^\forall t \in]0,\zeta_p[$, $\mathbf{P}_m$-a.s. We assume that $\varphi(x,y) + \varphi(y,x) = 0$ for $x, y \in E_\partial$, which includes the case $M = M^u$ for $u \in \mathcal{F}_e$. Under this assumption, we have $\mathbf{P}_m$-a.s.

$$\Lambda(M)_t = -\frac{1}{2}\left(M_t + M_t \circ r_t + \varphi(X_t, X_{t-})\right) \qquad \text{for } t \in]0,\zeta[.$$

We further assume that $\mu_{\langle M\rangle}$, the energy measure of quadratic variation process $\langle M\rangle$, is of Hardy class and satisfies

$$\delta_0 := [2\delta(\langle M^c\rangle)]^{1/2} + [2\delta(\langle M^j\rangle)]^{1/2} + \frac{1}{2}[\delta(\langle M^k\rangle)]^{1/2} < 1, \tag{5.1}$$

where M^j is the purely jump part and M^k is the killing part of M, respectively. Actually, M^d, M^j and M^k can be expressed as follows:

$$M_t^d := \lim_{k\to\infty}\lim_{\varepsilon\to 0}\left(\sum_{s\in]0,t]} 1_{G_k}(X_{s-})(\varphi 1_{\{|\varphi|>\varepsilon\}})(X_{s-}, X_s) - (1_{G_k} N(\varphi 1_{\{|\varphi|>\varepsilon\}}) * H)_t\right),$$

$$M_t^j := \lim_{k\to\infty}\lim_{\varepsilon\to 0}\left(\sum_{s\in]0,t]} 1_{G_k}(X_{s-})(\varphi 1_{\{|\varphi|>\varepsilon\}})(X_{s-}, X_s)1_{\{s<\zeta\}} - (1_{G_k} N(1_{E\times E}\varphi 1_{\{|\varphi|>\varepsilon\}}) * H)_t\right),$$

$$M_t^k := \lim_{k\to\infty}\lim_{\varepsilon\to 0}\left(1_{G_k}(X_{\zeta-})(\varphi 1_{\{|\varphi|>\varepsilon\}})(X_{\zeta-},\partial)1_{\{t\ge\zeta_i\}} - \int_0^t 1_{G_k}(X_s)(1_{\{|\varphi|>\varepsilon\}}\varphi)(X_s,\partial)N(X_s,\{\partial\})dH_s\right).$$

Note $M^d = M^j + M^k$. Here $\{G_k\}$ is a nest of finely open Borel sets such that $1_{G_k} * M \in \overset{\circ}{\mathcal{M}}$ (see Proposition 2.17 in [3]). We also see $\langle M^j\rangle_t = \int_0^t \left(\int_E \varphi(X_s,y)^2 N(X_s,dy)\right) dH_s$, $\langle M^k\rangle_t = \int_0^t \varphi(X_s,\partial)^2 N(X_s,\{\partial\})dH_s$ and $\mu_{\langle M^j\rangle}(dx) = N(1_{E\times E}\varphi^2)\mu_H(dx)$, $\mu_{\langle M^k\rangle}(dx) = \varphi(x,\partial)^2 N(x,\{\partial\})\mu_H(dx)$.

We consider the following quadratic form $\mathcal{Q}$ on $\mathcal{F}$: For $f,g\in\mathcal{F}$,

$$\begin{aligned}\mathcal{Q}(f,g):=\mathcal{E}(f,g)&+\frac{1}{2}\int_E g\,d\mu_{\langle M^{f,c},M^c\rangle}+\frac{1}{2}\int_E f\,d\mu_{\langle M^{g,c},M^c\rangle}\\ &+\frac{1}{2}\int_E g(x)\left(\int_E (f(y)-f(x))\varphi(x,y)N(x,dy)\right)\mu_H(dx)\\ &\quad+\frac{1}{2}\int_E f(x)\left(\int_E (g(y)-g(x))\varphi(x,y)N(x,dy)\right)\mu_H(dx)\\ &\quad-\int_E f(x)g(x)\varphi(x,\partial)N(x,\{\partial\})\mu_H(dx)\end{aligned}\tag{5.2}$$

From (5.1), we have (3.4) and (3.5) with

$$\alpha_0:=\gamma(\langle M^c\rangle)\left(2/\delta(\langle M^c\rangle)\right)^{1/2}+\gamma(\langle M^j\rangle)\left(2/\delta(\langle M^j\rangle)\right)^{1/2}+\frac{1}{2}\gamma(\langle M^k\rangle)\left(1/\delta(\langle M^k\rangle)\right)^{1/2}.\tag{5.3}$$

Note that for $f,g\in\mathcal{F}_b$

$$\mathcal{Q}(f,g)=\mathcal{E}(f,g)+\frac{1}{2}\nu_{\langle M^{fg},M\rangle}(E),\tag{5.4}$$

where the signed measure $\nu_{\langle M,N\rangle}$ for locally square-integrable MAFs M,N on $[\![0,\zeta[\![$ (more strongly, on $I(\zeta)$) is defined by $\nu_{\langle M,N\rangle}:=\mu_{\langle M^c,N^c\rangle}+\mu_{\langle M^j,N^j\rangle}+2\mu_{\langle M^k,N^k\rangle}$. We have the following result extending Theorem 1.2 in [8].

Theorem 5.1 *Suppose that $\mu_{\langle M\rangle}\in\mathbf{H}(X)$ and satisfies (5.1). Then $\overline{P}_tf(x):=\mathbf{E}_x[e^{\Lambda(M)_t}f(X_t)]$ is the semigroup associated with $(\mathcal{Q},\mathcal{F})$.*

Proof. We consider an adequate nest $\{F_n\}$ such that all objects $\mu_{\langle M^c\rangle}$, $N(\varphi^2)\mu_H$ restricted to F_n are of Kato class as in the proof of Theorem 3.1 and recognize the fine interior E_n of F_n as the whole space for each n.

First we assume the boundedness of $|\varphi|$, that is, $1/L\le|\varphi|\le L$ for some $L>0$. Note that $|e^x-1|\le|x|\frac{e^{L/2}-1}{L/2}$ for $|x|\le L/2$. Let J_t be the purely discontinuous locally square-integrable MAF on $I(\zeta)$ satisfying $\Delta J_t=\exp[-\frac{1}{2}\varphi(X_{t-},X_t)]-1$, $t\in]0,\zeta[$ and set $\overline{M}_t:=-\frac{1}{2}M_t^c+J_t$ for $t\in I(\zeta)$. Then we see $\Delta\overline{M}_t=\Delta J_t>-1$ for $t\in]0,\zeta[$.

We then see on $\{t<\zeta\}$

$$e^{-\frac{1}{2}M_t}=\mathrm{Exp}\left(\overline{M}_t\right)e^{-A_t^\mu},$$

where

$$\mu(dx):=\int_{E_\partial}\left(1-\frac{\varphi(x,y)}{2}-e^{-\frac{\varphi(x,y)}{2}}\right)N(x,dy)\mu_H(dx)-\frac{1}{8}\mu_{\langle M^c\rangle}(dx).$$

Then we have on $\{t<\zeta\}$

$$e^{\Lambda(M)_t}=\mathrm{Exp}\left(\overline{M}_t\right)\circ r_t\mathrm{Exp}\left(\overline{M}_t\right)e^{-2A_t^\mu}e^{-\frac{\varphi(X_t,X_{t-})}{2}}.$$

Recall that $1_{E_n}\mu_{\langle\overline{M}\rangle}$ and $1_{E_n}|\mu|$ are of Kato class. Hence δ_0 for $\overline{M}$, A^μ, $(e^{-\varphi/2}-1)^2$ and $(e^{\varphi/2}-1)^2$ as in (3.2) strictly less than 1 over $(\mathcal{Q}^{(n)},\mathcal{F}^{(n)})$. Here $\mathcal{Q}^{(n)}:=\mathcal{Q}$ on $\mathcal{F}^{(n)}:=\mathcal{F}_{E_n}$. Therefore, by Theorem 3.1, we have the desired result for $(\mathcal{Q}^{(n)},\mathcal{F}^{(n)})$ on $L^2(E_n;m)$, that is, $\overline{P}_t^{(n)}f(x):=\mathbf{E}_x[e^{\Lambda(M)_t}f(X_t):t<\tau_n]$ is the semigroup of $(\mathcal{Q}^{(n)},\mathcal{F}^{(n)})$ on $L^2(E_n;m)$. Let $\overline{R}_\alpha^{(n)}:=\int_0^\infty e^{-\alpha t}\overline{P}_t^{(n)}dt$ be the resolvent of $\{\overline{P}_t^{(n)},t>0\}$.

For general φ, we approximate it by $\varphi_\ell:=1_{\{1/\ell<|\varphi|<\ell\}}\varphi$. We let $M_t^\ell:=M_t^c+M_t^{d,\ell}$, where $M_t^{d,\ell}$ is the purely discontinuous locally square-integrable MAF on $I(\zeta)$ satisfying

$$\Delta M_t^{d,\ell}=(1_{\{1/\ell<|\varphi|<\ell\}}\varphi)(X_{t-},X_t)\qquad\text{for } t\in]0,\zeta[.$$

We have $\mathbf{e}(1_{E_n}(M-M^\ell))=\frac{1}{2}\int_{E_n}N(1_{\{1/\ell<|\varphi|<\ell\}^c}\varphi^2)d\mu_H\to 0$ as $\ell\to\infty$. Consequently, $\Lambda(M^\ell)_t$ converges uniformly to $\Lambda(M)_t$ on each compact subinterval of $[0,\tau_n[$ under $\mathbf{P}_m$ by taking some subsequence of ℓ. We consider $\left(\mathcal{Q}^{\ell,(n)},\overline{P}_t^{\ell,(n)},\overline{R}_\alpha^{\ell,(n)}\right)$ instead of $\left(\mathcal{Q}^{(n)},\overline{P}_t^{(n)},\overline{R}_\alpha^{(n)}\right)$ by replacing M with M^ℓ. In particular, $\overline{P}_t^{\ell,(n)}f(x):=\mathbf{E}_x[e^{\Lambda(M^\ell)_t}f(X_t):t<\tau_n]$. Then by the above argument $\mathcal{Q}^{\ell,(n)}(\overline{R}_\alpha^{\ell,(n)}f,g)=(f,g)_{L^2(E_n;m)}$ for $\alpha>\alpha_0$, $f\in L^2(E_n;m)$, $g\in\mathcal{F}^{(n)}$.

The proof of the uniform integrability of $e^{\Lambda(M^\ell)_t}$ under the law $1_{\{t<\tau_n\}}f(X_t)g(X_0)\mathbf{P}_m$ for nonnegative Borel $f,g\in L^2(E_n;m)$ is very similar to the analogous part of the proof of Theorem 3.1. Actually, we can prove that there exists a constant $\beta_0>0$ independent of ℓ (possibly depending on n) such that for any nonnegative Borel $f,g\in L^2(E_n;m)$

$$\sup_{\ell\in\mathbb{N}}\mathbf{E}_m\left[f(X_t)g(X_0)e^{2\Lambda(M^\ell)_t}:t<\tau_n\right]\le e^{\beta_0 t}\|f\|_2\|g\|_2. \tag{5.5}$$

Then we have that $\overline{P}_t^{(n)}$ maps $L^2(E_n;m)$ to itself with the bound $e^{\alpha_0 t}$, where α_0 independent of ℓ comes from (5.3) and the weak convergence of $\overline{P}_t^{\ell,(n)}\to\overline{P}_t^{(n)}$. As in the proof of Theorem 3.1(3), for each Borel $f\in L^2(E_n;m)$, $\overline{R}_\alpha^{\ell,(n)}f$ is weakly convergent to $\overline{R}_\alpha^{(n)}f$ for some subsequence of ℓ and $\overline{R}_\alpha^{(n)}f\in\mathcal{F}^{(n)}$. We only show that $\overline{R}_\alpha^{\ell,(n)}$ strongly converges to $\overline{R}_\alpha^{(n)}$ on $L^2(E_n;m)$. Note that $\mathcal{Q}^{(n)}$ has the following expression: For $u,v\in\mathcal{F}^{(n)}$,

$$\begin{aligned}\mathcal{Q}^{(n)}(u,v)=\mathcal{E}(u,v)&+\frac{1}{2}\int_{E_n}v\,d\mu_{\langle M^{u,c},M^c\rangle}+\frac{1}{2}\int_{E_n}u\,d\mu_{\langle M^{v,c},M^c\rangle}\\&+\frac{1}{2}\int_{E_n}v(x)\left(\int_{E_n}(u(y)-u(x))\varphi(x,y)N(x,dy)\right)\mu_H(dx)\\&+\frac{1}{2}\int_{E_n}u(x)\left(\int_{E_n}(v(y)-v(x))\varphi(x,y)N(x,dy)\right)\mu_H(dx)\\&-\int_{E_n}u(x)v(x)\left(\int_{E_n^c}\varphi(x,y)N(x,dy)+\varphi(x,\partial)N(x,\{\partial\})\right)\mu_H(dx).\end{aligned}$$

Similarly, $\mathcal{Q}^{\ell,(n)}$ has the same expression by replacing φ with φ_ℓ. Since

$$\begin{aligned}
&\mathcal{Q}^{\ell,(n)}(u,v)-\mathcal{Q}^{(n)}(u,v)\\
&=\frac{1}{2}\int_{E_n}v(x)\left(\int_{E_n}(u(y)-u(x))1_{\{1/\ell<|\varphi|<\ell\}^c}\varphi(x,y)N(x,dy)\right)\mu_H(dx)\\
&\quad+\frac{1}{2}\int_{E_n}u(x)\left(\int_{E_n}(v(y)-v(x))1_{\{1/\ell<|\varphi|<\ell\}^c}\varphi(x,y)N(x,dy)\right)\mu_H(dx)\\
&\quad-\int_{E_n}u(x)v(x)\left(\int_{E_n^c\cup\{\partial\}}1_{\{1/\ell<|\varphi|<\ell\}^c}\varphi(x,y)N(x,dy)\right)\mu_H(dx),
\end{aligned}$$

we have

$$\begin{aligned}
&|\mathcal{Q}^{\ell,(n)}(u,v)-\mathcal{Q}^{(n)}(u,v)|\\
&\le\frac{1}{2}\left(\int_{E_n}v^2N(1_{E_n\times E_n}\varphi^2)d\mu_H\right)^{1/2}\left(\int_{E_n\times E_n}(u(x)-u(y))^21_{\{1/\ell<|\varphi|<\ell\}^c}N(x,dy)\mu_H(dx)\right)^{1/2}\\
&+\frac{1}{2}\left(\int_{E_n}u^2N(1_{E_n\times E_n}(\varphi_\ell-\varphi)^2)d\mu_H\right)^{1/2}(2\mathcal{E}(v,v))^{1/2}\\
&+\left(\int_{E_n}u^2N(1_{E_n\times E_n^c\cup\{\partial\}}1_{\{1/\ell<|\varphi|<\ell\}^c}\varphi^2)d\mu_H\right)^{1/2}\left(\int_{E_n}v(x)^2N(x,E_n^c\cup\{\partial\})\mu_H(dx)\right)^{1/2}.
\end{aligned}$$

Taking $u=\overline{R}_\alpha^{(n)}f$ and $v:=g_\ell:=\overline{R}_\alpha^{\ell,(n)}f-\overline{R}_\alpha^{(n)}f$ and noting $\sup_{\ell\in\mathbb{N}}\mathcal{E}_1(g_\ell,g_\ell)<\infty$, we have

$$\begin{aligned}
\mathcal{E}_1(\overline{R}_\alpha^{\ell,(n)}f-\overline{R}_\alpha^{(n)}f,\overline{R}_\alpha^{\ell,(n)}f-\overline{R}_\alpha^{(n)}f)&\\
&\le M\mathcal{Q}_\alpha^{\ell,(n)}(\overline{R}_\alpha^{\ell,(n)}f-\overline{R}_\alpha^{(n)}f,\overline{R}_\alpha^{\ell,(n)}f-\overline{R}_\alpha^{(n)}f)\\
&=M\left(\mathcal{Q}_\alpha^{(n)}(\overline{R}_\alpha^{(n)}f,g_\ell)-\mathcal{Q}_\alpha^{\ell,(n)}(\overline{R}_\alpha^{(n)}f,g_\ell)\right)\\
&=M\left|\mathcal{Q}^{\ell,(n)}(\overline{R}_\alpha^{(n)}f,g_\ell)-\mathcal{Q}^{(n)}(\overline{R}_\alpha^{(n)}f,g_\ell)\right|\\
&\to 0\text{ as }\ell\to\infty.
\end{aligned}$$

Therefore, we obtain the desired result for $(\mathcal{Q}^{(n)},\mathcal{F}^{(n)})$. The rest is the same as before. □

6 Examples

Example 5.1 Let $X=(X_t,\mathbf{P}_x)_{x\in\mathbb{R}^d}$ be the symmetric α-stable process with $\alpha\in]0,2[$ and $(\mathcal{E},\mathcal{F})$ the corresponding Dirichlet form on $L^2(\mathbb{R}^d)$, and K a compact subset of $\mathbb{R}^d$. It is well-known that X is transient and has Green function given by $G(x,y):=\mathcal{A}(d,\alpha)|x-y|^{-(d-\alpha)}$ under $d>\alpha$, where

$$\mathcal{A}(d,\beta):=\frac{|\beta|\Gamma(\frac{d-\beta}{2})}{2^{1+\beta}\pi^{d/2}\Gamma(1+\frac{\beta}{2})},\quad\beta<d$$

and $(\mathcal{E},\mathcal{F})$ is given by

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{\mathcal{A}(d,-\alpha)}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}dxdy,\\ \mathcal{F} &= \Big\{u\in L^2(\mathbb{R}^d)\;\Big|\;\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy<\infty\Big\}.\end{aligned}$$

X has a Lévy system (N,H), where $N(x,dy):=\mathcal{A}(d,-\alpha)|x-y|^{-(d+\alpha)}dy$ and $H_t=t$. So $\mu_H(dx)=dx$.

We say that a measurable function f on $\mathbb{R}^d$ is of Kato class (resp. Hardy class) if the measure $|f(x)|dx$ is of Kato class (resp. Hardy class) and write $f\in\mathbf{K}(X)$ (resp. $f\in\mathbf{H}(X)$). Since there exists $C>0$ such that $p_t(x,y)\le Ct^{-d/\alpha}$ for all $x,y\in\mathbb{R}^d$, we have $L^p(\mathbb{R}^d)\subset\mathbf{K}(X)\subset\mathbf{H}(X)$ if $p>d/\alpha$ (resp. $p\ge 1$) for the case $d>\alpha$ (resp. for $d=\alpha=1$) (see [13]). For $d>\alpha$, we have the following Sobolev inequality

$$\|u\|_{\frac{2d}{d-\alpha}}\le C_{d,\alpha}\mathcal{E}(u,u).$$

In this case we see $L^{d/\alpha}(\mathbb{R}^d)\subset\mathbf{H}(X)$ and the coefficient $\delta(|f|)$ for $f\in L^{d/\alpha}(\mathbb{R}^d)$ can be taken to be arbitrarily small.

We assume one of the following:

- Let $\varphi_i>-1$ be a Borel function on $\mathbb{R}^d\times\mathbb{R}^d$ such that $|\varphi_i(x,y)|\le C_i 1_K(x)1_K(y)|x-y|^{\gamma_i}$ with $\gamma_i>\alpha/2$ and $C_i>0$, $i=1,2$. Then $N(\varphi_i^2)$ is bounded, hence in $\mathbf{K}(X)$.
- Take $f_i\in L^p_+(\mathbb{R}^d)$ with $p\ge d/\alpha$ for $d>\alpha$, or with $p\ge 1$ for $d=\alpha=1$ and set $\gamma_i>\alpha/2$, $i=1,2$. We let $\varphi_i(x,y):=f_i(x)^{1/2}1_K(x)1_K(y)|x-y|^{\gamma_i}$ (resp. $\varphi_i(x,y):=f_i(x)^{1/2}1_K(y)|x-y|^{\gamma_i}$) when $p>d/\alpha$ (resp. $p=d/\alpha$), $i=1,2$. Then $N(\varphi_i^2)\in L^{d/\alpha}(\mathbb{R}^d)\subset\mathbf{H}(X)$.

Let M, $\widehat{M}$ be locally square-integrable MAFs with $\Delta M_t=\varphi_1(X_{t-},X_t)$ and $\Delta\widehat{M}_t=\varphi_2(X_{t-},X_t)$ for $t>0$. Then $T_tf(x):=\mathbf{E}_x[Z_tf(X_t)]$ with $Z_t:=\mathrm{Exp}\Big(\widehat{M}_t\Big)\circ r_t\mathrm{Exp}(M_t)(1+\varphi_2(X_t,X_{t-}))$ is associated with

$$\begin{aligned}\mathcal{Q}(f,g) &:= \mathcal{E}(f,g)-\mathcal{A}(d,-\alpha)\int_{\mathbb{R}^d}g(x)\left(\int_{\mathbb{R}^d}(f(y)-f(x))\varphi_1(x,y)\frac{dy}{|x-y|^{d+\alpha}}\right)dx\\ &\quad -\mathcal{A}(d,-\alpha)\int_{\mathbb{R}^d}f(x)\left(\int_{\mathbb{R}^d}(g(y)-g(x))\varphi_2(x,y)\frac{dy}{|x-y|^{d+\alpha}}\right)dx\\ &\quad -\mathcal{A}(d,-\alpha)\int_{\mathbb{R}^d\times\mathbb{R}^d}f(y)g(x)\varphi_1(x,y)\varphi_2(y,x)\frac{dxdy}{|x-y|^{d+\alpha}}.\end{aligned}$$

Example 5.2 Let $X=(X_t,\mathbf{P}_x)_{x\in D}$ be the symmetric censored α-stable process on a bounded Lipschitz domain D with $d\ge 2>\alpha>1$ and $(\mathcal{E},\mathcal{F})$ the corresponding Dirichlet form on $L^2(D)$. $(\mathcal{E},\mathcal{F})$ on $L^2(D)$ is given by

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{\mathcal{A}(d,-\alpha)}{2}\int_{D\times D}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}dxdy,\\ \mathcal{F} &= W_0^{\alpha/2,2}(D),\end{aligned}$$

where $W_0^{\alpha/2,2}(D)$ is the completion of $C_0^\infty(D)$ with respect to the $\mathcal{E}_1^{1/2}$-norm and $\mathcal{A}(d,-\alpha)$ is the constant appear in the previous example. X has Lévy system (N,H), where $H_t=t$, $N(x,dy):=\mathcal{A}(d,-\alpha)|x-y|^{-(d+\alpha)}dy$ on D and $N(x,\{\partial\})=0$, $x\in D$.

Then the following Hardy inequality holds by Chen-Song [7]

$$\int_D \frac{u(x)^2}{d(x,\partial D)^\alpha}dx \le C_{D,\alpha}\mathcal{E}(u,u), \quad u\in\mathcal{F}.$$

Let $\varepsilon>2(2+\sqrt{6})C_{D,\alpha}$ and for each $i=1,2$ and $\gamma_i>\alpha$ set

$$\varphi_i(x,y):=\left(\sup_{x\in D}\int_{D-x}|z|^{\gamma_i-\alpha-1}dz\right)^{-1/2}\left(\frac{1}{\mathcal{A}(d,-\alpha)}\cdot\frac{1}{(C_{D,\alpha}+\varepsilon)}\frac{|x-y|^{\gamma_i}}{d(x,\partial D)^\alpha}\right)^{1/2}.$$

Then $N(\varphi_i^2)(x)=(C_{D,\alpha}+\varepsilon)^{-1}d(x,\partial D)^{-\alpha}$, hence $\delta(\varphi_i^2)\le C_{D,\alpha}/(C_{D,\alpha}+\varepsilon)$ We have

$$\delta_0:=\sqrt{2\delta(\varphi_1^2)}+\sqrt{2\delta(\varphi_2^2)}+\sqrt{\delta(\varphi_1^2)\delta(\varphi_2^2)}<1.$$

Let M, $\widehat{M}$ be locally square-integrable MAFs with $\Delta M_t=\varphi_1(X_{t-},X_t)$ and $\Delta\widehat{M}_t=\varphi_2(X_{t-},X_t)$ for $t\in]0,\zeta[$. Then $T_tf(x):=\mathbf{E}_x[Z_tf(X_t)]$ with $Z_t:=\mathrm{Exp}\left(\widehat{M}_t\right)\circ r_t\mathrm{Exp}(M_t)(1+\varphi_2(X_t,X_{t-}))$ is associated with

$$\begin{aligned}\mathcal{Q}(f,g) \;:=\; & \mathcal{E}(f,g)-\mathcal{A}(d,-\alpha)\int_D g(x)\left(\int_D(f(y)-f(x))\varphi_1(x,y)\frac{dy}{|x-y|^{d+\alpha}}\right)dx\\ & -\mathcal{A}(d,-\alpha)\int_D f(x)\left(\int_D(g(y)-g(x))\varphi_2(x,y)\frac{dy}{|x-y|^{d+\alpha}}\right)dx\\ & -\mathcal{A}(d,-\alpha)\int_{D\times D}f(y)g(x)\varphi_1(x,y)\varphi_2(y,x)\frac{dxdy}{|x-y|^{d+\alpha}}.\end{aligned}$$

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