Large deviations for perturbed reflected diffusion processes

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Abstract

In this article, we establish a large deviation principle for the solutions of perturbed reflected diffusion processes. The key is to prove a uniform Freidlin-Ventzell estimates of perturbed diffusion processes.

Keywords: Large deviations; perturbed diffusion processes; uniform Freidlin-Ventzell estimates.

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1 Introduction

In [6], Doney and Zhang obtained the existence and uniqueness of the solutions for the following perturbed diffusion and perturbed reflected diffusion equations:

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \alpha \sup_{0\le s\le t} X_s, \quad t \in [0,1], \tag{1.1}$$

and

$$Y_t = y + \int_0^t a(Y_s)dB_s + \alpha \sup_{0\le s\le t} Y_s + L_t, \quad t \in [0,1], \tag{1.2}$$

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where $\alpha \in (0,1)$, $x \in \mathbf{R}$, $y \in \mathbf{R}_+$ are deterministic, b, $\sigma : \mathbf{R} \to \mathbf{R}$ and $a : \mathbf{R}_+ \to \mathbf{R}$ are bounded Lipschitz continuous function, $\{L_t, t \in [0,1]\}$ is non-decreasing with $L_0 = 0$ and

$$\int_0^t \chi_{\{Y_s=0\}} dL_s = L_t, \quad t \in [0,1],$$

$\{B_t, t \in [0,1]\}$ is a 1-dimensional standard Brownian motion on a completed probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. The perturbed reflected Brownian motion was first introduced by Le Gall and M. Yor, in [8, 9], and subsequently studied by Carmona, Petit and Yor in [2, 3], Perman and Werner in [12] and Chaumont and Doney in [4]. Consider the small noise perturbations

$$X_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(X_s^\varepsilon) dB_s + \int_0^t b(X_s^\varepsilon) ds + \alpha \sup_{0\leq s\leq t} X_s^\varepsilon, \quad t \in [0,1], \tag{1.3}$$

and

$$Y_t^\varepsilon = y + \sqrt{\varepsilon} \int_0^t a(Y_s^\varepsilon) dB_s + \alpha \sup_{0\leq s\leq t} Y_s^\varepsilon + L_t^\varepsilon, \quad t \in [0,1]. \tag{1.4}$$

The aim of this paper is to establish a large deviation principle for the laws of $X_\cdot^\varepsilon$ and $Y_\cdot^\varepsilon$ on the space of continuous functions equipped with uniform topology. Our approach will be based on a classical result of Azencott [1], which can be stated as the follows:

Proposition 1.1 *Let (E_i, d_i) $(i = 1, 2)$ be two Polish spaces and $\Phi_i^\varepsilon : \Omega \to E_i$, $\varepsilon > 0$ $(i = 1, 2)$ be two families of random variables. Assume that*

(1) $\{\Phi_1^\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with the rate function $I_1 : E_1 \to [0, \infty]$.

(2) There exists a map $K : \{I_1 < \infty\} \to E_2$ such that for every $a < \infty$, $K : \{I_1 \leq a\} \to E_2$ is continuous.

(3) For any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in E_1$ satisfying $I_1(h) \leq a$ and $\varepsilon \leq \varepsilon_0$,

$$\mathbf{P}\left(d_2(\Phi_2^\varepsilon, K(h)) \geq \delta, d_1(\Phi_1^\varepsilon, h) \leq \rho\right) \leq \exp\left(-\frac{R}{\varepsilon}\right). \tag{1.5}$$

Then $\{\Phi_2^\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with the rate function

$$I(g) = \inf\{I_1(h);\ K(h) = g\}.$$

The estimates (1.5) is also known as the uniform Freidlin-Ventzell estimates.

The rest of the paper is organized as follows. Section 2 is for the large deviation principle of perturbed diffusion processes. The large deviation estimates for the perturbed reflected diffusion processes is in Section 3.

2 LDP for perturbed diffusion processes

In this section, we will give a large deviation principle of the perturbed diffusion processes (1.3). Let H denote the Cameron-Martin space, i.e.

$$H = \left\{ h(t) = \int_0^t \dot{h}(s)ds : [0,1] \to \mathbf{R};\ \int_0^1 |\dot{h}(s)|^2 ds < +\infty \right\},$$

and $\{\nu_\varepsilon, \varepsilon > 0\}$ be the probability measure induced by $X^\varepsilon_\cdot$ on $C_x([0,1], \mathbf{R})$, the space of all continuous functions $f : [0,1] \to \mathbf{R}$ such that $f(0) = x$, equipped with the supremum norm topology. For $h \in C_0([0,1], \mathbf{R})$, define $\widetilde{I} : C_0([0,1], \mathbf{R}) \to [0, \infty]$ by

$$\widetilde{I}(h) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{h}(s)|^2 ds, & \text{if } h \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

The well known Schilder theorem states that the laws μ_ε of $\{\sqrt{\varepsilon} B_t, t \in [0,1]\}$ satisfies a large deviation principle on $C_0([0,1], \mathbf{R})$ with the rate function $\widetilde{I}(\cdot)$, that is,

(a) for any closed subset $C \subset C_0([0,1], \mathbf{R})$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(C) \le - \inf_{g \in C} \widetilde{I}(g),$$

(b) for any open subset $G \subset C_0([0,1], \mathbf{R})$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \ge - \inf_{g \in G} \widetilde{I}(g).$$

Let $F(h)$ be the solution of the perturbed equation

$$\begin{aligned} F(h)(t) &= x + \int_0^t \sigma(F(h)(s))\dot{h}(s)ds + \int_0^t b(F(h)(s))ds \\ &\quad + \alpha \sup_{0 \le s \le t} F(h)(s), \quad h \in H,\ t \in [0,1]. \end{aligned} \tag{2.1}$$

Then we have the following main result.

Theorem 2.1 *If $\alpha \in (0,1)$, then the family $\{\nu_\varepsilon, \varepsilon > 0\}$ obeys a large deviation principle with the rate function*

$$I(g) = \inf \left\{ \widetilde{I}(h);\ F(h) = g \right\}, \quad g \in C_x([0,1], \mathbf{R}).$$

Before giving the proof of Theorem 2.1, recall the following lemma (see e.g. [13]).

Lemma 2.1 *Let* $Z_t := \int_s^t C(u)dB_u + \int_s^t D(u)du$ *be an Itô process, where* $0 \leq s < t < \infty$ *and* $C, D : [0,\infty) \times \Omega \to \mathbf{R}$ *are* $(\mathcal{F}_t)_{t\geq 0}$ *progressively measurable random processes. If* $|C(\cdot)| \leq M_1$ *and* $|D(\cdot)| \leq M_2$*, then for* $T > s$ *and* $R > 0$ *satisfying* $R > M_2(T-s)$*, we have*

$$\mathbf{P}\left(\sup_{s\leq t\leq T} |Z_t| \geq R\right) \leq \exp\left(-\frac{(R - M_2(T-s))^2}{2M_1^2(T-s)}\right). \tag{2.2}$$

Proof of Theorem 2.1. For $h \in H$, let $F(h)$ be defined as in (2.1). First we prove that for any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in C_0([0,1], \mathbf{R})$ satisfying $\widetilde{I}(h) \leq a$ and $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned}
&\mathbf{P}\left(\sup_{0\leq t\leq 1} |X_t^\varepsilon - F(h)(t)| \geq \delta, \sup_{0\leq t\leq 1} |\sqrt{\varepsilon}B_t - h(t)| \leq \rho\right) \\
\leq\ & \exp\left(-\frac{R}{\varepsilon}\right).
\end{aligned} \tag{2.3}$$

By (1.3) and (2.1),

$$\begin{aligned}
X_t^\varepsilon - F(h)(t) \ =\ & \int_0^t \sigma(X_s^\varepsilon)\left(\sqrt{\varepsilon}dB_s - \dot{h}(s)ds\right) \\
& + \int_0^t \left(\sigma(X_s^\varepsilon) - \sigma(F(h)(s))\right)\dot{h}(s)ds \\
& + \int_0^t \left(b(X_s^\varepsilon) - b(F(h)(s))\right)ds \\
& + \alpha\left(\sup_{0\leq s\leq t} X_s^\varepsilon - \sup_{0\leq s\leq t} F(h)(s)\right).
\end{aligned} \tag{2.4}$$

Consequently,

$$\begin{aligned}
|X_t^\varepsilon - F(h)(t)| \ \leq\ & \left|\int_0^t \sigma(X_s^\varepsilon)\left(\sqrt{\varepsilon}dB_s - \dot{h}(s)ds\right)\right| \\
& + L\int_0^t |X_s^\varepsilon - F(h)(s)|\left(1 + |\dot{h}(s)|\right)ds \\
& + \alpha\sup_{0\leq s\leq t} |X_s^\varepsilon - F(h)(s)|,
\end{aligned} \tag{2.5}$$

where $L > 0$ is the Lipschitz coefficient, and we also used the fact that

$$\left|\sup_{0\leq s\leq t} u(s) - \sup_{0\leq s\leq t} v(s)\right| \leq \sup_{0\leq s\leq t} |u(s) - v(s)|,$$

for any two continuous functions u and v on $\mathbf{R}_+$. Thus it follows from (2.5) that, for $t \in [0,1]$,

$$
\begin{aligned}
\sup_{0\le u\le t} |X_u^\varepsilon - F(h)(u)| \quad &\le \quad \frac{1}{(1-\alpha)} \sup_{0\le u\le t} \left| \int_0^u \sigma(X_s^\varepsilon) \left(\sqrt{\varepsilon} dB_s - \dot{h}(s) ds \right) \right| \\
&\quad + \frac{L}{(1-\alpha)} \int_0^t \sup_{0\le u\le s} |X_u^\varepsilon - F(h)(u)| \left(1 + |\dot{h}(s)| \right) ds.
\end{aligned}
\tag{2.6}
$$

By the Gronwall lemma this yields that

$$
\begin{aligned}
\sup_{0\le t\le 1} |X_t^\varepsilon - F(h)(t)| \quad &\le \quad \frac{1}{(1-\alpha)} \sup_{0\le t\le 1} \left| \int_0^t \sigma(X_s^\varepsilon) \left(\sqrt{\varepsilon} dB_s - \dot{h}(s) ds \right) \right| \\
&\quad \times \exp\left(\int_0^1 \frac{L}{(1-\alpha)} (1 + |\dot{h}(s)|) ds \right) \\
&\le \quad C_1 \sup_{0\le t\le 1} \left| \int_0^t \sigma(X_s^\varepsilon) \left(\sqrt{\varepsilon} dB_s - \dot{h}(s) ds \right) \right|, \qquad (2.7)
\end{aligned}
$$

where $C_1 := \frac{1}{1-\alpha} \exp\left(\frac{L(1+\|h\|_H)}{1-\alpha} \right) < \infty$ with $\|h\|_H := \left(\int_0^1 |\dot{h}(s)|^2 ds \right)^{\frac{1}{2}}$ for $h \in H$. Thus to prove (2.3), it suffices to prove that, for any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in C_0([0,1], \mathbf{R})$ satisfying $\widetilde{I}(h) \le a$ and $\varepsilon \le \varepsilon_0$,

$$
\begin{aligned}
&\mathbf{P} \left(\sup_{0\le t\le 1} \left| \int_0^t \sigma(X_s^\varepsilon) \left(\sqrt{\varepsilon} dB_s - \dot{h}(s) ds \right) \right| \ge \delta, \sup_{0\le t\le 1} |\sqrt{\varepsilon} B_t - h(t)| \le \rho \right) \\
\le \quad & \exp\left(-\frac{R}{\varepsilon} \right).
\end{aligned}
\tag{2.8}
$$

For $\varepsilon > 0$, define a probability measure $\mathbf{P}^\varepsilon$ on Ω by

$$
d\mathbf{P}^\varepsilon = Z_\varepsilon d\mathbf{P} := \exp\left(\frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{h}(s) dB_s - \frac{1}{2\varepsilon} \int_0^1 |\dot{h}(s)|^2 ds \right) d\mathbf{P}.
$$

Then by Girsanov theorem, $\{B_t^\varepsilon := B_t - \frac{1}{\sqrt{\varepsilon}} h(t), t \in [0,1]\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P}^\varepsilon)$. If we let

$$
\begin{aligned}
A^\varepsilon \quad &= \quad \left\{ \sup_{0\le t\le 1} \left| \int_0^t \sigma(X_s^\varepsilon) \left(\sqrt{\varepsilon} dB_s - \dot{h}(s) ds \right) \right| \ge \delta, \sup_{0\le t\le 1} |\sqrt{\varepsilon} B_t - h(t)| \le \rho \right\} \\
&= \quad \left\{ \sup_{0\le t\le 1} \left| \int_0^t \sigma(X_s^\varepsilon) \sqrt{\varepsilon} dB_s^\varepsilon \right| \ge \delta, \sup_{0\le t\le 1} |\sqrt{\varepsilon} B_t^\varepsilon| \le \rho \right\},
\end{aligned}
$$

then by the Hölder inequality

$$\begin{aligned}\mathbf{P}(A^\varepsilon) &= \int_\Omega Z_\varepsilon^{-1}\chi_{A^\varepsilon}(\omega)\mathbf{P}^\varepsilon(d\omega)\\ &\le \left(\int_\Omega Z_\varepsilon^{-2}(\omega)\mathbf{P}^\varepsilon(d\omega)\right)^{\frac{1}{2}}(\mathbf{P}^\varepsilon(A^\varepsilon))^{\frac{1}{2}}.\end{aligned}$$

Note that

$$\begin{aligned}\int_\Omega Z_\varepsilon^{-2}(\omega)\mathbf{P}^\varepsilon(d\omega) &= \mathbf{E}_{\mathbf{P}^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h(s)dB_s + \frac{1}{\varepsilon}\int_0^1 |\dot h(s)|^2 ds\right)\right]\\ &= \mathbf{E}_{\mathbf{P}^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h(s)dB_s^\varepsilon - \frac{1}{\varepsilon}\int_0^1 |\dot h(s)|^2 ds\right)\right]\\ &= \mathbf{E}_{\mathbf{P}^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h(s)dB_s^\varepsilon - \frac{2}{\varepsilon}\int_0^1 |\dot h(s)|^2 ds\right)\right]\\ &\quad \times\exp\left(\frac{1}{\varepsilon}\int_0^1 |\dot h(s)|^2 ds\right)\\ &= \exp\left(\frac{1}{\varepsilon}\|h\|_H^2\right).\end{aligned}$$

Therefore if $\widetilde{I}(h) \le a$, then

$$\mathbf{P}\left(A^\varepsilon\right) \le \exp\left(\frac{a}{\varepsilon}\right)(\mathbf{P}^\varepsilon(A^\varepsilon))^{\frac{1}{2}}. \tag{2.9}$$

Note that uner the probability $\mathbf{P}^\varepsilon$, X^ε satisfies the following stochastic differential equation.

$$U_t^\varepsilon = x + \int_0^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon}dB_s + \int_0^t (b(U_s^\varepsilon) + \sigma(U_s^\varepsilon)\dot h(s))ds + \alpha \sup_{0\le s\le t} U_s^\varepsilon. \tag{2.10}$$

Therefore,

$$\begin{aligned}\mathbf{P}^\varepsilon(A^\varepsilon) &= \mathbf{P}^\varepsilon\left(\sup_{0\le t\le 1}\left|\int_0^t \sigma(X_s^\varepsilon)\sqrt{\varepsilon}dB_s^\varepsilon\right| \ge \delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}B_t^\varepsilon| \le \rho\right)\\ &= \mathbf{P}\left(\sup_{0\le t\le 1}\left|\int_0^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon}dB_s\right| \ge \delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}B_t| \le \rho\right),\end{aligned}$$

Thus, in view of (2.9), to prove (2.8), the proof of (2.8) is reduced to show that, for any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in C_0([0,1], \mathbf{R})$ satisfying $\widetilde{I}(h) \le a$ and $\varepsilon \le \varepsilon_0$,

$$\mathbf{P}\left(\sup_{0\le t\le 1}\left|\int_0^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon}dB_s\right| \ge \delta,\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}B_t| \le \rho\right) \le \exp\left(-\frac{R}{\varepsilon}\right). \tag{2.11}$$

For $n \in \mathbf{N}$ fixed, set $t_k = \frac{k}{n}$ for $k \in \{0, 1, 2, \cdots, n\}$. Define

$$U_t^{\varepsilon,n} := U_{t_k}^\varepsilon, \quad \text{if } \ t_k \le t < t_{k+1}, \ k \in \{0, 1, \cdots, n-1\}.$$

Then for $\delta_1 > 0$

$$\begin{aligned} A^\varepsilon \quad &\subset \quad \left\{ \sup_{0 \le t \le 1} \left| \int_0^t (\sigma(U_s^\varepsilon) - \sigma(U_s^{\varepsilon,n})) \sqrt{\varepsilon} dB_s \right| \ge \frac{\delta}{2}, \sup_{0 \le t \le 1} |U_t^\varepsilon - U_t^{\varepsilon,n}| \le \delta_1 \right\} \\ &\quad \cup \left\{ \sup_{0 \le t \le 1} |U_t^\varepsilon - U_t^{\varepsilon,n}| > \delta_1 \right\} \\ &\quad \cup \left\{ \sup_{0 \le t \le 1} \left| \int_0^t \sigma(U_s^{\varepsilon,n}) \sqrt{\varepsilon} dB_s \right| \ge \frac{\delta}{2}, \sup_{0 \le t \le 1} |\sqrt{\varepsilon} B_t| \le \rho \right\} \\ &:= \quad B^\varepsilon \cup C^\varepsilon \cup D^\varepsilon. \end{aligned}$$

On the set $\left\{ \sup_{0 \le t \le 1} |U_t^\varepsilon - U_t^{\varepsilon,n}| \le \delta_1 \right\}$,

$$\sup_{0 \le s \le 1} \varepsilon |\sigma(U_s^\varepsilon) - \sigma(U_s^{\varepsilon,n})|^2 \le L^2 \varepsilon \delta_1^2.$$

By Lemma 2.1, it follows that

$$\mathbf{P}(B^\varepsilon) \le \exp\left(-\frac{\delta^2}{8L^2 \varepsilon \delta_1^2} \right) \le \exp\left(-\frac{R}{\varepsilon} \right), \tag{2.12}$$

if $\delta_1 \le \frac{\delta}{2L\sqrt{2R}}$.

On the other hand, on the set $\left\{ \sup_{0 \le t \le 1} |\sqrt{\varepsilon} B_t| \le \rho \right\}$, for any $t \in [0, 1]$

$$\begin{aligned} \left| \int_0^t \sigma(U_s^{\varepsilon,n}) \sqrt{\varepsilon} dB_s \right| \quad &= \quad \sqrt{\varepsilon} \left| \sum_{j=0}^{n-1} \sigma(U_{t_j}^\varepsilon)(B_{t_{j+1} \wedge t} - B_{t_j \wedge t}) \right| \\ &\le \quad 2 \sup_{0 \le t \le 1} |\sqrt{\varepsilon} B_t| \sum_{j \in \{j \in \{0, \cdots, n-1\}: t_j \le t\}} |\sigma(U_{t_j}^\varepsilon)| \\ &\le \quad 2nM\rho, \end{aligned}$$

where $M > 0$ is a common bound of b and σ. Therefore if $\rho < \frac{\delta}{4nM}$, then

$$D^\varepsilon = \emptyset.$$

To treat C^ε, we note that for $t \in [t_k, t_{k+1})$, $k \in \{0, 1, \cdots, n-1\}$,

$$\begin{aligned} |U_t^\varepsilon - U_t^{\varepsilon,n}| \quad &\le \quad \left| \int_{t_k}^t \sigma(U_s^\varepsilon) \sqrt{\varepsilon} dB_s + \int_{t_k}^t \left(b(U_s^\varepsilon) + \sigma(U_s^\varepsilon) \dot{h}(s) \right) ds \right| \\ &\quad + \alpha \left(\sup_{0 \le s \le t} U_s^\varepsilon - \sup_{0 \le s \le t_k} U_s^\varepsilon \right). \end{aligned} \tag{2.13}$$

Consider two possible cases:

• The case I: $\sup_{0\le s\le t} U_s^\varepsilon = \sup_{0\le s\le t_k} U_s^\varepsilon$.

We have

$$|U_t^\varepsilon - U_t^{\varepsilon,n}| \le \left| \int_{t_k}^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon} dB_s + \int_{t_k}^t \left(b(U_s^\varepsilon) + \sigma(U_s^\varepsilon)\dot{h}(s) \right) ds \right|. \quad (2.14)$$

• The case II: $\sup_{0\le s\le t} U_s^\varepsilon > \sup_{0\le s\le t_k} U_s^\varepsilon$.

Then there exists $u \in (t_k, t]$ such that $U_u^\varepsilon = \sup_{0\le s\le t} U_s^\varepsilon$, which yields that

$$\begin{aligned}
\left| U_t^\varepsilon - U_{t_k}^\varepsilon \right| &\le \left| \int_{t_k}^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon} dB_s + \int_{t_k}^t \left(b(U_s^\varepsilon) + \sigma(U_s^\varepsilon)\dot{h}(s) \right) ds \right| \\
&\quad + \alpha \left(U_u^\varepsilon - \sup_{0\le s\le t_k} U_s^\varepsilon \right) \\
&\le \left| \int_{t_k}^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon} dB_s + \int_{t_k}^t \left(b(U_s^\varepsilon) + \sigma(U_s^\varepsilon)\dot{h}(s) \right) ds \right| \\
&\quad + \alpha \left(U_u^\varepsilon - U_{t_k}^\varepsilon \right) \\
&\le \left| \int_{t_k}^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon} dB_s + \int_{t_k}^t \left(b(U_s^\varepsilon) + \sigma(U_s^\varepsilon)\dot{h}(s) \right) ds \right| \\
&\quad + \alpha \sup_{t_k\le t< t_{k+1}} \left| U_t^\varepsilon - U_{t_k}^\varepsilon \right|.
\end{aligned}$$

This implies

$$\begin{aligned}
& \sup_{t_k\le t<t_{k+1}} |U_t^\varepsilon - U_t^{\varepsilon,n}| \\
= & \sup_{t_k\le t<t_{k+1}} \left| U_t^\varepsilon - U_{t_k}^\varepsilon \right| \\
\le & \; C_2 \sup_{t_k\le t<t_{k+1}} \left| \int_{t_k}^t \sigma(U_s^\varepsilon)\sqrt{\varepsilon} dB_s + \int_{t_k}^t \left(b(U_s^\varepsilon) + \sigma(U_s^\varepsilon)\dot{h}(s) \right) ds \right|,
\end{aligned} \quad (2.15)$$

where $C_2 := \frac{1}{1-\alpha}$.

Since $\sup_{t_k\le s<t_{k+1}} \varepsilon |\sigma(U_s^\varepsilon)|^2 \le \varepsilon M^2$, $\sup_{t_k\le s<t_{k+1}} |b(U_s^\varepsilon)| \le M$ and

$$\sup_{t_k\le t<t_{k+1}} \left| \int_{t_k}^t \sigma(U_s^\varepsilon)\dot{h}(s) ds \right| \le M\sqrt{\frac{2a}{n}},$$

it follows from (2.15) and Lemma 2.1 that for $n \ge n_1 := \left[\frac{2MC_2}{\delta_1}\right] + 1 \vee n_2 :=$

$\left[\frac{8C_2^2M^2a}{\delta_1^2}\right]+1$,

$$\begin{aligned}
&\mathbf{P}\left(\sup_{t_k\le t<t_{k+1}}|U_t^\varepsilon-U_t^{\varepsilon,n}|>\delta_1\right)\\
\le\ &\mathbf{P}\left(\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t\sigma(U_s^\varepsilon)\sqrt{\varepsilon}dB_s+\int_{t_k}^t b(U_s^\varepsilon)ds\right|>\frac{\delta_1}{2C_2}\right)\\
&+\mathbf{P}\left(\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t\sigma(U_s^\varepsilon)\dot{h}(s)ds\right|>\frac{\delta_1}{2C_2}\right)\\
\le\ &\exp\left(-n\frac{\delta_1^2}{8C_2^2M^2\varepsilon}\right).
\end{aligned}\tag{2.16}$$

Therefore

$$\begin{aligned}
\mathbf{P}(C^\varepsilon)\ &=\ \mathbf{P}\left(\max_{k\in\{0,1,\cdots,n-1\}}\sup_{t_k\le t<t_{k+1}}|U_t^\varepsilon-U_t^{\varepsilon,n}|>\delta_1\right)\\
&\le\ \sum_{k=0}^{n-1}\mathbf{P}\left(\sup_{t_k\le t<t_{k+1}}|U_t^\varepsilon-U_t^{\varepsilon,n}|>\delta_1\right)\\
&\le\ n\exp\left(-n\frac{\delta_1^2}{8C_2^2M^2\varepsilon}\right).
\end{aligned}\tag{2.17}$$

Now given $R>0$, choose first $\delta_1>0$ such that (2.12) holds. Then choose n large enough so that

$$n\exp\left(-n\frac{\delta_1^2}{8C_2^2M^2\varepsilon}\right)\le\exp\left(-\frac{R}{\varepsilon}\right).$$

Finally for the fixed n, there exists $\rho>0$ such that $D^\varepsilon=\emptyset$. Combining above inequalities proves (2.11).

Next, we prove the map $F:H\to C_x([0,1],\mathbf{R})$ is continuous on $\{h:\widetilde{I}(h)\le a\}$ for $a\in[0,\infty)$. Let $h_n,h\in\{h\in C_0([0,1],\mathbf{R}):\widetilde{I}(h)\le a\}$ with $h_n\to h$ in $C_0([0,1],\mathbf{R})$. Since $\{h:\widetilde{I}(h)\le a\}$ is weakly compact in H, we conclude that h_n also weakly converges to h in H. By the Lipschitz continuity,

$$\begin{aligned}
&\sup_{0\le s\le t}|F(h_n)(s)-F(h)(s)|\\
\le\ &\xi_n(t)+\alpha\sup_{0\le s\le t}|F(h_n)(s)-F(h)(s)|\\
&+L\int_0^t\sup_{0\le u\le s}|F(h_n)(u)-F(h(u))|\,(1+|\dot{h}_n(s)|)ds,
\end{aligned}$$

where

$$\xi_n(t) = \sup_{0\le s\le t} \left| \int_0^s \sigma(F(h)(u))(\dot{h}_n(u) - \dot{h}(u))du \right| \tag{2.18}$$

Applying the Gronwall's lemma,

$$\begin{aligned} & \sup_{0\le t\le 1} |F(h_n)(t) - F(h)(t)| \\ \le \; & \frac{2M}{1-\alpha}\xi_n(1) \times \exp\left(\frac{L\left(1+\sqrt{2\widetilde{I}(h_n)}\right)}{1-\alpha} \right) \\ \le \; & \frac{2M}{1-\alpha}\exp\left(\frac{L\left(1+\sqrt{2a}\right)}{1-\alpha} \right)\xi_n(1). \end{aligned}$$

Since $h_n \to h$ weakly in H, for every $s \in [0,1]$,

$$\int_0^s \sigma(F(h)(u))(\dot{h}_n(u) - \dot{h}(u))du \to 0, \quad \text{as } n \to \infty.$$

On the other hand, it is easy to see that there exists a constant C_a such that

$$\left| \int_s^t \sigma(F(h)(u))(\dot{h}_n(u) - \dot{h}(u))du \right| \le C_a |t-s|^{\frac{1}{2}}.$$

Therefore, we conclude that $\xi_n(1) \to 0$, as $n \to \infty$. Consequently,

$$\sup_{0\le t\le 1} |F(h_n)(t) - F(h)(t)| \to 0, \quad \text{as } n \to \infty.$$

which proves the continuity of the mapping F. Now, letting $\Phi_1^\varepsilon = \sqrt{\varepsilon}B.$ and $K(\cdot) = F(\cdot)$, Theorem 2.1 follows from Proposition 1.1. □

3 LDP for perturbed reflected diffusion processes

In this section, we will prove the large deviation principle for the solution of the perturbed reflected diffusion equation (1.4).

For $y \ge 0$ and $f \in C_y([0,1], \mathbf{R})$, define two operators $\Gamma : C_y([0,1], \mathbf{R}) \to C_y([0,1], \mathbf{R}_+)$ and $K : C_y([0,1], \mathbf{R}) \to C_0([0,1], \mathbf{R}_+)$ by

$$\Gamma f = f + \widetilde{f} \quad \text{and} \quad Kf = \widetilde{f}, \quad \text{where} \quad \widetilde{f}(t) := -\inf_{s\le t}\left(f(s) \wedge 0\right), \; t \in [0,1].$$

By the reflection principle, the solution Y^ε of (1.4) is given by

$$Y_t^\varepsilon = (\Gamma Z^\varepsilon)(t) \quad \text{and} \quad L_t^\varepsilon = (KZ^\varepsilon)(t), \quad t \in [0,1], \tag{3.1}$$

where Z^ε is a solution of the following stochastic equation

$$Z_t^\varepsilon = y + \sqrt{\varepsilon}\int_0^t \sigma((\Gamma Z^\varepsilon)(s))dB_s + \alpha \sup_{0\le s\le t}(\Gamma Z^\varepsilon)(s), \quad t\in[0,1].$$

For $h \in H$, let $G(h)$ be the solution of the following equation

$$G(h)(t) = y + \int_0^t \sigma(G(h)(s))\dot{h}_s ds + \alpha \sup_{0\le s\le t} G(h)(s) + \eta_t, \quad t\in[0,1], \quad (3.2)$$

where $G(h)$ is continuous, nonnegative, and η is an increasing continuous function satisfying $\eta_t = \int_0^t \chi_{\{G(h)(s)=0\}} d\eta_s$.

Similar as (3.1), $G(h)$ can also be written as

$$G(h)(t) = (\Gamma V(h))(t) \quad \text{and} \quad \eta_t = (KV(h))(t), \quad t\in[0,1], \qquad (3.3)$$

where $V(h)$ satisfies

$$V(h)(t) = y + \int_0^t \sigma((\Gamma V(h))(s))\dot{h}_s ds + \alpha \sup_{0\le s\le t}(\Gamma V(h))(s), \quad t\in[0,1].$$

Let ν_ε^1 be the law of $Y_\cdot^\varepsilon$ on $C_y([0,1],\mathbf{R}_+)$. Our main result in this section is

Theorem 3.1 *If $\alpha\in(0,\frac{1}{2})$, then the family $\{\nu_\varepsilon^1, \varepsilon>0\}$ obeys a large deviation principle with the rate function*

$$I(g) = \inf\left\{\widetilde{I}(h);\ G(h)=g\right\}, \quad g\in C_y([0,1],\mathbf{R}_+).$$

By Proposition 1.1, Theorem 3.1 is the consequence of the following two propositions.

Proposition 3.1 *If $\alpha\in(0,\frac{1}{2})$, then for any $R,\delta,a>0$, there exist $\rho>0$ and $\varepsilon_0>0$ such that, for any $h\in C_0([0,1],\mathbf{R})$ satisfying $\widetilde{I}(h)\le a$ and $\varepsilon\le\varepsilon_0$,*

$$\mathbf{P}\left(\sup_{0\le t\le 1}|Y_t^\varepsilon - G(h)(t)| + \sup_{0\le t\le 1}|L_t^\varepsilon - \eta_t| \ge \delta, \sup_{0\le t\le 1}\left|\sqrt{\varepsilon}B_t - h(t)\right| \le \rho\right)$$
$$\le \exp\left(-\frac{R}{\varepsilon}\right).$$

Proof. The proof is similar to that of Theorem 2.1. We only highlight the differences. Note that for $f_1, f_2 \in C_y([0,1], \mathbf{R})$ and $t \in [0,1]$,

$$\sup_{0\le s\le t} |(\Gamma f_1)(s) - (\Gamma f_2)(s)| \le 2 \sup_{0\le s\le t} |f_1(s) - f_2(s)|. \tag{3.4}$$

Then by (3.1) and (3.3),

$$\sup_{0\le t\le 1} |Y_t^\varepsilon - G(h)(t)| + \sup_{0\le t\le 1} |L_t^\varepsilon - \eta_t| \le 3 \sup_{0\le t\le 1} |Z_t^\varepsilon - V(h)(t)|.$$

Therefore, the proof of Proposition 3.1 is reduced to show that for any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in C_0([0,1], \mathbf{R})$ satisfying $\widetilde{I}(h) \le a$ and $\varepsilon \le \varepsilon_0$,

$$\mathbf{P}\left(\sup_{0\le t\le 1} |Z_t^\varepsilon - V(h)(t)| \ge \delta, \sup_{0\le t\le 1} \left|\sqrt{\varepsilon} B_t - h(t)\right| \le \rho \right) \le \exp\left(-\frac{R}{\varepsilon}\right). \tag{3.5}$$

Using (3.4) and the Gronwall's lemma,

$$\sup_{0\le t\le 1} |Z_t^\varepsilon - V(h)(t)| \le C_3 \sup_{0\le t\le 1} \left| \int_0^t \sigma((\Gamma Z^\varepsilon)(s)) \sqrt{\varepsilon} dB_s^\varepsilon \right|,$$

where $C_3 := \frac{\exp\left(\frac{2L}{1-2\alpha}\|h\|_H\right)}{1-2\alpha}$. Thus as in section 2, by the Girsanov's theorem, to prove (3.5), it suffices to prove

$$\mathbf{P}\left(\sup_{0\le t\le 1} \left| \int_0^t \sigma((\Gamma \bar{Z}^\varepsilon)(s)) \sqrt{\varepsilon} dB_s \right| \ge \delta, \sup_{0\le t\le 1} \left|\sqrt{\varepsilon} B_t\right| \le \rho \right) \le \exp\left(-\frac{R}{\varepsilon}\right),$$

where $\bar{Z}^\varepsilon$ satisfies

$$\begin{aligned} \bar{Z}_t^\varepsilon &= y + \sqrt{\varepsilon} \int_0^t \sigma((\Gamma \bar{Z}^\varepsilon)(s)) dB_s + \int_0^t \sigma((\Gamma \bar{Z}^\varepsilon)(s)) \dot{h}(s) ds \\ &\quad + \alpha \sup_{0\le s\le t} (\Gamma \bar{Z}^\varepsilon)(s), \quad t \in [0,1]. \end{aligned}$$

For a function f, put $\Gamma(f)^{(n)}(t) = \Gamma(f)(t_k)$, if $t_k \le t < t_{k+1}$, $k \in \{0, 1, \cdots, n-1\}$. Then

$$\begin{aligned} \widetilde{A}^\varepsilon &:= \left\{ \sup_{0\le t\le 1} \left| \int_0^t \sigma((\Gamma \bar{Z}^\varepsilon)(s)) \sqrt{\varepsilon} dB_s \right| \ge \delta, \sup_{0\le t\le 1} \left|\sqrt{\varepsilon} B_t\right| \le \rho \right\} \\ &\subset \left\{ \sup_{0\le t\le 1} \left| \int_0^t (\sigma((\Gamma \bar{Z}^\varepsilon)(s)) - \sigma((\Gamma \bar{Z}^\varepsilon)^{(n)}(s))) \sqrt{\varepsilon} dB_s \right| \ge \frac{\delta}{2}, \right. \end{aligned}$$

$$
\begin{aligned}
& \left.\sup_{0\le t\le 1}|\Gamma(\bar Z^\varepsilon)(t)-\Gamma(\bar Z^\varepsilon)^{(n)}(t)|\le\delta_1\right\}\\
& \cup\left\{\sup_{0\le t\le 1}|\Gamma(\bar Z^\varepsilon)(t)-\Gamma(\bar Z^\varepsilon)^{(n)}(t)|>\delta_1\right\}\\
& \cup\left\{\sup_{0\le t\le 1}\left|\int_0^t\sigma((\Gamma(\bar Z^\varepsilon)^{(n)}(s))\sqrt{\varepsilon}dB_s\right|\ge\frac{\delta}{2},\ \sup_{0\le t\le 1}|\sqrt{\varepsilon}B_t|\le\rho\right\}\\
:= \;& \widetilde{B}^\varepsilon\cup\widetilde{C}^\varepsilon\cup\widetilde{D}^\varepsilon,
\end{aligned}
$$

Let us just sketch the proof of

$$
\mathbf{P}(\widetilde{C}^\varepsilon)\le\exp\left(-\frac{R}{\varepsilon}\right). \tag{3.6}
$$

For $t\in[t_k,t_{k+1})$, it is easy to see that

$$
\begin{aligned}
& \sup_{t_k\le s\le t}|\Gamma(\bar Z^\varepsilon)(s)-\Gamma(\bar Z^\varepsilon)^{(n)}(s)|\\
= \;& \sup_{t_k\le s\le t}|\Gamma(\bar Z^\varepsilon)(s)-\Gamma(\bar Z^\varepsilon)(t_k)|\\
\le \;& 2\sup_{t_k\le s\le t}|\bar Z^\varepsilon(s)-\bar Z^\varepsilon(t_k)|.
\end{aligned}\tag{3.7}
$$

But,

$$
\begin{aligned}
\left|\bar Z^\varepsilon_t-\bar Z^\varepsilon_{t_k}\right| \le\;& \left|\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\sqrt{\varepsilon}dB_s+\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\dot h(s)ds\right|\\
& +\alpha\left|\sup_{0\le s\le t}(\Gamma\bar Z^\varepsilon)(s)-\sup_{0\le s\le t_k}(\Gamma\bar Z^\varepsilon)(s)\right|\\
\le\;& \left|\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\sqrt{\varepsilon}dB_s+\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\dot h(s)ds\right|\\
& +\alpha\sup_{t_k\le s\le t}|\Gamma(\bar Z^\varepsilon)(s)-\Gamma(\bar Z^\varepsilon)(t_k)|\\
\le\;& \left|\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\sqrt{\varepsilon}dB_s+\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\dot h(s)ds\right|\\
& +2\alpha\sup_{t_k\le s\le t}\left|\bar Z^\varepsilon(s)-\bar Z^\varepsilon(t_k)\right|.
\end{aligned}\tag{3.8}
$$

Therefore

$$
\begin{aligned}
& \sup_{t_k\le t<t_{k+1}}\left|\bar Z^\varepsilon_t-\bar Z^\varepsilon_{t_k}\right|\\
\le\;& \frac{1}{1-2\alpha}\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\sqrt{\varepsilon}dB_s+\int_{t_k}^t\sigma((\Gamma\bar Z^\varepsilon)(s))\dot h(s)ds\right|.
\end{aligned}
$$

The rest of the proof is the same as the proof of (2.17) of Theorem 2.1. We omit it. □

Proposition 3.2 *If* $\alpha \in (0, \frac{1}{2})$, *then* $G : H \to C_y([0,1], \mathbf{R}_+)$ *and* $\eta : H \to C_0([0,1], \mathbf{R}_+)$ *defined by (3.3) are continuous on the compact set* $\{h \in C_0([0,1], \mathbf{R}) : \widetilde{I}(h) \leq a\}$ *for* $a \in [0, \infty)$.

Proof. By (3.4), it suffices to prove that the conclusion holds for $V : H \to C_y([0,1], \mathbf{R})$. Let $h_n, h \in \{h \in C_0([0,1], \mathbf{R}) : \widetilde{I}(h) \leq a\}$ with $h_n \to h$ in $C_0([0,1], \mathbf{R})$. Then we have

$$\begin{aligned} &\sup_{0 \leq s \leq t} |V(h_n)(t) - V(h)(t)| \\ \leq\ & \frac{2L}{1-2\alpha} \int_0^t \sup_{0 \leq u \leq s} |V(h_n)(s) - V(h)(s)|\, |\dot{h}_n(s)| ds + \frac{1}{1-2\alpha} \xi_n(t), \end{aligned}$$

where

$$\xi_n(t) \ := \ \sup_{0 \leq s \leq t} \left| \int_0^s \sigma((\Gamma V(h))(s))(\dot{h}_n(s) - \dot{h}(s)) ds \right|.$$

The rest of the proof is similar to the last part of the proof of Theorem 2.1. □

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References

[1] R. Azencott (1980): Grandes déviations et applications. *Ecole d'Eté de Probabilitiés de Saint-Flour VIII, 1978. Lecture Notes in Math.* **779** 1-176. Springer, New York.

[2] P. Carmona, F. Petit and M. Yor (1994): Some extentions of the arc-sine law as (partial) consequences of the scaling property of Brownian motion. *Probab. Theory Relat. Fields* **100** 1-29.

[3] P. Carmona, F. Petit and M. Yor (1998): Beta variables as times spent in $[0, \infty)$ by certain perturbed Brownian motions. *J. London Math. Soc.* **58** 239-256.

[4] L. Chaumont and R. Doney (1999) Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion. *Probab. Theory Relat. Fields* **113** 519-534.

[5] R. Doney (1998): Some calculations for perturbed Brownian motion. *Sém. Probab. XXXII, Lecture Notes Math.* **1686** 231-236. Springer, New York.

[6] R. Doney and T. Zhang (2005): Perturbed Skorohod equations and perturbed reflected diffusion processes. *Ann. I. H. Poincaré-PR* **41** 107-121.

[7] H. Doss and P. Priouret (1983): Petites perturbations de systems dynamiques avec reflexion. *Lecture Notes in Math.* No. 986, Springer, New York, 1983.

[8] J.F. Le Gall and M. Yor (1986): Excursions browniennes et carrés de processus de Bessel. *C. R. Acad. Sci. Paris Sér. I* **303** 73-76.

[9] J.F. Le Gall and M. Yor (1990): Enlacements du mouvement brownien autour des courbes de l'espace. *Trans. Amer. Math. Soc.* **317** 687-722.

[10] A. Millet, D. Nualart and M. Sanz (1992): Large deviations for a class of anticipating stochastic differential equations. *Ann. Probab.* **20**(4) 1902-1931.

[11] A. Mohammed and T. Zhang (2006): Large deviations for stochastic systems with memory. *Disc. and Contin. Dyn. Sys.-Series B* **6**(4) 881-893.

[12] M. Perman and W.Werner(1997): Perturbed Brownian motions. *Probability Theory and Related Fields* **108** 357-383.

[13] W. Stroock (1983): Some applications of stochastic calculus to partial differential equations. *Ecole d'Eté de Probabilités de Saint-Flour XI, 1981. Lecture Notes in Math.* **976** Springer, New York.