The Itô-Ventzell Formula and Forward Stochastic Differential Equations Driven by Poisson Random Measures

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Abstract

In this paper we obtain existence and uniqueness of solutions of forward stochastic differential equations driven by compensated Poisson random measures. To this end, an Itô-Ventzell formula for jump processes is proved and the flow properties of solutions of stochastic differential equations driven by compensated Poisson random measures are studied.

Key words and phrases: Itô-Ventzell formula, Lévy processes, Poisson random measures, Skorohod integrals, forward integrals, forward differential equations, Sobolev imbeding theorems.

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1 Introduction.

In recent years, there has been growing interests on jump processes, especially Lévy processes, partly due to the applications in mathematical finance. In [7] a Malliavin calculus was developed for Lévy processes. Among other things, the authors in [7] introduced a forward integral with respect to compensated Poisson random measures and showed that the forward integrals coincide with the Itô integrals when the integrands are non-anticipating. The purpose of this paper is to solve the following forward stochastic differential equation

$$X_t = X_0 + \int_0^t b(\omega, s, X_s)ds + \int_0^t \int_R \sigma(X_{s-}, z)\widetilde{N}(d^-s, dz) \tag{1.1}$$

with possibly anticipating coefficients and anticipating initial values, where $\widetilde{N}(d^-s, dz)$ indicates a forward integral. To this end, we adopt a same strategy as in [21] where anticipating stochastic differential equations driven by Brownian motion were studied. We first prove an Itô-Ventzell formula for jump processes and then go on to study the properties of the solution of the stochastic differential equation:

$$\phi_t(x) = x + \int_0^t \int_R \sigma(\phi_{s-}, z)\widetilde{N}(ds, dz) \tag{1.2}$$

Surprisingly little is known in the literature about the flow properties of $\phi_t(x)$ (see, however, [6] for the case of multidimensional Lévy processes). We obtain bounds on

$\phi_t(x)$, $\phi_t'(x)$ and $(\phi_t'(x))^{-1}$ under reasonable conditions on σ, where $\phi_t'(x)$ stands for the derivative of $\phi_t(x)$ with respect to the space variable x. Finally we show that the composition of ϕ_t with a solution of a random differential equation gives rise to a solution to our equation (1.1). We also mention that a pathwise approach to forward stochastic differential equations driven by Poisson processes is considered in [13].

The rest of the paper is organized as follows. Section 2 is the preliminaries. In Section 3, we prove the Itô-Ventzell formula. The flow properties of solutions of stochastic differential equations driven by compensated Poisson random measures are studied in Section 4, where the main result is also presented.

2 Preliminaries.

In this section, we recall some of the framework and preliminary results from [7], which we will use later. Let $\Omega = \mathcal{S}'(\mathbb{R})$ be the Schwartz space of tempered distributions equipped with its Borel σ-algebra $\mathcal{F} = \mathfrak{B}(\Omega)$. The space $\mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions on $\mathbb{R}$. We denote the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R})$ on $f \in \mathcal{S}(\mathbb{R})$ by $\langle \omega, f\rangle = \omega(f)$.

Thanks to the Bochner-Milnos-Sazonov theorem, the white noise probability measure P can be defined by the relation

$$\int_\Omega e^{i\langle \omega, f\rangle} dP(\omega) = e^{\int_\mathbb{R} \psi(f(x))dx - i\alpha \int_\mathbb{R} f(x)dx}, \qquad f \in \mathcal{S}(\mathbb{R}),$$

where the real constant α and

$$\psi(u) = \int_\mathbb{R} \big(e^{iuz} - 1 - iuz1_{\{|z|<1\}}\big)\nu(dz)$$

are the elements of the exponent in the characteristic functional of a pure jump Lévy process with the Lévy measure $\nu(dz)$, $z \in \mathbb{R}$, which, we recall, satisfies

$$\int_\mathbb{R} 1 \wedge z^2 \nu(dz) < \infty. \tag{2.1}$$

Assuming that

$$M := \int_\mathbb{R} z^2 \nu(dz) < \infty, \tag{2.2}$$

we can set $\alpha = \int_\mathbb{R} z1_{\{|z|>1\}}\nu(dz)$ and then we obtain that

$$E\big[\langle \cdot, f\rangle\big] = 0 \quad \text{and} \quad E\big[\langle \cdot, f\rangle^2\big] = M \int_\mathbb{R} f^2(x)dx, \qquad f \in \mathcal{S}(\mathbb{R}).$$

Accordingly the *pure jump Lévy process with no drift*

$$\eta = \eta(\omega, t), \qquad \omega \in \Omega, t \in \mathbb{R}_+,$$

that we do consider here and in the following, is the cadlag modification of $\langle \omega, \chi_{(0,t]}\rangle$, $\omega \in \Omega$, $t > 0$, where

$$\chi_{(0,t]}(x) = \begin{cases} 1, & 0 < x \le t \\ 0, & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}, \tag{2.3}$$

with $\eta(\omega, 0) := 0$, $\omega \in \Omega$. We remark that, for all $t \in \mathbb{R}_+$, the values $\eta(t)$ belong to $L_2(P) := L_2(\Omega, \mathcal{F}, P)$.

The Lévy process η can be expressed by

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(ds, dz), \qquad t \in \mathbb{R}_+, \tag{2.4}$$

where $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is the *compensated Poisson random measure* associated with η.

Let $\mathcal{F}_t$, $t \in \mathbb{R}_+$, be the completed filtration generated by the Lévy process in (2.4). We fix $\mathcal{F} = \mathcal{F}_\infty$.

Let $L_2(\lambda) = L_2(\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+), \lambda)$ denote the space of the square integrable functions on $\mathbb{R}_+$ equipped with the Borel σ-algebra and the standard Lebesgue measure $\lambda(dt)$, $t \in \mathbb{R}_+$. Denote by $L_2(\nu) := L_2(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \nu)$ the space of the square integrable functions on $\mathbb{R}$ equipped with the Borel σ-algebra and the Lévy measure ν. Write $L_2(P) := L_2(\Omega, \mathcal{F}, P)$ for the space of the square integrable random variables.

For the symmetric function $f \in L_2\big((\lambda \times \nu)^m\big)$ $(m = 1, 2, ...)$, define $I_0(f) := f$ for $f \in \mathbb{R}$.

$$I_m(f) := m! \int_0^\infty \int_{\mathbb{R}} ... \int_0^{t_2} \int_{\mathbb{R}} f(t_1, x_1, ..., t_m, x_m)\widetilde{N}(dt_1, dx_1)...\widetilde{N}(dt_m, dx_m) \qquad (m = 1, 2, ...)$$

and set $I_0(f) := f$ for $f \in \mathbb{R}$. We have

Theorem 2.1 (Chaos expansion). *Every $F \in L_2(P)$ admits the (unique) representation*

$$F = \sum_{m=0}^\infty I_m(f_m) \tag{2.5}$$

via the unique sequence of symmetric functions $f_m \in L_2\big((\lambda \times \nu)^m\big)$, $m = 0, 1, ...$

Let $X(t, z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be a random field taking values in $L_2(P)$. Then, for all $t \in \mathbb{R}_+$ and $z \in \mathbb{R}$, Theorem 2.1 provides the chaos expansion via symmetric functions

$$X(t, z) = \sum_{m=0}^\infty I_m\big(f_m(t_1, z_1, ..., t_m, z_m; t, z)\big)$$

Let $\widehat{f}_m = \widehat{f}_m(t_1, z_1, ..., t_{m+1}, z_{m+1})$ be the symmetrization of $f_m(t_1, z_1, ..., t_m, z_m; t, z)$ as a function of the $m + 1$ variables $(t_1, z_1), ..., (t_{m+1}, z_{m+1})$ with $t_{m+1} = t$ and $z_{m+1} = z$.

Definition 2.1 [11], [12] The random field $X(t, z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, is *Skorohod integrable* if
$\sum_{m=0}^\infty (m + 1)! \|\widehat{f}_m\|^2_{L^2((\lambda\times\nu)^{m+1})} < \infty$. Then its *Skorohod integral with respect to $\widetilde{N}$*, i.e.

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z)\widetilde{N}(\delta t, dz),$$

is defined by

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z)\widetilde{N}(\delta t, dz) := \sum_{m=0}^\infty I_{m+1}(\widehat{f}_m). \tag{2.6}$$

The Skorohod integral is an element of $L_2(P)$ and

(2.7)
$$\left\| \int_{I\!R_+} \int_{I\!R} X(t,z)\widetilde{N}(\delta t, dz) \right\|^2_{L^2(P)} = \sum_{m=0}^{\infty} (m+1)! \, \|\widehat{f}_m\|^2_{L^2((\lambda\times\nu)^{m+1})}.$$

Moreover,

(2.8)
$$E \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t,z)\widetilde{N}(\delta t, dz) = 0.$$

The Skorohod integral can be regarded as an extension of the Itô integral to *anticipating* integrands. In fact, the following result can be proved. Cf. [11],[12], [5], [7], [18] and [21].

Proposition 2.2 *Let $X(t,z)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}$, be a non-anticipating (adapted) integrand. Then the Skorohod integral and the Itô integral coincide in $L_2(P)$, i.e.*

$$\int_{I\!R_+} \int_{I\!R} X(t,z)\widetilde{N}(\delta t, dz) = \int_{I\!R_+} \int_{I\!R} X(t,z)\widetilde{N}(dt, dz).$$

Definition 2.2 The space $I\!D_{1,2}$ is the set of all the elements $F \in L^2(P)$ whose chaos expansion: $F = E[F] + \sum_{m=1}^{\infty} I_m(f_m)$, satisfies

$$\|F\|^2_{I\!D_{1,2}} := \sum_{m=1}^{\infty} m \cdot m! \, \|f_m\|^2_{L^2((\lambda\times\nu)^m)} < \infty.$$

The *Malliavin derivative* $D_{t,z}$ is an operator defined on $I\!D_{1,2}$ with values in the standard L_2-space $L_2(P \times \lambda \times \nu)$ given by

(2.9)
$$D_{t,z}F := \sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, t, z)),$$

where $f_m(\cdot, t, z) = f_m(t_1, z_1, ..., t_{m-1}, z_{m-1}; t, z)$.

Note that the operator $D_{t,z}$ is proved to be closed and to coincide with a certain difference operator defined in [22].

We recall the *forward integral* with respect to the Poisson random measure $\widetilde{N}$ defined in [7].

Definition 2.3 The *forward integral*

$$J(\theta) := \int_0^T \int_{\mathbb{R}} \theta(t,z)\widetilde{N}(d^-t, dz)$$

with respect to the Poisson random measure $\widetilde{N}$, of a caglad stochastic function $\theta(t,z)$, $t \in \mathbb{R}_+, z \in \mathbb{R}$, with

$$\theta(t,z) := \theta(t,z,\omega), \qquad \omega \in \Omega,$$

is defined as

(2.10)
$$\int_0^T \int_{\mathbb{R}} \theta(t,z)\widetilde{N}(d^-t, dz) := \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}} \theta(t,z) I_{U_m} \widetilde{N}(d^-t, dz)$$

if the limit exists in $L^2(P)$. Here $U_m, m = 1, 2, ...,$ is an increasing sequence of compact sets $U_m \subset \mathbb{R} \setminus \{0\}$ with $\nu(U_m) < \infty$ such that $\lim_{m\to\infty} U_m = \mathbb{R} \setminus \{0\}$

The relation between the forward integral and the Skorohod integral is the following.

Lemma 2.1 *[7] If $\theta(t,z)+D_{t^+,z}\theta(t,z)$ is Skorohod integrable and $D_{t^+,z}\theta(t,z) := \lim_{s\to t^+} D_{s,z}\theta(t,z)$ exists in $L^2(P\times\lambda\times\nu)$, then the forward integral exists in $L_2(P)$ and*

$$\int_0^T\int_{\mathbb{R}}\theta(t,z)\widetilde{N}(d^-t,dz) = \int_0^T\int_{\mathbb{R}} D_{t^+,z}\theta(t,z)\nu(dz)dt + \int_0^T\int_{\mathbb{R}}\big(\theta(t,z)+D_{t^+,z}\theta(t,z)\big)\widetilde{N}(\delta t,dz),$$

3 The Itô-Ventzell formula.

Consider the following two forward processes depending on a parameter $x\in\mathbb{R}$:

$$F_t(x) = F_0(x) + \int_0^t G_s(x)ds + \int_0^t\int_{\mathbb{R}} H_s(z,x)\widetilde{N}(d^-s,dz),$$

$$Y_t(x) = Y_0(x) + \int_0^t K_s(x)ds + \int_0^t\int_{\mathbb{R}} J_s(z,x)\widetilde{N}(d^-s,dz),$$

where the integrands are such that the above integrals belong to $L^2(\Omega\times\mathbb{R}, P\times dx)$. Let $<,>$ denote the inner product in the space $L^2(\mathbb{R},dx)$.

Lemma 3.1 *It holds that*

$$< F_t, Y_t > = < Y_0, F_0 > + \int_0^t < F_s, K_s > ds + \int_0^t < Y_s, G_s > ds + \int_0^t\int_{\mathbb{R}} < H_s(z,\cdot), J_s(z,\cdot) > \nu(dz)ds$$

$$+\int_0^t\int_{\mathbb{R}}[< F_{s-}, J_s(z,\cdot) > + < H_s(z,\cdot), Y_{s-} > + < H_s(z,\cdot), J_s(z,\cdot) >]\widetilde{N}(d^-s,dz) \tag{3.11}$$

Proof. Let $e_i, i\geq 1$ be an orthornormal basis of $L^2(\mathbb{R},dx)$. For each $i\geq 1$, we have

$$< F_t, e_i > = < F_0, e_i > + \int_0^t < G_s, e_i > ds + \int_0^t\int_{\mathbb{R}} < H_s(z,\cdot), e_i > \tilde{N}(d^-s,dz)$$

$$< Y_t, e_i > = < Y_0, e_i > + \int_0^t < K_s, e_i > ds + \int_0^t\int_{\mathbb{R}} < J_s(z,\cdot), e_i > \tilde{N}(d^-s,dz)$$

By the Itô's formula for forward processes in [7],

$$< F_t, e_i >< Y_t, e_i > = < F_0, e_i >< Y_0, e_i > + \int_0^t < F_s, e_i >< K_s, e_i > ds + \int_0^t < Y_s, e_i >< G_s, e_i > ds$$

$$+\int_0^t\int_{\mathbb{R}}[< F_{s-}, e_i >< J_s(z,\cdot), e_i > + < H_s(z,\cdot), e_i >< Y_{s-}, e_i >$$

$$+ < H_s(z,\cdot), e_i >< J_s(z,\cdot), e_i >]\widetilde{N}(d^-s,dz) + \int_0^t\int_{\mathbb{R}} < H_s(z,\cdot), e_i >< J_s(z,\cdot), e_i > \nu(dz)ds \tag{3.12}$$

Taking the summation over i, we get (3.11). ■

We now state and prove an Itô-Ventzell formula for forward processes. Let X_t be a forward process given by

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t\int_{\mathbb{R}}\gamma(s,z)\widetilde{N}(d^-s,dz) \tag{3.13}$$

Theorem 3.1 *Assume that $F_t(x)$ is C^1 w. r. t. the space variable $x \in \mathbb{R}$. Then*

$$F_t(X_t) = F_0(X_0)+\int_0^t F'_s(X_s)\alpha_s ds+\int_0^t\int_{\mathbb{R}}[F_s(X_s+\gamma(s,z))-F_s(X_s)-F'_s(X_s)\gamma(s,z)]\nu(dz)ds$$

$$+\int_0^t G_s(X_s)ds+\int_0^t\int_{\mathbb{R}}[H_s(z,X_s+\gamma(s,z))-H_s(z,X_s)]\nu(dz)ds$$

$$+\int_0^t\int_{\mathbb{R}}[F_{s-}(X_{s-}+\gamma(s,z))-F_{s-}(X_{s-})+H_s(z,X_{s-}+\gamma(s,z))]\widetilde{N}(d^-s,dz). \tag{3.14}$$

Here, and in the following, $F'_s(x)$ denotes the derivative of $F_s(x)$ with repect to x.

Proof. We are using the same method as in [21]. Let $\phi \in C_0^\infty(\mathbb{R},\mathbb{R}_+)$ with $\int_{\mathbb{R}}\phi(x)dx = 1$. Define for $\varepsilon > 0$, $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(\frac{x}{\varepsilon})$. It follows from Theorem 4.6 in [7] that

$$\phi_\varepsilon(X_t-x) = \phi_\varepsilon(X_0-x)+\int_0^t \phi'_\varepsilon(X_s-x)\alpha_s ds$$

$$+\int_0^t\int_{\mathbb{R}}[\phi_\varepsilon(X_s+\gamma(s,z)-x)-\phi_\varepsilon(X_s-x)-\phi'_\varepsilon(X_s-x)\gamma(s,z)]\nu(dz)ds$$

$$+\int_0^t\int_{\mathbb{R}}[\phi_\varepsilon(X_{s-}+\gamma(s,z)-x)-\phi_\varepsilon(X_{s-}-x)]\widetilde{N}(d^-s,dz) \tag{3.15}$$

Using Lemma 3.1 we get that

$$\int_{\mathbb{R}} F_t(x)\phi_\varepsilon(X_t-x)dx = \int_{\mathbb{R}} F_0(x)\phi_\varepsilon(X_0-x)dx+\int_0^t\int_{\mathbb{R}} F_s(x)\alpha_s\phi'_\varepsilon(X_s-x)dx$$

$$+\int_0^t ds\int_{\mathbb{R}} F_s(x)dx\int_{\mathbb{R}}[\phi_\varepsilon(X_s+\gamma(s,z)-x)-\phi_\varepsilon(X_s-x)-\phi'_\varepsilon(X_s-x)\gamma(s,z)]\nu(dz)$$

$$+\int_0^t ds\int_{\mathbb{R}} G_s(x)\phi_\varepsilon(X_s-x)dx+\int_0^t ds\int_{\mathbb{R}}\nu(dz)\int_{\mathbb{R}} H_s(z,x)[\phi_\varepsilon(X_s+\gamma(s,z)-x)-\phi_\varepsilon(X_s-x)]dx$$

$$+\int_0^t\int_{\mathbb{R}}\{\int_{\mathbb{R}} F_{s-}(x)[\phi_\varepsilon(X_{s-}+\gamma(s,z)-x)-\phi_\varepsilon(X_{s-}-x)]dx$$

$$+\int_{\mathbb{R}} H_s(z,x)\phi_\varepsilon(X_{s-}+\gamma(s,z)-x)dx\}\widetilde{N}(d^-s,dz) \tag{3.16}$$

Integrating by parts,

$$\int_{\mathbb{R}} F_t(x)\phi_\varepsilon(X_t-x)dx = \int_{\mathbb{R}} F_0(x)\phi_\varepsilon(X_0-x)dx+\int_0^t\int_{\mathbb{R}} F'_s(x)\alpha_s\phi_\varepsilon(X_s-x)dx$$

$$+\int_0^t ds\int_{\mathbb{R}} F_s(x)dx\int_{\mathbb{R}}[\phi_\varepsilon(X_s+\gamma(s,z)-x)-\phi_\varepsilon(X_s-x)]\nu(dz)-\int_0^t ds\int_{\mathbb{R}} F'_s(x)dx\int_{\mathbb{R}}\phi_\varepsilon(X_s-x)\gamma(s,z)]\nu(dz)$$

$$+\int_0^t ds\int_{\mathbb{R}} G_s(x)\phi_\varepsilon(X_s-x)dx+\int_0^t ds\int_{\mathbb{R}}\nu(dz)\int_{\mathbb{R}} H_s(z,x)[\phi_\varepsilon(X_s+\gamma(s,z)-x)-\phi_\varepsilon(X_s-x)]dx$$

$$+\int_0^t \int_{\mathbb{R}} \{ \int_{\mathbb{R}} F_{s-}(x)[\phi_\varepsilon(X_{s-}+\gamma(s,z)-x)-\phi_\varepsilon(X_{s-}-x)]dx$$

$$+\int_{\mathbb{R}} H_s(z,x)\phi_\varepsilon(X_{s-}+\gamma(s,z)-x)dx\}\widetilde{N}(d^-s,dz) \tag{3.17}$$

Since ϕ_ε approximates to identity as $\varepsilon \to 0$, letting $\varepsilon \to 0$ we obtain that

$$F_t(X_t) = F_0(X_0)+\int_0^t F_s'(X_s)\alpha_s ds+\int_0^t \int_{\mathbb{R}}[F_s(X_s+\gamma(s,z))-F_s(X_s)-F_s'(X_s)\gamma(s,z)]\nu(dz)ds$$

$$+\int_0^t G_s(X_s)ds + \int_0^t \int_{\mathbb{R}}[H_s(z,X_s+\gamma(s,z))-H_s(z,X_s)]\nu(dz)ds$$

$$+\int_0^t \int_{\mathbb{R}}[F_{s-}(X_{s-}+\gamma(s,z))-F_{s-}(X_{s-})+H_s(z,X_{s-}+\gamma(s,z))]\widetilde{N}(d^-s,dz) \tag{3.18}$$

■

Next we are going to deduce an Itô-Ventzell formula for Skorohod integrals using the relation between the forward integral and the Skorohod integral. Consider

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\widetilde{N}(\delta s,dz). \tag{3.19}$$

$$F_t(x) = F_0(x) + \int_0^t G_s(x)ds + \int_0^t \int_{\mathbb{R}} H_s(z,x)\widetilde{N}(\delta s,dz).$$

The stochastic integrals here are understood as Skorohod integrals. Let $\hat{H}_s(z,x) = S_{s,z}H_s(z,x)$, $\hat{\gamma}(s,z) = S_{s,z}\gamma(s,z)$, where $S_{s,z}$ is an operator satisfying

$$S_{s,z}G + D_{t^+,z}\Big(S_{s,z}G\Big) = G$$

for any smooth random variable G. See [7] for details.

Theorem 3.2 *Assume that $F_t(x)$ is C^1 w. r. t. the space variable $x \in R$. Then*

$$F_t(X_t) = F(X_0) + \int_0^t F_s'(X_s)[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\nu(dz)]ds + \int_0^t G_s(X_s)ds$$

$$+\int_0^t ds \int_{\mathbb{R}}[F_s(X_s+\hat{\gamma}(s,z))-F_s(X_s)-F_s'(X_s)\hat{\gamma}(s,z)]\nu(dz)$$

$$+\int_0^t ds \int_{\mathbb{R}}[\hat{H}_s(z,X_s+\hat{\gamma}(s,z))-\hat{H}_s(z,X_s)]\nu(dz)$$

$$+\int_0^t ds \int_{\mathbb{R}} D_{s^+,z}[F_{s-}(X_{s-}+\hat{\gamma}(s,z))-F_{s-}(X_{s-})+\hat{H}_s(z,X_s+\hat{\gamma}(s,z))]\nu(dz)ds$$

$$+\int_0^t ds \int_{\mathbb{R}}\{[F_{s-}(X_{s-}+\hat{\gamma}(s,z))-F_{s-}(X_{s-})+\hat{H}_s(z,X_s+\hat{\gamma}(s,z))]$$

$$+D_{s^+,z}[F_{s-}(X_{s-}+\hat{\gamma}(s,z))-F_{s-}(X_{s-})+\hat{H}_s(z,X_s+\hat{\gamma}(s,z))]\}\widetilde{N}(\delta s,dz) \tag{3.20}$$

Proof. Using the relation between forward integrals and Skorohod integrals, we rewrite X_t and $F_t(x)$ as

$$X_t = X_0 + \int_0^t [\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\nu(dz)]ds + \int_0^t \int_{\mathbb{R}} \hat{\gamma}(s,z)\widetilde{N}(d^- s, dz)$$

$$F_t(x) = F_0(x) + \int_0^t [G_s(x) - \int_{\mathbb{R}} D_{s^+,z}\hat{H}_s(z,x)\nu(dz)]ds + \int_0^t \int_{\mathbb{R}} \hat{H}_s(z,x)\widetilde{N}(d^- s, dz)$$

It follows from Theorem 3.1 that

$$\begin{aligned}
F_t(X_t) = F_0(X_0) + \int_0^t F_s'(X_s)[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\nu(dz)]ds \\
+ \int_0^t ds \int_{\mathbb{R}} [F_s(X_s + \hat{\gamma}(s,z)) - F_s(X_s) - F_s'(X_s)\hat{\gamma}(s,z)]\nu(dz) + \int_0^t G_s(X_s)ds \\
+ \int_0^t ds \int_{\mathbb{R}} [\hat{H}_s(z, X_s + \hat{\gamma}(s,z)) - \hat{H}_s(z, X_s)]\nu(dz) \\
+ \int_0^t ds \int_{\mathbb{R}} [F_{s-}(X_{s-} + \hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s + \hat{\gamma}(s,z))]\widetilde{N}(d^- s, dz) \\
= F(X_0) + \int_0^t F_s'(X_s)[\alpha_s - \int_{\mathbb{R}} D_{s^+,z}\hat{\gamma}(s,z)\nu(dz)]ds + \int_0^t G_s(X_s)ds \\
+ \int_0^t ds \int_{\mathbb{R}} [F_s(X_s + \hat{\gamma}(s,z)) - F_s(X_s) - F_s'(X_s)\hat{\gamma}(s,z)]\nu(dz) \\
+ \int_0^t ds \int_{\mathbb{R}} [\hat{H}_s(z, X_s + \hat{\gamma}(s,z)) - \hat{H}_s(z, X_s)]\nu(dz) \\
+ \int_0^t ds \int_{\mathbb{R}} D_{s^+,z}[F_{s-}(X_{s-} + \hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s + \hat{\gamma}(s,z))]\nu(dz)ds \\
+ \int_0^t ds \int_{\mathbb{R}} \{[F_{s-}(X_{s-} + \hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s + \hat{\gamma}(s,z))] \\
+ D_{s^+,z}[F_{s-}(X_{s-} + \hat{\gamma}(s,z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s + \hat{\gamma}(s,z))]\}\widetilde{N}(\delta s, dz)
\end{aligned}$$

■

Example 3.1 (Stock price influenced by a large investor with inside information)

Suppose the price $S_t = S_t(x)$ at time t of a stock is modelled by a geometric Lévy process of the form

$$dS_t(x) = S_{t-}(x)[\mu(t,x)dt + \int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz)], S_0 > 0 \quad (constant) \tag{3.21}$$

(See e. g. [2] for more information about the use of this type of process in financial modelling) Here $x \in \mathbb{R}$ is a parameter and for each x and z the processes $\mu(t) = \mu(t,x,\omega)$ and $\theta(t,z) = \theta(t,z,\omega)$ are $\mathcal{F}_t$-adapted , where $\mathcal{F}_t$ is the filtration generated by the driving Lévy process

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(ds,dz).$$

Suppose the value of this " hidden parameter" x is influenced by a large investor with inside information, so that x can be represented by a stochastic process X_t of the form

$$x = X_t = X_0 + \int_0^t \alpha(s)ds + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\widetilde{N}(d^- s, dz); \quad X_0 \in \mathbb{R} \tag{3.22}$$

where $\alpha(t)$ and $\gamma(t,z)$ are processes adapted to a larger insider filtration $\mathcal{G}_t$, satisfying $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$. (For a justification of the use of forward integrals in the modelling of insider trading, see e. g. [7]).

Combing (3.21) and (3.22) and using Theorem 3.1 we see that the dynamics of the corresponding stock price $S_t(X_t)$ is, with $S_t'(x) = \frac{\partial}{\partial x} S_t(x)$,

$$\begin{aligned} d(S_t(X_t)) = S_t'(X_t)\alpha(t)dt \\ + \int_{\mathbb{R}} \{S_t(X_t + \gamma(t,z)) - S_t(X_t) - \gamma(t,z)S_t'(X_t)\}\nu(dz)dt \\ + S_t(X_t)\mu(t, X_t)dt \\ + \int_{\mathbb{R}} \{S_t(X_t + \gamma(t,z)) - S_t(X_t)\}\theta(t,z)\nu(dz)dt \end{aligned}$$

$$+ \int_{\mathbb{R}} \{S_{t-}(X_{t-} + \gamma(t,z)) - S_{t-}(X_{t-}) + S_{t-}(X_{t-} + \gamma(t,z))\theta(t,z)\}\widetilde{N}(d^- t, dz) \tag{3.23}$$

By the Itô formula

$$S_t(x) = S_0 exp\{\int_0^t \mu(s,x)ds + \int_0^t \int_{\mathbb{R}} (ln(1+\theta(s,z)) - \theta(s,z))\nu(dz)ds$$

$$+ \int_0^t \int_{\mathbb{R}} ln(1+\theta(s,z))\widetilde{N}(ds,dz)\}, \tag{3.24}$$

and hence

$$S_t'(x) = S_t(x) \int_0^t \mu'(s,x)ds,$$

where

$$\mu'(s,x) = \frac{\partial}{\partial x}\mu(s,x).$$

Substituted into (3.23) this gives

$$dS_t(X_t)) = S_t(X_t))[\alpha(t) + \mu(t,X_t) + \int_0^t \mu'(s,X_t)ds]dt$$

$$+ \int_{\mathbb{R}} \{S_t(X_t + \gamma(t,z))(1+\theta(t,z)) - S_t(X_t)(1 + \theta(t,z) + \gamma(t,z)\int_0^t \mu'(s,X_t)ds)\}\nu(dz)dt$$

$$+ \int_{\mathbb{R}} \{S_{t-}(X_{t-} + \gamma(t,z))(1+\theta(t,z)) - S_{t-}(X_{t-})\}\widetilde{N}(d^- t, dz). \tag{3.25}$$

4 Forward SDEs Driven by Poisson Random Measures

Let $b(\omega, s, x) : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\sigma(x, z) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be measurable mappings (possibly anticipating). Let X_0 be a random variable. In this section, we are going to solve the following forward sde:

$$X_t = X_0 + \int_0^t b(\omega, s, X_s)ds + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}, z)\widetilde{N}(d^- s, dz) \tag{4.26}$$

Let $\phi_t(x), t \geq 0$ be the stochastic flow determined by the following non-anticipating SDE:

$$\phi_t(x) = x + \int_0^t \int_{\mathbb{R}} \sigma(\phi_{s-}(x), z)\widetilde{N}(ds, dz) \tag{4.27}$$

Define

$$\hat{b}(\omega, s, x) = (\phi'_s)^{-1}(x)b(\omega, s, \phi_s(x))$$

Consider the differential equation:

$$\frac{dY_t}{dt} = \hat{b}(\omega, t, Y_t), \quad Y_0 = X_0. \tag{4.28}$$

Theorem 4.1 *If $Y_t, t \geq 0$ is the unique solution to equation (4.28), then $X_t = \phi_t(Y_t), t \geq 0$ is the unique solution to equation (4.26).*

Proof. It follows from Theorem 3.1 that

$$\begin{aligned} X_t = \phi_t(Y_t) &= X_0 + \int_0^t \phi'_s(Y_s)\hat{b}(\omega, s, Y_s)ds + \int_0^t \int_{\mathbb{R}} \sigma(\phi_{s-}(Y_{s-}), z)\widetilde{N}(d^- s, dz) \\ &= X_0 + \int_0^t b(\omega, s, X_s)ds + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}, z)\widetilde{N}(d^- s, dz) \end{aligned}$$

■

Next we are going to provide appropriate conditions under which (4.28) has a unique solution. To this end, we need to study the flow generated by the solution of the following equation:

$$X_t(x) = x + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}(x), z)\widetilde{N}(ds, dz) \tag{4.29}$$

Let (p, D_p) denote the point process generating the Poisson random measure $N(dt, dz)$, where D_p, called the domain of the point process p, is a countable subset of $[0, \infty)$ depending on the random parameter ω.

Proposition 4.1 *Let $k \geq 1$. Assume that for $l = 1, 2, ..., 2k$,*

$$\int_{\mathbb{R}} |\sigma(y, z)|^l \nu(dz) \leq C(1 + |y|^l). \tag{4.30}$$

Let $X_t(x), t \geq 0$ be the unique solution to equation (4.29). Then, we have

$$E[\sup_{0 \leq t \leq T} |X_t(x)|^{2k}] \leq C_{T,k}(1 + |x|^{2k}). \tag{4.31}$$

Proof. It follows from Itô's formula that

$$(X_t(x))^{2k} = x^{2k} + \int_0^t \int_{\mathbb{R}} [(X_{s-}(x) + \sigma(X_{s-}(x), z))^{2k} - (X_{s-}(x))^{2k}] \widetilde{N}(ds, dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k} - 2k(X_s(x))^{2k-1} \sigma(X_s(x), z)] \nu(dz) ds. \tag{4.32}$$

Denote by M_t the martingale part in the above equation. We have

$$[M]_t^{\frac{1}{2}} = \left(\sum_{0 \leq s \leq t} (\Delta M_s)^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{0 \leq s \leq t, s \in D_p} [(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}]^2 \right)^{\frac{1}{2}}$$

$$\leq \sum_{0 \leq s \leq t, s \in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}|. \tag{4.33}$$

By Burkholder's inequality,

$$E[\sup_{0 \leq s \leq t} |M_s|] \leq CE([M]_t^{\frac{1}{2}})$$

$$\leq E[\sum_{0 \leq s \leq t, s \in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}|]$$

$$= E[\int_0^t \int_{\mathbb{R}} |(X_{s-}(x) + \sigma(X_{s-}(x), z))^{2k} - (X_{s-}(x))^{2k}| N(ds, dz)]$$

$$= E[\int_0^t \int_{\mathbb{R}} |(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k}| ds \nu(dz)].$$

By the Mean-Value Theorem, there exists $\theta(s, z, \omega) \in [0, 1]$ such that

$$(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k}$$

$$= 2k(X_s(x) + \theta(s, z, \omega)\sigma(X_s(x), z))^{2k-1} \sigma(X_s(x), z).$$

Therefore,

$$E[\sup_{0 \leq s \leq t} |M_s|]$$

$$\leq C_k E[\int_0^t ds |X_s(x)|^{2k-1} \int_{\mathbb{R}} |\sigma(X_s(x), z)| \nu(dz)]$$

$$+ C_k E[\int_0^t ds \int_{\mathbb{R}} |\sigma(X_s(x), z)|^{2k} \nu(dz)$$

$$\leq C_k + C_k \int_0^t E[|X_s(x)|^{2k}] ds. \tag{4.34}$$

By Taylor expansion, there exists $\eta(s,z,\omega)\in[0,1]$ such that

$$E[\int_0^t\int_{\mathbb{R}}|(X_s(x)+\sigma(X_s(x),z))^{2k}-(X_s(x))^{2k}-2k(X_s(x))^{2k-1}\sigma(X_s(x),z)|\nu(dz)ds]$$

$$=2k(2k-1)E[\int_0^t\int_{\mathbb{R}}|(X_s(x)+\eta(s,z,\omega)\sigma(X_s(x),z))^{2k-2}||\sigma(X_s(x),z)|^2 ds\nu(dz)]$$

$$\leq C_kE[\int_0^t ds|X_s(x)|^{2k-2}\int_{\mathbb{R}}|\sigma(X_s(x),z)|^2\nu(dz)]$$

$$+C_kE[\int_0^t ds\int_{\mathbb{R}}|\sigma(X_s(x),z)|^{2k}\nu(dz)]$$

$$\leq C_k+C_k\int_0^t E[|X_s(x)|^{2k}]ds. \tag{4.35}$$

(4.32), (4.34) and (4.35) imply that

$$E[\sup_{0\leq s\leq t}|X_s(x)|^{2k}]\leq C_k+C_k\int_0^t E[|X_s(x)|^{2k}]ds.$$

Applying Gronwall's lemma we get

$$E[\sup_{0\leq t\leq T}|X_t(x)|^{2k}]\leq C_{T,p}(1+|x|^{2k}).$$

■

Proposition 4.2 *Assume that* $\frac{\partial\sigma(y,z)}{\partial y}$ *exists and*

$$\sup_y\int_{\mathbb{R}}|\frac{\partial\sigma(y,z)}{\partial y}|^l\nu(dz)<\infty, \tag{4.36}$$

for $l=1,2,...,2k$. *Let* $X'_t(x)$ *denote the derivative of* $X_t(x)$ *w.r.t.* x. *Then there exists a constant* $C_{T,k}$ *such that*

$$E[\sup_{0\leq t\leq T}|X'_t(x)|^{2k}]\leq C_{T,k}. \tag{4.37}$$

Proof. Differentiating both sides of the equation (4.29) we get

$$X'_t(x)=1+\int_0^t\int_{\mathbb{R}}\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}X'_{s-}(x)\widetilde{N}(ds,dz). \tag{4.38}$$

Put

$$h(s,z)=\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}X'_{s-}(x).$$

By Itô's formula,

$$(X'_t(x))^{2k}=1+\int_0^t\int_{\mathbb{R}}[(X'_{s-}(x)+h(s,z))^{2k}-(X'_{s-}(x))^{2k}]\widetilde{N}(ds,dz)$$

$$+\int_0^t\int_{\mathbb{R}}[(X'_s(x)+h(s,z))^{2k}-(X'_s(x))^{2k}-2k(X'_s(x))^{2k-1}h(s,z)]\nu(dz)ds. \tag{4.39}$$

Denote the martingale part of the above equation by M. Reasoning as in the proof of Proposition 4.1 we have that

$$E[\sup_{0\leq s\leq t}|M_s|]\leq CE([M]_t^{\frac{1}{2}})$$

$$\leq CE[\int_0^t\int_{\mathbb{R}}|(X'_{s-}(x)+h(s,z))^{2k}-(X'_{s-}(x))^{2k}|N(ds,dz)]$$

$$=E[\int_0^t\int_{\mathbb{R}}|(X'_{s-}(x)+h(s,z))^{2k}-(X'_{s-}(x))^{2k}|ds\nu(dz)].$$

$$\leq C_kE[\int_0^t ds|X'_s(x)|^{2k-1}\int_{\mathbb{R}}|h(s,z)|\nu(dz)]$$

$$+C_kE[\int_0^t ds\int_{\mathbb{R}}|h(s,z)|^{2k}\nu(dz)$$

$$\leq C_kE[\int_0^t ds|X'_s(x)|^{2k}\int_{\mathbb{R}}|\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}|\nu(dz)]$$

$$+C_kE[\int_0^t ds|X'_s(x)|^{2k}\int_{\mathbb{R}}|\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}|^{2k}\nu(dz)]$$

$$\leq\hat{C}_k+\hat{C}_k\int_0^t E[|X'_s(x)|^{2k}]ds, \tag{4.40}$$

where

$$\hat{C}_k=C_k(\sup_y\int_{\mathbb{R}}|\frac{\partial\sigma(y,z)}{\partial y}|\nu(dz)+\sup_y\int_{\mathbb{R}}|\frac{\partial\sigma(y,z)}{\partial y}|^{2k}\nu(dz)).$$

A similar treatment applied to the second term in (4.39) yields

$$E[|\int_0^t\int_{\mathbb{R}}[(X'_s(x)+h(s,z))^{2k}-(X'_s(x))^{2k}-2k(X'_s(x))^{2k-1}h(s,z)]\nu(dz)ds|]$$

$$\leq C_k+C_k\int_0^t E[|X'_s(x)|^{2k}]ds. \tag{4.41}$$

Combining (4.39), (4.40)and (4.41) we get

$$E[\sup_{0\leq s\leq t}|X'_s(x)|^{2k}]\leq C_k(1+\int_0^t E[|X'_s(x)|^{2k}]ds).$$

An application of the Gronwall's inequality completes the proof. ■

Our next step is to give estimates for $(X'_t(x))^{-1}$. Define

$$Z_t=\int_0^t\int_{\mathbb{R}}\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\widetilde{N}(ds,dz).$$

Then we see that

$$X'_t(x)=1+\int_0^t X'_{s-}(x)dZ_s.$$

Define

$$W_t =: -Z_t + \int_0^t \int_{\mathbb{R}} \frac{\left(\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}\right)^2}{1+\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}} N(ds,dz).$$

Let $Y_t(x), t \geq 0$ be the solution to the equation:

$$Y_t(x) = 1 + \int_0^t Y_{s-}(x) dW_s. \tag{4.42}$$

An application of Itô's formula shows that $Y_t(x) = (X'_t(x))^{-1}$.

Proposition 4.3 *Assume*

$$\sup_y \int_{\mathbb{R}} \left| \frac{\frac{\partial \sigma(y,z)}{\partial y}^2}{1+\frac{\partial \sigma(y,z)}{\partial y}} \right|^l \nu(dz) < \infty, \tag{4.43}$$

for $l = 1, ..., 2k$. Then there exists a constant $C_{T,k}$ such that

$$E[\sup_{0\leq t\leq T} |Y_t(x)|^{2k}] \leq C_{T,k}. \tag{4.44}$$

Proof. Note that

$$Y_t(x) = 1 - \int_0^t Y_{s-}(x) \int_{\mathbb{R}} \frac{\partial \sigma(X_{s-}(x),z)}{\partial y} \widetilde{N}(ds,dz)$$

$$+ \int_0^t Y_{s-}(x) \int_{\mathbb{R}} \frac{\left(\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}\right)^2}{1+\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}} N(ds,dz). \tag{4.45}$$

Set

$$f(s,z) = Y_{s-}(x) \frac{\left(\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}\right)^2}{1+\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}},$$

$$h(s,z) = -Y_{s-}(x) \frac{\partial \sigma(X_{s-}(x),z)}{\partial y}.$$

By Itô's formula,

$$(Y_t(x))^{2k} = 1 + \int_0^t \int_{\mathbb{R}} [(Y_{s-}(x) + h(s,z))^{2k} - (Y_{s-}(x))^{2k}] \widetilde{N}(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(Y_{s-}(x) + f(s,z))^{2k} - (Y_{s-}(x))^{2k}] N(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(Y_s(x) + h(s,z))^{2k} - (Y_s(x))^{2k} - 2k(Y_s(x))^{2k-1} h(s,z)] \nu(dz) ds. \tag{4.46}$$

Denote the three terms on the right hand side of (4.46) by I_t, II_t, III_t respectively. Similar arguments as in the proof of Proposition 4.3 show that there exists a constant C_1 such that

$$E[\sup_{0\leq s\leq t} |I_s|] \leq C_1 (1 + \int_0^t E[|Y_s(x)|^{2k}] ds). \tag{4.47}$$

$$E[\sup_{0\le s\le t}|III_s|] \le C_1(1+\int_0^t E[|Y_s(x)|^{2k}]ds). \tag{4.48}$$

By the Mean Value Theorem, we have

$$\begin{aligned}
E[\sup_{0\le s\le t}|II_s|] &\le E[\int_0^t\int_{\mathbb{R}} |(Y_{s-}(x)+f(s,z))^{2k}-(Y_{s-}(x))^{2k}|N(ds,dz)] \\
&= E[\int_0^t\int_{\mathbb{R}} |(Y_{s-}(x)+f(s,z))^{2k}-(Y_{s-}(x))^{2k}|ds\nu(dz)] \\
&\le CE[\int_0^t ds|Y_{s-}(x)|^{2k}\int_{\mathbb{R}}\left|\frac{\left(\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\right)^2}{1+\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}}\right|\nu(dz) \\
&+CE[\int_0^t ds|Y_{s-}(x)|^{2k}\int_{\mathbb{R}}\left|\frac{\left(\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}\right)^2}{1+\frac{\partial\sigma(X_{s-}(x),z)}{\partial y}}\right|^{2k}\nu(dz)
\end{aligned}$$

$$\le CE[\int_0^t ds|Y_s(x)|^{2k}], \tag{4.49}$$

where we have used the fact that

$$\sup_y\int_{\mathbb{R}}\left|\frac{\frac{\partial\sigma(y,z)}{\partial y}^2}{1+\frac{\partial\sigma(y,z)}{\partial y}}\right|^l\nu(dz)<\infty,$$

for $l=1,...,2k$. It follows from (4.46), (4.47),(4.48) and (4.49) that

$$E[\sup_{0\le s\le t}|Y_s(x)|^{2k}] \le C_k(1+\int_0^t E[|Y_s(x)|^{2k}]ds).$$

The desired result follows from the Gronwall's lemma. ■

Finally, we need some estimates for the derivative of $Y_t(x)$. Define

$$K(s,z) =: -Y'_{s-}(x)\frac{\partial\sigma(X_{s-}(x),z)}{\partial y} - Y_{s-}(x)X'_{s-}(x)\frac{\partial^2\sigma(X_{s-}(x),z)}{\partial y^2},$$

$$J(y,z) =: \frac{\left(\frac{\partial\sigma(y,z)}{\partial y}\right)^2}{1+\frac{\partial\sigma(y,z)}{\partial y}},$$

$$\begin{aligned}
L(y,z) =: &\frac{2\frac{\partial\sigma(y,z)}{\partial y}(1+\frac{\partial\sigma(y,z)}{\partial y})\frac{\partial^2\sigma(y,z)}{\partial y^2}}{(1+\frac{\partial\sigma(y,z)}{\partial y})^2} \\
&-\frac{\frac{\partial^2\sigma(y,z)}{\partial y^2}(\frac{\partial\sigma(y,z)}{\partial y})^2}{(1+\frac{\partial\sigma(y,z)}{\partial y})^2},
\end{aligned}$$

$$m(s,z) =: Y'_{s-}(x)J(X_{s-}(x),z)+Y_{s-}(x)X'_{s-}(x)L(X_{s-}(x),z).$$

Proposition 4.4 *Assume*

$$\sup_y \int_{\mathbb{R}} |\frac{\partial^2 \sigma(y,z)}{\partial y^2}|^l \nu(dz) < \infty, \tag{4.50}$$

and

$$\sup_y \int_{\mathbb{R}} |L(y,z)|^l \nu(dz) < \infty, \quad \sup_y \int_{\mathbb{R}} |J(y,z)|^l \nu(dz) < \infty, \tag{4.51}$$

for $l = 1, ..., 2k$. *Then there exists a constant* C_k *such that* $E[\sup_{0\le s\le t} |Y'_s(x)|^{2k}] \le C_k$.

Proof. The proof is in the same nature as the proofs of previous propositions. We only sketch it. Differentiating (4.45) we see that

$$Y'_t(x) = \int_0^t \int_R K(s,z)\widetilde{N}(ds,dz) + \int_0^t \int_R m(s,z)N(ds,dz) \tag{4.52}$$

By Itô's formula,

$$(Y'_t(x))^{2k} = \int_0^t \int_{\mathbb{R}} [(Y'_{s-}(x) + K(s,z))^{2k} - (Y'_{s-}(x))^{2k}]\widetilde{N}(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(Y'_{s-}(x) + m(s,z))^{2k} - (Y'_{s-}(x))^{2k}]N(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(Y'_s(x) + K(s,z))^{2k} - (Y'_s(x))^{2k} - 2k(Y'_s(x))^{2k-1}K(s,z)]\nu(dz)ds \tag{4.53}$$

Let us denote the three terms on the right side by I_t, II_t and III_t. Reasoning in the same way as in the proof of Proposition 4.3, we have

$$E[\sup_{0\le s\le t} |I_s|] \le E[\int_0^t \int_{\mathbb{R}} |(Y'_s(x) + K(s,z))^{2k} - (Y'_s(x))^{2k}| ds\nu(dz)]$$

$$\le CE[\int_0^t ds|Y'_{s-}(x)|^{2k} \int_{\mathbb{R}} (|\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}| + |\frac{\partial \sigma(X_{s-}(x),z)}{\partial y}|^{2k})\nu(dz)$$

$$+CE[\int_0^t ds|Y'_{s-}(x)|^{2k-1}|Y_s(x)X'_s(x)| \int_{\mathbb{R}} |\frac{\partial^2 \sigma(X_{s-}(x),z)}{\partial y^2}|\nu(dz)]$$

$$+CE[\int_0^t ds|Y_s(x)X'_s(x)|^{2k} \int_{\mathbb{R}} |\frac{\partial^2 \sigma(X_{s-}(x),z)}{\partial y^2}|^{2k}\nu(dz)] \tag{4.54}$$

Since

$$\sup_y \int_{\mathbb{R}} |\frac{\partial \sigma(y,z)}{\partial y}|^l \nu(dz) < \infty, \text{for} \quad l = 1, ..., 2k,$$

and

$$\sup_y \int_{\mathbb{R}} |\frac{\partial^2 \sigma(y,z)}{\partial y^2}|^l \nu(dz) < \infty, \text{for} \quad l = 1, ..., 2k,$$

(4.54) is less than

$$CE[\int_0^t ds|(Y'_{s-}(x)|^{2k}] + CE[\int_0^t ds|(Y'_{s-}(x)|^{2k-1}|Y_s(x)X'_s(x)|] + CE[\int_0^t ds|Y_s(x)X'_s(x)|^{2k}] \tag{4.55}$$

Note that

$$E[|Y'_{s-}(x)|^{2k-1}|Y_s(x)X'_s(x)|] \le C_k(E[|(Y'_{s-}(x)|^{2k}] + E[|Y_s(x)X'_s(x)|^{2k}]),$$

and from Propostion 4.3 ,

$$E[\sup_{0\le s\le T}|Y_s(x)X'_s(x)|^{\alpha}] < \infty, \quad \text{for} \quad \alpha \le 2k.$$

It follows from (4.55) that

$$E[\sup_{0\le s\le t}|I_s|] \le C(1 + E[\int_0^t |Y'_{s-}(x)|^{2k}ds]. \tag{4.56}$$

By a similar argument, we can show that

$$E[\sup_{0\le s\le t}|III_s|] \le C(1 + E[\int_0^t |Y'_{s-}(x)|^{2k}ds]. \tag{4.57}$$

For the second term, we have

$$E[\sup_{0\le s\le t}|II_s|] \le E[\int_0^t \int_{\mathbb{R}} |(Y'_{s-}(x) + m(s,z))^{2k} - (Y'_{s-}(x))^{2k}|ds\nu(dz)]$$

$$\le C_k E[\int_0^t \int_{\mathbb{R}} (|Y'_{s-}(x)|^{2k-1}|m(s,z)| + |m(s,z)|^{2k})ds\nu(dz)]$$

$$\le C_k E[\int_0^t \int_{\mathbb{R}} |Y'_{s-}(x)|^{2k}(|J(X_{s-}(x),z)| + |J(X_{s-}(x),z)|^{2k})ds\nu(dz)]$$

$$+C_k E[\int_0^t \int_{\mathbb{R}} (|Y'_{s-}(x)|^{2k-1}|Y_{s-}(x)X'_{s-}(x)||L(X_{s-}(x),z)|)ds\nu(dz)]$$

$$+C_k E[\int_0^t \int_{\mathbb{R}} |Y_{s-}(x)X'_{s-}(x)|^{2k}|L(X_{s-}(x),z)|^{2k}ds\nu(dz)]$$

$$\le C_k E[\int_0^t |Y'_{s-}(x)|^{2k}ds] + C_k E[\int_0^t |Y_{s-}(x)X'_{s-}(x)|^{2k}ds],$$

$$\le C(1 + E[\int_0^t |Y'_{s-}(x)|^{2k}ds]) \tag{4.58}$$

where we have used the assumptions (4.51) and the fact that

$$E[\sup_{0\le s\le T}|Y_{s-}(x)X'_{s-}(x)|^{2k}] < \infty.$$

Now (4.53), (4.56), (4.57) imply

$$E[\sup_{0\le s\le t}|Y'_s(x)|^{2k}] \le C_k(1 + \int_0^t E[|Y'_s(x)|^{2k}]ds),$$

which yields the desired result by Gronwall's inequality. ■

Theorem 4.2 *Assume that $b(\omega, s, x)$ is locally Lipschitz in x uniformly with respect to (ω, s) and*

$$|b(\omega, s, x)| \le C(1 + |x|^\delta), \tag{4.59}$$

for some constants $C > 0$ and $\delta < 1$. Moreover assume that (4.30),(4.36),(4.43), (4.50) and (4.51) hold for some $k > \frac{1+\delta}{1-\delta}$. Then the equation (4.28) admits a unique solution. So does the equation (4.26).

Proof. Recall the Sobolev imbedding theorem: if $p > 1$, then

$$\sup_{x\in\mathbb{R}} |h(x) \le c_p ||h||_{1,p}, \tag{4.60}$$

where $||h||_{1,p}^p = \int_{\mathbb{R}} \left(|h(x)|^p + |h'(x)|^p \right) dx$. Let $\beta > 0$, $\alpha > 0$ and $p > 1$ be any parameters with $2\alpha p > 1$ and $(2\beta - 1)p > 1$. Set

$$f_s(x) = (1 + x^2)^{-\beta} X_s(x), \quad g_s(x) = (1 + x^2)^{-\alpha} Y_s(x),$$

where $Y_s(x) = (X_s'(x))^{-1}$. For any $T > 0$, using Proposition 4.2 ,

$$E[\sup_{0\le s\le T} ||f_s||_{1,p}^p]$$

$$\le C_{\beta,p} \int_{\mathbb{R}} E[\sup_{0\le s\le T} |X_s(x)|^p] \Big[(1 + x^2)^{-\beta p} + |x|^p (1 + x^2)^{-(\beta+1)p} \Big] dx$$

$$+ C_{\beta,p} \int_{\mathbb{R}} E[\sup_{0\le s\le T} |X_s'(x)|^p] (1 + x^2)^{-\beta p} dx$$

$$\le \int_{\mathbb{R}} \{ |x|^p \left((1 + x^2)^{-\beta p} + |x|^p (1 + x^2)^{-(\beta+1)p} \right) + (1 + x^2)^{-\beta p} \} dx < \infty. \tag{4.61}$$

Similarly, by Proposition 4.4 ,

$$E[\sup_{0\le s\le T} ||g_s||_{1,p}^p]$$

$$\le C_{\alpha,p} \int_{\mathbb{R}} E[\sup_{0\le s\le T} |Y_s(x)|^p] \Big[(1 + x^2)^{-\alpha p} + |x|^p (1 + x^2)^{-(\alpha+1)p} \Big] dx$$

$$+ C_{\alpha,p} \int_{\mathbb{R}} E[\sup_{0\le s\le T} |Y_s'(x)|^p] (1 + x^2)^{-\alpha p} dx$$

$$\le \int_{\mathbb{R}} \{ \{ \left((1 + x^2)^{-\alpha p} + |x|^p (1 + x^2)^{-(\alpha+1)p} \right) + (1 + x^2)^{-\alpha p} \} dx < \infty. \tag{4.62}$$

By the Sobolev imbedding theorem there exist random constants $C_{\beta,T}(\omega)$ and $C_{\alpha,T}(\omega)$ such that

$$\sup_{0\le s\le T} |X_s(x)| \le C_{\beta,T}(\omega)(1 + x^2)^\beta,$$

and

$$\sup_{0\le s\le T} |Y_s(x)| \le C_{\alpha,T}(\omega)(1 + x^2)^\alpha.$$

The assumption (4.59) together with the above two inequalities gives

$$\sup_{0\le s\le T} |\hat{b}(\omega,s,x)| = \sup_{0\le s\le T}\{|Y_s(x)||b(\omega,s,X_s(x))|$$

$$\le C(\omega)(1+x^2)^{\alpha}\left(1+|X_s(x)|^{\delta}\right)$$

$$\le M_{\alpha,\beta,T}(\omega)(1+x^2)^{\alpha+\beta\delta} \tag{4.63}$$

If $p > \frac{1+\delta}{1-\delta}$, it is possible to choose $\beta > 0$ and $\alpha > 0$ such that $2\alpha p > 1$, $(2\beta - 1)p > 1$ and $2\alpha + 2\beta\delta \le 1$. Therefore, there exists a random constant $C_T(\omega)$ such that

$$\sup_{0\le s\le T} |\hat{b}(\omega,s,x)| \le C_T(\omega)(1+|x|) \tag{4.64}$$

It now follows from the general theory of ordinary differential equations that (4.28) admits a unique global solution. ■

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