A study of a class of stochastic differential equations with non-Lipschitzian coefficients

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Abstract

We study a class of stochastic differential equations with non-Lipschitz coefficients. A unique strong solution is obtained. The dependence with respect to the initial values is investigated. We establish the non confluence of the solutions of the stochastic differential equations under our general conditions. To obtain a continuous version of the solution, the modules of continuity of coefficients is assumed to be less than $|x-y|\log\frac{1}{|x-y|}$. In this case, it will give rise to a flow of homeomorphisms if the coefficients are compactly supported. A large deviation principle of Freidlin-Wentzell type also has been established in the paper.

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1 Introduction

Let $\sigma: R^d \to R^d \otimes R^m$ and $b: R^d \to R^d$ be continuous functions. It is well-known that the following Itô s.d.e:

$$dX(t) = \sigma(X(t))\, dW_t + b(X(t))\, dt, \quad X(0) = x_o \tag{1}$$

has a weak solution up to a lifetime ζ (see [SV], [IW, p.155-163]), where $t \to W_t$ is a R^m-valued standard Brownian motion. It is also known that under the assumption of linear growth of coefficients σ and b, the lifetime $\zeta = +\infty$ almost surely. The celebrated result of Yamada and Watanable says that if the s.d.e (1) has the pathwise uniqueness, then it admits a strong solution (see [IW, p.149], [RY, p.341]). So the study of pathwise uniqueness is of great interest. It is a classical result that under the Lipschitz conditions, the pathwise uniqueness holds and the solution of s.d.e. (1) can be constructed using Picard interation; moreover the solution depends continuousely on the initial data. The main tool to these studies is the Gronwall lemma. When the coefficients σ and b do not satisfy the Lipschitz conditions, the use of Gronwall lemma meets serious difficulties. Therefore, there are few results of pathwise uniqueness of solutions of s.d.e. beyond the Lipschitz (or local Lipschitz) conditions in the literature except in the one dimensional case (see [IW, p.168], [RY, CH IX-3]). In the case of ordinary differential equations, the Gronwall lemma was generalized to establish the uniqueness result (see e.g. [La]). However the method is not applicable to s.d.e.. For s.d.e. with non-Lipschitz coefficients, a new concept of solutions has been introduced recently and many interesting phenomena have been obtained in [LJR1,2]. In this work, we shall study a class of stochastic differential equations with non-Lipschitzian coefficients, which are not covered in [LJR1,2]. Our basic assumptions are $(H1)$ and $(H2)$ in section 2, which can be seen as the critical conditions to guarantee the existence of non confluent flow.

Now let's describe briefly the content of this paper. First we prove the pathwise uniqueness of solutions under $(H1.b)$ and non explosion of the solution under $(H2.b)$. For the pathwise uniqueness, we impose a condition on the modules of the continuity of the coefficients only in an arbitrarily small neigbourhood of the diagonal. This seems the first time in the literature to notice this phenomena. Our method is dimension-free and does not require that the control function of the modules of the continuity of the coefficients is concave in contrast to the existing literature . The crucial idea is to construct a family of positive increasing functions $(\Phi_\rho)_{\rho>0}$ on R_+ so that the Gronwall lemma can be applied to the composition of the functions (Φ_ρ) with appropriate processes. Secondly, we study the dependence of the solutions on the initial data. Let $X_t(x_o, w)$ be the solution of the s.d.e. (1) with initial value x_0. It is well-known that the solution admits a continuous version $\tilde{X}_t(x_o, w)$ if the coefficients σ and b are locally Lipschitzian (see [Pr]), and it gives rise to a flow of homeomorphisms if the coefficients are globally Lipschitzian (see [Ku]). We refer also to [Ma2], [IW] and [El] for the study of stochastic flow of diffeomorphisms, to [Ma1] and [F] for the non-Lipschitz feature in the study of homeomorphism of the circle S^1. In the last part of the paper, we will establish a Freidlin-Wentzell type large deviation principle for the solutions of the s.d.e's (see [FW]). As a by-product, it is seen that the unique solution of the s.d.e. can be obtained by Euler approximation. Our strategy for the large deviation is to prove that the Euler approximations to the s.d.e. is exponentially fast. The method of estimating moments used in the literature ([DS],[DZ], [S]) wouldn't work here because of the non-Lipschitzian feature of the coefficients. We again appeal to a family of positive functions $(\Phi_{\rho,\lambda})_{\rho>0}$. The proof of the uniform convergence of solutions of the corresponding skeleton equations over compact level sets is also tricky due to the non-Lipschitzian feature.

The organization of the paper is as follows. Section 2 is the framework. We formulate the precise assumptions and present some examples. In Section 3, we discuss the case of ordinary differential equations; although the results about non explosion and uniqueness are not new, but our method can be used to study the dependence of initial values and the non confluence of the equations. In Section 4, we consider the s.d.e. case. The pathwise uniqueness and the criterion of non-explosion will be established. In Section 5, we prove that the non confluence (or non-contact) property holds under $(H1.b)$. Such properties were also studied by M. Emery in an early work [Em] for Lipschitz case, and by T.Yamada and Y. Ogura for a non-Lipschitz case in [YO]. The mixing condition imposed in [YO] for σ and b seems difficult to check in general. In Section 6, we establish a continuous version of the solutions. The main tool for doing this is the Kolmogorov's modification theorem, which will work in the case where $r(\xi) = \log \frac{1}{\xi}$. In Section 7, we shall prove that the continuous version of solutions will give rise to a flow of homeomorphisms if the coefficients are compactly supported (for simplicity). The ordinary differential equation

$$\frac{dX(t)}{dt} = b(X(t)), \quad X(0) = x_o \tag{2}$$

gives rise to a dynamical system on R^d. In section 8, we shall consider its small random perturbation:

$$dX^\varepsilon(t) = \varepsilon^{\frac{1}{2}}\,\sigma(X^\varepsilon(t))\,dW_t + b(X^\varepsilon(t))\,dt, \quad X^\varepsilon(0) = x_o \tag{3}$$

and state a large deviation principle for $(X^\varepsilon(t))_{t\in[0,1]}$. Section 9 and 10 are devoted to the proof of the large deviation principle. Section 9 is for the case of bounded coefficients. Section 10 is for the general case.

2 Framework

Let $\{W_t, t \geq 0\}$ be a R^m-valued standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying usual conditions. Let $\sigma :\ R^d \to R^d \otimes R^m$ and $b :\ R^d \to R^d$ be continuous functions. Consider the following Itô stochastic differential equation:

$$dX(t) = \sigma(X(t))\,dW_t + b(X(t))\,dt, \quad X(0) = x_o. \tag{4}$$

We first introduce the basic assumptions throughout the paper.

(H1.a): There exists $\delta_0 > 0$ such that

$$\begin{cases} ||\sigma(x) - \sigma(y)||^2 & \leq \quad C\,|x-y|^2\, r(|x-y|^2), \\ |b(x) - b(y)| & \leq \quad C\,|x-y|\, r(|x-y|^2), \end{cases} \tag{5}$$

for $|x-y| \leq \delta_o$, where $r :]0, \delta_o] \to R_+$ is a strictly positive, continuous function satisfying (i) $\liminf_{s\to 0} r(s) > 0$, (ii) $\int_0^{\delta_0} \frac{ds}{sr(s)} = +\infty$.

(H1.b): There exists $\delta_0 > 0$ such that

$$\begin{cases} ||\sigma(x) - \sigma(y)||^2 & \leq & C\,|x-y|^2\, r(|x-y|^2), \\ |b(x) - b(y)| & \leq & C\,|x-y|\, r(|x-y|^2), \end{cases} \tag{6}$$

for $|x - y| \leq \delta_o$, where $r :]0, \delta_o] \to R_+$ is a strictly positive function of class C^1 satisfying (i) $\liminf_{s\to 0} r(s) > 0$, (ii) $\int_0^{\delta_0} \frac{ds}{sr(s)} = +\infty$ and (iii) $\sup_{0\leq s\leq \delta_0} \frac{sr'(s)}{r(s)} < +\infty$.

(H2.a): There exists $K > 0$ such that for $|x| \geq K$,

$$\begin{cases} ||\sigma(x)||^2 & \leq & C\,(|x|^2\, \rho(|x|^2) + 1), \\ |b(x)| & \leq & C\,(|x|\, \rho(|x|^2) + 1), \end{cases} \tag{7}$$

where ρ is a positive continuous function satisfying $(i)\liminf_{s\to\infty} \rho(s) > 0$, (ii) $\int_K^\infty \frac{ds}{s\rho(s)+1} = +\infty$.

(H2.b): There exists $K > 0$ such that for $|x| \geq K$,

$$\begin{cases} ||\sigma(x)||^2 & \leq & C\,(|x|^2\, \rho(|x|^2) + 1), \\ |b(x)| & \leq & C\,(|x|\, \rho(|x|^2) + 1), \end{cases} \tag{8}$$

where $\rho : [K, +\infty[\to R_+$ is a $\mathcal{C}^1$ function such that (i) $\liminf_{s\to\infty} \rho(s) > 0$, (ii) $\int_K^\infty \frac{ds}{s\rho(s)+1} = +\infty$, (iii) $\sup_{s\geq K} \frac{|s\rho'(s)|}{\rho(s)} < \infty$.

Remark 2.1. Functions $r(s) = \log(\frac{1}{s})$, $\log(\frac{1}{s}) \log\log(\frac{1}{s}), \cdots$ and $\rho(s) = \log(s), \log(s)\log\log(s), \cdots$ are typical examples satisfying the conditions in the above assumptions. If $r(s) \equiv 1$, the assumption (H1.a), (H1.b) reduces to a Lipschitz condition in the neigbourhood of the diagnoal. If $\rho \equiv 1$, the assumption (H2.a), (H2.b) reduces to the well known linear growth condition.

Remark 2.2. In s.d.e. (4), we can consider a sequence of noises $(W_t^i;\ i \geq 1)$. The covariance function $C(x, y)$ of (4) satisfies

$$\sum_{i=1}^d (C^{ii}(x, x) + C^{ii}(y, y) - 2C^{ii}(x, y)) = \sum_{i=1}^d \sum_{j=1}^\infty |\sigma^{ij}(x) - \sigma^{ij}(y)|^2.$$

The same results hold under the assumption (H1.b), (H2.b).

Example 2.3 Define

$$f(x_1, x_2) = \sum_{k\geq 1} \frac{\sin kx_1 \cdot \sin kx_2}{k^2}.$$

Obviously the function f is continuous on R^2. We claim

$$|f(X) - f(Y)| \leq C\ |X - Y| \log \frac{1}{|X - Y|} \quad \text{for } |X - Y| < \frac{1}{e}, \tag{9}$$

where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. Now,

$$f(X)-f(Y) = \sum_{k=1}^{\infty}\Big\{\frac{(\sin kx_1 - \sin ky_1)\sin kx_2}{k^2} + \frac{(\sin kx_2 - \sin ky_2)\sin ky_1}{k^2}\Big\}.$$

It follows that

$$|f(X) - f(Y)| \leq 2\sum_{k=1}^{\infty}\Big\{\frac{|\sin(k\frac{x_1-y_1}{2})|}{k^2} + \frac{|\sin(k\frac{x_2-y_2}{2})|}{k^2}\Big\}.$$

To prove (9), we need the following lemma.

Lemma 2.4 *For $0 < \theta < 1/e$, we have*

$$V(\theta) := \sum_{k=1}^{\infty}\frac{|\sin k\theta|}{k^2} \leq C_1\ \theta\ \log\frac{1}{\theta}. \tag{10}$$

Proof. Consider $\phi(s) = \frac{\sin s\theta}{s^2}$. We have

$$\phi'(s) = \frac{s^2\theta\cos s\theta - 2s\sin s\theta}{s^4}.$$

Then $|\phi'(s)| \leq \frac{3\theta}{s^2}$. Let $W(\theta) = \int_1^{+\infty}\frac{|\sin s\theta|}{s^2}\,ds$. We have

$$\begin{aligned} |V(\theta) - W(\theta)| &\leq \sum_{k=1}^{+\infty}\int_k^{k+1}|\phi(s) - \phi(k)|\,ds \\ &\leq 3\theta\sum_{k=1}^{+\infty}\frac{1}{k^2} = \frac{\pi^2\theta}{2}. \end{aligned} \tag{11}$$

Now

$$W(\theta) = \theta\int_\theta^{+\infty}\frac{|\sin t|}{t^2}\,dt \leq \theta\int_\theta^1\frac{\sin t}{t}\frac{dt}{t} + \theta\int_1^{+\infty}\frac{ds}{s^2},$$

which is dominated by

$$\theta\Big(\log\frac{1}{\theta} + 1\Big).$$

Therefore, according to (11)

$$V(\theta) \leq \theta\Big(\log\frac{1}{\theta} + 1 + \frac{\pi^2}{2}\Big),$$

which is less that $2(\frac{\pi^2}{2} + 1)\ \theta\log\frac{1}{\theta}$ for $0 < \theta < \frac{1}{e}$. ∎

Now applying (10), for $|x_1 - y_1| + |x_2 - y_2| < 1/e$,

$$\begin{aligned} &|f(X) - f(Y)| \\ \leq\ & 4C_1\Big(\tfrac{|x_1-y_1|}{2}\log\tfrac{1}{|x_1-y_1|} + \tfrac{|x_2-y_2|}{2}\log\tfrac{1}{|x_2-y_2|}\Big) \\ \leq\ & 4C_1\Big(\tfrac{|x_1-y_1|+|x_2-y_2|}{2}\log\tfrac{2}{|x_1-y_1|+|x_2-y_2|}\Big) \end{aligned} \tag{12}$$

where the last inequality was due to the concavity of the function $\xi\log\frac{1}{\xi}$ over $]0,1[$. Therefore (9) holds for some constant $C > 0$. ∎

3 Ordinary differential equations

Let $b : R^d \to R^d$ be a continuous function. It is essentially due to Ascoli-Arzela theorem that the following differential equation:

$$\frac{dX(t)}{dt} = b(X(t)), \quad X(0) = x_0 \tag{13}$$

admits a solution up to a lifetime ζ. The following result weakens the linear growth condition for non explosion.

Theorem 3.1 *Assume (H2.a). Then the lifetime of any solution of (13) is infinite: $\zeta = +\infty$.*

Proof. Extend ρ to $[0, \infty)$ by setting $\rho(s) = \rho(K)$ for $0 \le s \le K$. We may assume that (H2.a) holds for all x. Define for $\xi \ge 0$,

$$\psi(\xi) = \int_0^\xi \frac{ds}{s\rho(s) + 1} \quad \text{and } \Phi(\xi) = e^{\psi(\xi)}.$$

We have

$$\Phi'(\xi) = \frac{\Phi(\xi)}{\xi\rho(\xi) + 1}. \tag{14}$$

Let $\xi_t = |X_t|^2$, where X_t is a solution to (13). Then

$$\frac{d}{dt}\Phi(\xi_t) = 2\Phi'(\xi_t)\langle X_t, b(X_t)\rangle, \tag{15}$$

where $\langle\ ,\ \rangle$ denotes the inner product in R^d. By assumption $(H2.a)$, we have

$$\Big|\frac{d}{dt}\Phi(\xi_t)\Big| \le 2C\Phi'(\xi_t)|X_t|\,(|X_t|\rho(\xi_t) + 1). \tag{16}$$

It follows from (H2.a)(i) that

$$\sup_{s\ge 0} \frac{s^2\rho(s^2) + s}{s^2\rho(s^2) + 1} < +\infty.$$

Therefore there exits a constant $C_2 > 0$,

$$\Big|\frac{d}{dt}\Phi(\xi_t)\Big| \le 2C_2\Phi(\xi_t). \tag{17}$$

It follows that for $t < \zeta$,

$$\Phi(\xi_t) \le \Phi(|x_o|^2) + 2C_2\int_0^t \Phi(\xi_s)\,ds.$$

By the Gronwall lemma, we have

$$\Phi(\xi_t) \le \Phi(|x_o|^2)\,e^{2C_2\,t}. \tag{18}$$

If $\zeta < +\infty$, letting $t \uparrow \zeta$ in (18), we get $\Phi(\xi_\zeta) \le \Phi(|x_o|^2)\,e^{2C_2\zeta}$ which is impossible because $\xi_\zeta = +\infty$ and $\Phi(+\infty) = +\infty$. ∎

Remark 3.2. By considering the inequality $\frac{d}{dt}\Phi(\xi_t) \geq -2C_2\,\Phi(\xi_t)$, we have

$$\Phi(\xi_t) \geq \Phi(|x_o|^2) - 2C_2 \int_0^t \Phi(\xi_s)\,ds,$$

which yields

$$\Phi(\xi_t) \geq \Phi(|x_o|^2) e^{-2C_2 t}.$$

If we denote by $X_t(x_o)$ the solution to (13) with initial value x_o, then we get $\lim_{|x_o|\to+\infty} \Phi(|X_t(x_o)|^2) = +\infty$, which implies that

$$\lim_{|x_o|\to+\infty} |X_t(x_o)| = +\infty. \tag{19}$$

In what follows, for simplicity, we shall assume that solutions of (2) have no explosion.

Theorem 3.3 *Assume (H1.a). Then the differential equation* (13) *has a unique solution.*

Proof. Let $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ be two solutions of the equation (13). Set $\eta_t = X(t) - Y(t)$ and $\xi_t = |\eta_t|^2$. Let $\rho > 0$, consider

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s) + \rho} \quad \text{and} \quad \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.$$

We have

$$\Phi_\rho'(\xi) = \frac{\Phi_\rho(\xi)}{\xi\; r(\xi) + \rho}. \tag{20}$$

Let

$$\tau = \inf\{t > 0,\ \xi_t \geq \delta_0^2\,\}.$$

By assumption (H1.a), we have

$$|\langle \eta_t, b(X(t)) - b(Y(t))\rangle| \leq C\ \xi_t\, r(\xi_t), \quad t < \tau.$$

It follows by chain rule, for $t < \tau$,

$$\Phi_\rho(\xi_t) \leq 1 + 2C \int_0^t \Phi_\rho(\xi_s)\,ds,$$

which implies that $\Phi_\rho(\xi_t) \leq e^{2C\,t}$ for $t < \tau$. Letting $\rho \downarrow 0$, we get that $e^{\psi_0(\xi_t)} \leq e^{2Ct}$. Now by (H2.a)(ii), we obtain that $\xi_t = 0$ for all $t < \tau$. If $\tau < +\infty$, letting $t \uparrow \tau$, we get

$$\delta_0^2 = \xi_\tau = 0,$$

which is absurd. Therefore $\xi_t = 0$ for all $t \geq 0$. In other words, $X(t) = Y(t)$ for $t \geq 0$. ∎

In what follows, we shall study the dependence of the solutions on the initial data.

Theorem 3.4 *Assume (H1.a) and that the solution of (13) has no explosion. Then $x_o \rightarrow X_t(x_o)$ is continuous, uniformly with respect to t in any compact subset.*

Proof. Given any $0 < \varepsilon \leq \delta_0^2$, where δ_0 is the parameter in (H1.a), let δ be a small parameter such that $0 < \delta < \varepsilon$. Let $(x_o, y_o) \in R^d \times R^d$ such that $|x_o - y_o| < \delta$. Set $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t = |\eta_t|^2$. Define

$$\tau(x_o, y_o) = \inf\{t > 0,\ \xi_t \geq \varepsilon\ \}.$$

Consider

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s) + \rho} \quad \text{and} \quad \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.$$

As in proof of Theorem 3.3, we have for $t < \tau(x_o, y_o)$,

$$\Phi_\rho(\xi_t) \leq \Phi_\rho(\xi_o) e^{2C\,t}.$$

Taking $\rho = |x_o - y_o|$, we get

$$\Phi_\rho(\xi_t) \leq e^\rho\, e^{2C\,t}, \quad \text{for } t < \tau(x_o, y_o). \tag{21}$$

Fix the point x_o. If $\liminf_{y_o \rightarrow x_o} \tau(x_o, y_o) = \tau < +\infty$, we can choose $y_n \rightarrow x_o$ such that $\lim_{n \rightarrow +\infty} \tau(x_o, y_n) = \tau$. Applying (21) for (x_o, y_n) and letting $t \uparrow \tau(x_o, y_n)$, we get

$$\Phi_{\rho_n}(\varepsilon) = \Phi_{\rho_n}(\xi_{\tau(x_o, y_n)}) \leq e^{\rho_n}\, e^{2C\,\tau(x_o, y_n)}.$$

where $\rho_n = |x_o - y_n|$. Letting $n \rightarrow +\infty$, we have

$$+\infty = \Phi_o(\varepsilon) \leq e^{2C\,\tau}$$

which is absurd. Therefore

$$\lim_{y_o \rightarrow x_o} \tau(x_o, y_o) = +\infty,$$

which means that for any $t > 0$, there exists $\delta > 0$ such that for $|y_o - x_o| < \delta$, $\tau(x_o, y_o) > t$. In other words,

$$|X_t(x_o) - X_t(y_o)| \leq \varepsilon. \ \blacksquare$$

Proposition 3.5 *Assume (H1.a) and that the solution of (13) has no explosion. Then for $x_o \neq y_o$, we have $X_t(x_o) \neq X_t(y_o)$ for all $t \geq 0$.*

Proof. Let $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t = |\eta_t|^2$. Without loss of generality, assume that $0 < \xi_o < 1/2\delta_0^2$. Let

$$\tau = \inf\{t > 0,\ \xi_t \geq \frac{3}{4}\delta_0^2\}.$$

By starting from τ again , it is enough to prove that $\xi_t > 0$ for $t < \tau$. Consider

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s) + \rho} \quad \text{and} \quad \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.$$

By (H1.a), for $t < \tau$, we have

$$\Big|\frac{d\Phi_\rho(\xi_t)}{dt}\Big| \leq 2C\Phi_\rho(\xi_t).$$

It follows that $\Phi_\rho(\xi_t) \geq \Phi_\rho(\xi_o) - 2C\int_0^t \Phi_\rho(\xi_s)ds$ or

$$\Phi_\rho(\xi_t) \geq \Phi_\rho(\xi_o)e^{-2Ct} \quad \text{for} \quad t < \tau. \tag{22}$$

For $\rho > 0$ small enough, $\Phi_\rho(\xi_o)e^{-2Ct} > 1$. It follows that $\Phi_\rho(\xi_t) > 1$ or $\xi_t > 0$. ∎

Remark 3.6. By reversing the time, the non confluent property is equivalent to the pathwise uniqueness. However, this method is delicat to be applied for s.d.e..

Now using $(H1.a)$ and Proposition 3.5, and by the standard arguments, we obtain

Theorem 3.7 *Assume (H1.a) and (H2.a). Then for any $t > 0$, $x_o \to X_t(x_o)$ defines a flow of homeomorphisms of R^d.*

4 Uniqueness and non-explosion for stochastic differential equations

Let σ, b be defined as in section 2. Let $X(t)$ be a solution of the following Itô stochastic differential equation:

$$dX(t) = \sigma(X(t))\,dW_t + b(X(t))\,dt, \quad X(0) = x_o \tag{23}$$

with the lifetime $\zeta(w)$.

Theorem 4.1 *Assume (H2.b). Then the s.d.e* (23) *has no explosion: $P(\zeta = +\infty) = 1$.*

Proof. Extend the function ρ in the assumption (H2.b) to a strictly positive C^1 function on the whole half line $[0, \infty[$. It is easy to see that there exists a constant $C > 0$ such that (H2.b) holds for any x. Consider

$$\psi(\xi) = \int_0^\xi \frac{ds}{s\rho(s) + 1} \quad \text{and} \quad \Phi(\xi) = e^{\psi(\xi)}, \quad \xi \geq 0.$$

We have

$$\Phi'(\xi)\,(\xi\,\rho(\xi) + 1) = \Phi(\xi), \tag{24}$$

$$\Phi''(\xi) = \frac{\Phi(\xi)\,(1 - \rho(\xi) - \xi\rho'(\xi))}{(\xi\rho(\xi) + 1)^2}. \tag{25}$$

Let $\xi_t = |X_t|^2$. We have

$$
\begin{aligned}
d\xi_t &= 2\langle X_t, \sigma(X(t))\, dW_t\rangle + 2\langle X_t, b(X(t))\rangle\, dt \\
&\qquad +||\sigma(X(t))||^2\, dt,
\end{aligned} \tag{26}
$$

and the stochastic contraction $d\xi_t \cdot d\xi_t$ is given by

$$
d\xi_t \cdot d\xi_t = 4|\sigma^*(X_t)X_t|^2\, dt, \tag{27}
$$

where σ^* denotes the transpose matrix of σ.
Define

$$
\tau_R = \inf\{t > 0,\ \xi_t \geq R\}, \quad R > 0.
$$

Then $\tau_R \uparrow \zeta$ as $R \uparrow +\infty$. Now by Itô formula, we have

$$
\begin{aligned}
\Phi(\xi_{t\wedge\tau_R}) &= \Phi(\xi_o) + 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\langle X(s), \sigma(X(s))dW_s\rangle \\
&+ 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\,\langle X(s), b(X(s))\rangle\, ds \\
&+ \int_0^{t\wedge\tau_R} \Phi'(\xi_s)||\sigma(X(s))||^2\, ds \\
&+ 2\int_0^{t\wedge\tau_R} \Phi''(\xi_s)\,|\sigma^*(X(s))X(s)|^2\, ds.
\end{aligned} \tag{28}
$$

Using (H2.b) it is straightforward to see that

$$
\Phi'(\xi_s)\,||\sigma(X(s))||^2 \leq C\,\Phi(\xi_s). \tag{29}
$$

Now,

$$
|\Phi'(\xi_s)\,\langle X(s), b(X(s))\rangle| \leq C\Phi(\xi_s)\frac{\xi_s\rho(\xi_s) + \xi_s^{\frac{1}{2}}}{\xi_s\rho(\xi_s) + 1}.
$$

Using the condition (i) on ρ, it is easy to see that

$$
\sup_{0\leq s<\infty} \frac{s^2\rho(s^2) + s}{s^2\rho(s^2) + 1} < \infty.
$$

Therefore there exists a constant C_1 such that

$$
|\Phi'(\xi_s)\,\langle X(s), b(X(s))\rangle| \leq C_1\Phi(\xi_s). \tag{30}
$$

By (H2.b) and (25),

$$
\Phi''(\xi_s)\,|\sigma^*(X(s))X(s)|^2 \leq C\Phi(\xi_s)\frac{\xi_s\,(1 + |\xi_s\rho'(\xi_s)|)}{\xi_s\,\rho(\xi_s) + 1}. \tag{31}
$$

Again by the conditions on ρ it follows that

$$
\sup_{0\leq s<\infty} \frac{s\,(1 + |s\rho'(s)|)}{s\,\rho(s) + 1} < \infty.
$$

Thus, there exists a constant C_2 such that

$$\Phi''(\xi_s)\,|\sigma^*(X(s))X(s)|^2 \le C_2\Phi(\xi_s). \tag{32}$$

Now combining (29), (30), (32), it follows from (28) that

$$E\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \le \Phi(\xi_o) + C_4\int_0^t E(\Phi(\xi_{s\wedge\tau_R}))\,ds,$$

which implies that

$$E\Big(\Phi(\xi_{t\wedge\tau_R})\Big) \le \Phi(\xi_o)e^{C_4\,t}.$$

Letting $R\to+\infty$, by Fatou lemma, we get

$$E\Big(\Phi(\xi_{t\wedge\zeta})\Big) \le \Phi(\xi_o)e^{C_5\,t}. \tag{33}$$

Now if $P(\zeta<+\infty)>0$, then for some $T>0$, $P(\zeta\le T)>0$. Taking $t=T$ in (33), we get

$$E\Big(\mathbf{1}_{(\zeta\le T)}\Phi(\xi_\zeta)\Big) \le \Phi(\xi_o)e^{C_4\,t},$$

which is impossible, because $\Phi(\xi_\zeta)=+\infty$. ∎

Theorem 4.2 *Assume (H1.b). Then the pathwise uniqueness holds for stochastic differential equation (23).*

Proof. Let X, Y be two solutions of equation (23) with life time ζ_1 and ζ_2. Let $\eta_t = X(t)-Y(t)$ and $\xi_t = |\eta_t|^2$. Let $\rho>0$. Define the function $\psi_\rho:[0,\delta_o]\to R$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s)+\rho}. \tag{34}$$

It is clear that for any $0<\xi<\delta_0$,

$$\psi_\rho(\xi)\uparrow\psi_0(\xi) = \int_0^\xi \frac{ds}{sr(s)} = +\infty, \quad \text{as } \rho\downarrow 0.$$

Define

$$\Phi_\rho(\xi) = e^{\psi_\rho(\xi)}. \tag{35}$$

Then we have

$$\Phi'_\rho(\xi)\,(\xi\,r(\xi)+\rho) = \Phi_\rho(\xi), \tag{36}$$

and

$$\Phi''_\rho(\xi) = \frac{\Phi_\rho(\xi)(1-\xi\,r'(\xi)-r(\xi))}{(\xi\,r(\xi)+\rho)^2}. \tag{37}$$

Let

$$\tau = \inf\{\,t>0, \quad \xi_t\ge\delta_0^2\,\}.$$

By Itô formula, we have

$$\begin{aligned}
\Phi_\rho(\xi_{t\wedge\tau\wedge\zeta_1\wedge\zeta_2}) &= 1+2\int_0^{t\wedge\tau\wedge\zeta_1\wedge\zeta_2}\Phi'_\rho(\xi_s)\langle\eta_s,(\sigma(X(s))-\sigma(Y(s)))dW_s\rangle\\
&+ 2\int_0^{t\wedge\tau\wedge\zeta_1\wedge\zeta_2}\Phi'_\rho(\xi_s)\,\langle\eta_s, b(X(s))-b(Y(s))\rangle\,ds\\
&+ \int_0^{t\wedge\tau\wedge\zeta_1\wedge\zeta_2}\Phi'_\rho(\xi_s)||\sigma(X(s))-\sigma(Y(s))||^2\,ds\\
&+ 2\int_0^{t\wedge\tau\wedge\zeta_1\wedge\zeta_2}\Phi''_\rho(\xi_s)\,|(\sigma^*(X(s))-\sigma^*(Y(s)))\eta_s|^2\,ds.
\end{aligned} \tag{38}$$

By (H1.b) and (37) we have for $s \le \tau$

$$\Phi''_\rho(\xi_s)\,|(\sigma^*(X(s)) - \sigma^*(Y(s)))\eta_s|^2$$

$$\le C\Phi_\rho(\xi_s)\Big[\frac{\xi_s^2|r'(\xi_s)|}{\xi_s r(\xi_s)+\rho} + \frac{\xi_s}{\xi_s r(\xi_s)+\rho}\Big] \tag{39}$$

It follows from the conditions on r in (H1.b) that

$$\sup_{0\le s\le\delta_0}\Big[\frac{s^2|r'(s)|}{sr(s)+\rho} + \frac{s}{sr(s)+\rho}\Big]$$

$$\le \sup_{0\le s\le\delta_0}\Big[\frac{s|r'(s)|}{r(s)} + \frac{1}{r(s)}\Big] < \infty$$

Therefore there exists a constant C_1 such that for $s \le \tau$

$$\Phi''_\rho(\xi_s)\,|(\sigma^*(X(s)) - \sigma^*(Y(s)))\eta_s|^2 \le C_1\Phi_\rho(\xi_s) \tag{40}$$

Keeping (H1.b) in mind it is straightforward to check that for $s \le \tau$

$$|\Phi'_\rho(\xi_s)\,\langle \eta_s, b(X(s)) - b(Y(s))\rangle\,| \le C_2\Phi_\rho(\xi_s) \tag{41}$$

$$|\Phi'_\rho(\xi_s)||\sigma(X(s)) - \sigma(Y(s))||^2| \le C_3\Phi_\rho(\xi_s) \tag{42}$$

So it follows from (40), (41), (42)and (38) that

$$E\Big(\Phi_\rho(\xi_{t\wedge\tau\wedge\zeta_1\wedge\zeta_2})\Big) \le 1 + 2C_4\,E\Big(\int_0^{t\wedge\tau\wedge\zeta_1\wedge\zeta_2}\Phi_\rho(\xi_s)\,ds\Big).$$

for some constant C_4. By Gronwall lemma, we get $E\Big(\Phi_\rho(\xi_{t\wedge\tau\wedge\zeta_1\wedge\zeta_2})\Big) \le e^{2ct}$ or

$$E\Big(e^{\Psi_\rho(\xi_{t\wedge\tau\wedge\zeta_1\wedge\zeta_2})}\Big) \le e^{2Ct}. \tag{43}$$

Letting $\rho \downarrow 0$ in (43),

$$E\Big(e^{\Psi_0(\xi_{t\wedge\tau\wedge\zeta_1\wedge\zeta_2})}\Big) \le e^{2Ct}$$

which implies that for any given t ,

$$\xi_{t\wedge\tau\wedge\zeta_1\wedge\zeta_2} = 0 \quad \text{almost surely.} \tag{44}$$

If $P(\tau < \zeta_1\wedge\zeta_2) > 0$, then there exists $T > 0$ such that $P(\tau \le T < \zeta_1\wedge\zeta_2) > 0$. It follows that on $\{\tau \le T < \zeta_1\wedge\zeta_2\}$,

$$\xi_\tau = 0$$

which is absurd by the definition of τ. Therefore $\tau \ge \zeta_1\wedge\zeta_2$ almost surely. So any given t , $\xi_{t\wedge\zeta_1\wedge\zeta_2} = 0$ almost surely. Now by the continuity of samples, the two solutions are indistinguishable. ∎

Remark 4.3 In the case of $d = m = 1$, stronger results on pathwise uniqueness have been established. Namely σ was allowed to be Hölder of exponent $\geq 1/2$ (see [RY, Ch. IX-3], [IW, p.168]). For related discussion, we refer also to [MX].

Theorem 4.4 *Assume (H1.b) and (H2.b). Let $X_t(x)$ be the solution of the s.d.e.* (23) *with initial value x. Then for any $\varepsilon > 0$, we have*

$$\lim_{y_o \to x_o} P(\sup_{0 \leq s \leq t} |X_s(y_o) - X_s(x_o)| > \varepsilon) = 0. \tag{45}$$

Proof. Let x_0, y_0 be such that $|y_o - x_o| < \varepsilon < \delta_0$, δ_0 being the parameter in (H1.b). Let $\xi_t = |X_t(y_o) - X_t(x_o)|^2$. Define

$$\tau_w(x_o, y_o) = \inf\{\, t > 0,\ \xi_t > \varepsilon^2 \,\}.$$

The same arguments as above lead to

$$E\Big(\Phi_\rho(\xi_{t\wedge\tau(x_o,y_o)})\Big) \leq \Phi_\rho(\xi_o)\, e^{2Ct}.$$

Taking $\rho = |x_o - y_o|$, we have $E\Big(\Phi_\rho(\xi_{t\wedge\tau(x_o,y_o)})\Big) \leq e^\rho\, e^{2Ct}$. Hence

$$P(\tau(x_o, y_o) < t)\Phi_\rho(\varepsilon^2) \leq E\Big(\Phi_\rho(\xi_{t\wedge\tau(x_o,y_o)})\Big) \leq e^\rho\, e^{2Ct}.$$

It follows

$$P(\sup_{0\leq s\leq t} |X_s(y_o) - X_s(x_o)| > \varepsilon) = P(\tau(x_o, y_o) < t) \leq e^{-\psi_\rho(\varepsilon)}\, e^\rho\, e^{2Ct} \to 0$$

as $\rho = |y_o - x_o| \to 0$. ∎

Corollary 4.5 *The diffusion process $(X_t(x))$ given by the solution of the s.d.e. is Feller, i.e., the associated semigroup $(T_t, t \geq 0)$ maps $C_b(R^d)$ into $C_b(R^d)$.*

Proof. It is a direct consequence of Theorem 4.4 and the definition

$$T_t f(x) = E[f(X_t(x))], \qquad f \in C_b(R^d).$$

5 Non contact property

Theorem 5.1 *Assume (H1.b) and the s.d.e* (23) *has no explosion. If $x_o \neq y_o$, then $X_t(x_o, w) \neq X_t(y_o, w)$ almost surely for all $t > 0$.*

Proof. Consider

$$\psi(\xi) = \int_\xi^{\delta_0} \frac{ds}{s r(s)} \quad \text{and} \quad \Phi(\xi) = e^{\psi(\xi)}, \quad \text{for } \delta_0 \geq \xi > 0.$$

We have

$$\Phi'(\xi) = -\frac{\Phi(\xi)}{\xi\, r(\xi)} \leq 0.$$

By condition (i) and (iii) on the function r, there exists a constant $C > 0$ such that

$$\Phi''(\xi) = \frac{\Phi(\xi)\,(1 + r(\xi) + \xi r'(\xi))}{(\xi r(\xi))^2} \leq C\Phi(\xi)\frac{1}{\xi^2 r(\xi)}, \tag{46}$$

for $0 < \xi < \delta_0$. Let $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t = |\eta_t|^2$. We have

$$\begin{aligned} d\xi_t = \; & 2\langle \eta_t, (\sigma(X_t(x_o)) - \sigma(X_t(y_o)))\,dW_t\rangle \\ & + 2\langle \eta_t, b(X_t(x_o)) - b(X_t(y_o))\rangle\,dt \\ & + ||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2\,dt, \end{aligned} \tag{47}$$

and the stochastic contraction $d\xi_t \cdot d\xi_t$ is given by

$$4|(\sigma^*(X_t(x_o)) - \sigma^*(X_t(y_o)))\eta_t|^2\,dt,$$

where σ^* denotes the transpose matrix of σ. Without loss of generality, we may assume $|x_o - y_o| < \frac{\delta_0}{2}$. Let $0 < \varepsilon < |x_o - y_o| < \frac{\delta_0}{2}$. Define

$$\tau_\varepsilon = \inf\{t > 0, \xi_t \leq \varepsilon\}, \quad \tau = \inf\{t > 0, \xi_t = 0\}.$$

We have $\tau_\varepsilon \uparrow \tau$ as $\varepsilon \downarrow 0$. Let

$$\zeta = \inf\{t > 0,\ \xi_t \geq \frac{3\delta_0^2}{4}\}.$$

Using Itô formula,

$$\begin{aligned} \Phi(\xi_{t\wedge\tau_\varepsilon\wedge\zeta}) \; &= \Phi(\xi_o) + 2\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi'(\xi_s)\langle \eta_s, (\sigma(X_s(x_o)) - \sigma(X_s(y_o)))\,dW_s\rangle \\ & + 2\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi'(\xi_s)\langle \eta_s, b(X_s(x_o)) - b(X_s(y_o))\rangle\,ds \\ & + \int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi'(\xi_s)||\sigma(X_s(x_o)) - \sigma(X_s(y_o))||^2\,ds \\ & + 2\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi''(\xi_s)|(\sigma^*(X_s(x_o)) - \sigma^*(X_s(y_o)))\eta_s|^2\,ds. \end{aligned} \tag{48}$$

By $(H1.b)$ we have

$$|(\sigma^*(X_s(x_o)) - \sigma^*(X_s(y_o)))\eta_s|^2 \leq \xi_s^2 r(\xi_s).$$

Combining with (46), we get

$$2\Phi''(\xi_s)|(\sigma^*(X_s(x_o)) - \sigma^*(X_s(y_o)))\eta_s|^2 \leq C\,\Phi(\xi_s).$$

Again by $(H1.b)$ and the expression of $\Phi'(\xi)$, we have

$$\Big|\Phi'(\xi_s)\langle \eta_s, b(X_s(x_o)) - b(X_s(y_o))\rangle\Big| \leq C\,\Phi(\xi_s).$$

Therefore,

$$\Phi(\xi_{t\wedge\tau_\varepsilon\wedge\zeta}) \leq \Phi(\xi_o) + \text{martingale} + C\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi(\xi_s)\,ds.$$

Taking expectation, we get

$$
\begin{aligned}
E\Big(\Phi(\xi_{t\wedge\tau_\varepsilon\wedge\zeta})\Big) &\le \Phi(\xi_o) + CE\Big(\int_0^{t\wedge\tau_\varepsilon\wedge\zeta}\Phi(\xi_s)ds\Big)\\
&\le \Phi(\xi_o) + C\int_0^t E(\Phi(\xi_{s\wedge\tau_\varepsilon\wedge\zeta}))\,ds \qquad (49)
\end{aligned}
$$

which implies that

$$E\Big(\Phi(\xi_{t\wedge\tau_\varepsilon\wedge\zeta})\Big) \le \Phi(\xi_o)e^{C\,t}, \quad \text{for all } \ t>0.$$

Consequently,

$$P(\tau_\varepsilon < t\wedge\zeta)\Phi(\varepsilon) \le \Phi(\xi_o)e^{C\,t}, \quad \text{for all } \ t>0.$$

Now first letting $\varepsilon \to 0$, we obtain

$$P(\tau < t\wedge\zeta) = 0, \quad \text{for all } \ t>0,$$

and then letting $t\to\infty$ we get $P(\tau<\zeta)=0$. Therefore, $\xi_.$ is positive almost surely on the interval $[0,\zeta]$. Now define $T_0 := 0$,

$$T_1 := \zeta, \qquad T_2 = \inf\{t>0,\ \xi_t \le \frac{\delta_0^2}{4}\}$$

and generally

$$T_{2n} = \inf\{t>T_{2n-1},\ \xi_t \le \frac{\delta_0^2}{4}\}, \qquad T_{2n+1} = \inf\{t>T_{2n},\ \xi_t \ge \frac{3\delta_0^2}{4}\}$$

Clearly $T_n \to \infty$ almost surely as $n\to\infty$. By definition, $\xi_.$ is positive on the interval $[T_{2n-1}, T_{2n}]$. By pathwise uniqueness of solutions, X enjoys the strong Markovian property. Starting again from T_{2n} and applying the same arguments as in the first part of the proof, one can show that $\xi_.$ is positive almost surely also on the interval $[T_{2n}, T_{2n+1}]$. This completes the proof. ∎

Theorem 5.2 *Assume (H2.b). Then* $\lim_{|x_o|\to+\infty}|X_t(x_o,w)| = +\infty$ *in probability.*

Proof. Let $f\in C^1(R_+)$ be a fixed, strictly positive C^1 function on R_+ that satisfies

$$f(s) = \rho(s) \qquad \text{for} \qquad s\ge 1.$$

Then it is easy to see that $(H2.b)$ holds for all $x\in R^d$, with ρ replaced by f. From now on, we will use C to denote a generic constant which may change from line to line. Define

$$\psi(\xi) = \int_0^\xi \frac{ds}{sf(s)+1}, \quad \xi\ge 0$$

and put

$$\Phi(\xi) = e^{-\psi(\xi)}.$$

Keeping the assumptions on ρ in mind it follows that

$$\Phi'(\xi) = -\frac{\Phi(\xi)}{\xi f(\xi)+1} \le 0,$$

and

$$\begin{aligned}\Phi''(\xi) &= \tfrac{\Phi(\xi)}{(\xi f(\xi)+1)^2}[1+f(\xi)+\xi f'(\xi)] \\ &\leq C\Phi(\xi)\tfrac{f(\xi)}{(\xi f(\xi)+1)^2}. \end{aligned} \tag{50}$$

Let $\xi(t) = |X_t(x_0)|^2$. For any constant $R > 0$, define

$$\tau_R = \inf\{t \geq 0,\ |X_t(x_0)| \leq R\}.$$

By Itô formula, we have

$$\begin{aligned}\Phi(\xi_{t\wedge\tau_R}) = & \quad \Phi(|x_0|^2) + 2\int_0^{t\wedge\tau_R}\Phi'(\xi_s)\langle X_s, \sigma(X_s)dW_s\rangle \\ + & \quad 2\int_0^{t\wedge\tau_R}\Phi'(\xi_s)\,\langle X_s, b(X_s)\rangle\,ds + \int_0^{t\wedge\tau_R}\Phi'(\xi_s)||\sigma(X_s)||^2\,ds \\ + & \quad 2\int_0^{t\wedge\tau_R}\Phi''(\xi_s)\,|\sigma^*(X_s)X_s|^2\,ds. \end{aligned} \tag{51}$$

By $(H2.b)$, it holds that

$$\frac{|\sigma^*(X_s)X_s|^2}{(\xi_s\, f(\xi_s)+1)^2} \leq C\ \frac{\xi_s\,(\xi_s\, f(\xi_s)+1)}{(\xi_s\, f(\xi_s)+1)^2}$$

Together with (50), we have

$$\int_0^{t\wedge\tau_R}\Phi''(\xi_s)\,|\sigma^*(X_s)X_s|^2\,ds \leq C\int_0^{t\wedge\tau_R}\Phi(\xi_s)\,ds.$$

Similarly, we have for some constant $C > 0$,

$$|\Phi'(\xi_s)\langle X_s, b(X_s)\rangle| \leq C\,\Phi(\xi_s), \quad s > 0.$$

Combining above inequalities, we get from (51)

$$E(\Phi(\xi_{t\wedge\tau_R})) \leq \Phi(|x_0|^2) + C\int_0^t E(\Phi(\xi_{s\wedge\tau_R}))\,ds,$$

which implies that

$$E(\Phi(\xi_{t\wedge\tau_R})) \leq \Phi(|x_0|^2)\,e^{Ct}.$$

This gives that

$$P(\tau_R \leq t)\Phi(R^2) \leq E(\Phi(\xi_{t\wedge\tau_R})) \leq \Phi(|x_0|^2)\,e^{Ct}.$$

Therefore,

$$P(\inf_{0\leq s\leq t}|X_s(x_o)| \leq R) \leq e^{Ct}\,\exp\Big\{-\int_{R^2}^{|x_o|^2}\frac{ds}{sf(s)+1}\Big\},$$

which tends to 0 when $|x_o| \to +\infty$. ∎

6 Continuous dependence of initial data

In this section, we shall show that the solution to (23) admits a version that is jointly continuous in (t, x_o). Let's begin with the following lemma.

Lemma 6.1 *Let $p \geq 1$. Assume (H1.b) with the function r $:]0, \delta_o] \to R_+$ being decreasing and that the coefficients σ and b are compactly supported, say,*

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for } |x| \geq R. \tag{52}$$

Then there exists a constant $C_p > 0$ such that for $|x| \leq R+1$ and $|y| \leq R+1$,

$$\begin{cases} ||\sigma(x) - \sigma(y)||^2 & \leq & C_p\,|x-y|^2\, r\Big((\frac{|x-y|^2}{M})^p\Big), \\ |b(x) - b(y)| & \leq & C_p\,|x-y|\, r\Big((\frac{|x-y|^2}{M})^p\Big) \end{cases}$$

where $M = \frac{4(R+1)^2}{\delta_o} \geq 1$.

Proof. Because of the similarity, we only prove the conclusion for b. Let $\delta = \inf\{\frac{\delta_o}{2}, \frac{1}{2}\}$. If $|x-y| \leq \delta$, by hypothesis $(H1.b)$,

$$|b(x) - b(y)| \leq C\ |x-y| r(|x-y|^2) \leq C\ |x-y|\, r\Big((\frac{|x-y|^2}{M})^p\Big), \tag{53}$$

since r is supposed to be decreasing. Remark that

$$\inf_{\delta \leq \xi \leq 2(R+1)} \xi\, r\Big(\Big[\frac{\xi^2}{M}\Big]^p\Big) \geq \delta\, r(\delta^p)$$

and $\sup_{x,y} |b(x) - b(y)| \leq 2||b||_\infty$. Therefore there exists a constant $C_p > 0$ such that

$$|b(x) - b(y)| \leq C_p\,|x-y|\, r\Big((\frac{|x-y|^2}{M})^p\Big) \quad \text{for} \quad |x-y| \geq \delta. \tag{54}$$

Combining (53) and (54), we get the result. ∎

Lemma 6.2 *Assume the same hypothesis as in lemma 6.1 and furthermore $\xi \to \xi r(\xi)$ is concave over $]0, \delta_o]$. Let $p \geq 1$ be an integer. For $|x_o| \leq R+1$ and $|y_o| \leq R+1$, set*

$$\eta_t = |X_t(x_o) - X_t(y_o)|^2 \quad \text{and} \quad \xi_t = (\frac{\eta_t}{M})^p$$

where M is the constant appeared in Lemma 6.1. Put $\phi(t) = E(\xi_t)$. Then we have

$$\varphi'(t) \leq C_p\, \varphi(t)\, r(\varphi(t)), \tag{55}$$

for some constant C_p.

Proof. We remark that under the assumptions (52), for any $|x_o| \leq R+1$, $|X_t(x_o)| \leq R+1$ almost surely for all $t \geq 0$. In fact, this can be seen as follows. Define

$$T = \inf\{t \geq 0;\ |X_t(x_o)| \geq R+1\}$$

Set $Y_t(x_0) = X_{t\wedge T}(x_o)$. Then

$$Y_t = x_0 + \int_0^{t\wedge T} \sigma(Y_s)dW_s + \int_0^{t\wedge T} b(Y_s)ds.$$

Since

$$E\Big(\int_0^t \sigma^2(Y_s)(\mathbf{1}_{(s<T)} - 1)^2\, ds\Big) = E\Big(\int_T^t \sigma^2(Y_s)\, ds\Big) = 0,$$

we have for $t \geq 0$,

$$\int_0^{t\wedge T} \sigma(Y_s)dW_s = \int_0^t \sigma(Y_s)dW_s \quad \text{and} \quad \int_0^{t\wedge T} b(Y_s)ds = \int_0^t b(Y_s)ds,$$

almost surely. We see that $\{Y_t,\ t \geq 0\}$ satisfies the same stochastic differential equation as $\{X_t,\ t \geq 0\}$. By the pathwise uniqueness in section 4, we conclude that $Y_t = X_t$ a.s. for all $t \geq 0$, which proves the claim. Now we shall proceed as in [F]. By Itô's formula,

$$\begin{aligned} d\eta_t = \;& 2\langle X_t(x_o) - X_t(y_o), (\sigma(X_t(x_o)) - \sigma(X_t(y_o)))\, dW_t\rangle \\ & +2\langle X_t(x_o) - X_t(y_o), b(X_t(x_o)) - b(X_t(y_o))\rangle\, dt \\ & +||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2\, dt, \end{aligned} \tag{56}$$

and

$$d\xi_t = \frac{1}{M^p}\Big(p\eta_t^{p-1} d\eta_t + \frac{1}{2}p(p-1)\eta_t^{p-2} d\eta_t \cdot d\eta_t\Big). \tag{57}$$

Now by Lemma 6.1,

$$\begin{aligned} & \tfrac{2p}{M^p}\eta_t^{p-1}\Big|\langle X_t(x_o) - X_t(y_o), b(X_t(x_o)) - b(X_t(y_0))\rangle\Big| \\ & \leq C_p \tfrac{2p}{M^p}\eta_t^{p-1}\eta_t\, r(\xi_t) = 2pC_p\xi_t r(\xi_t). \end{aligned} \tag{58}$$

Similarly , we get the same control for other terms in (57) except the martingale part. Now by expression of $d\xi_t$, we have

$$\xi_{t+\varepsilon} - \xi_t \leq M_{t+\varepsilon} - M_t + C_p \int_t^{t+\varepsilon} \xi_s r(\xi_s)\, ds$$

where M_t is the martingale part of ξ_t. It follows that

$$E(\xi_{t+\varepsilon}|\mathcal{F}_t) - \xi_t \leq C_p E\Big(\int_t^{t+\varepsilon} \xi_s r(\xi_s) ds|\mathcal{F}_t\Big)$$

where $\mathcal{F}_t$ is the filtration generated by $\{w(s);\ s \leq t\}$. Let $\varphi(t) = E(\xi_t)$. Then

$$\varphi(t+\varepsilon) - \varphi(t) \leq C_p \int_t^{t+\varepsilon} E(\xi_s\, r(\xi_s))\, ds.$$

It follows that

$$\varphi'(t) \leq C_p E(\xi_t\, r(\xi_t)).$$

Since

$$\xi \to \xi r(\xi) \quad \text{is concave over} \quad]0, \delta],$$

we have

$$\varphi'(t) \leq C_p\, \varphi(t)\, r(\varphi(t)). \quad \blacksquare \tag{59}$$

Theorem 6.3 *Assume (H1.b) with $r(\xi) = \log \frac{1}{\xi}$ and the s.d.e (23) has no explosion. Then there exists a version of $X_t(x_o, w)$ such that $(t, x_o) \to X_t(x_o, w)$ is continuous over $[0, +\infty[\times R^d$ almost surely.*

Proof. We split the proof into two steps.
Step 1. Assume that σ and b are compactly supported, say,

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for } |x| \geq R.$$

Let φ be defined as in Lemma 6.2. Solving (59), we get $\varphi(t) \leq (\varphi(0))^{e^{-C_p\, t}}$ or explicitly

$$E\Big(|X_t(x_o) - X_t(y_o)|^{2p}\Big) \leq C_p\, |x_o - y_o|^{2pe^{-C_p\, t}}.$$

On the other hand, it is easy to see that

$$E\Big(|X_t(x_o) - X_s(x_o)|^{2p}\Big) \leq C_p |t - s|^p.$$

Therefore,

$$E\Big(|X_t(x_o) - X_s(y_o)|^{2p}\Big) \leq C_p \Big[|t - s|^p + |x_o - y_o|^{2pe^{-C_p\, t}}\Big]. \tag{60}$$

Fix $p > d+1$. Choose a constant $T_o > 0$ small enough such that $2pe^{-C_p T_0} > d+1$. It follows from (60) and Kolmogorov's modification theorem that there exists a version of $X_t(x_o, w)$, denoted by $\tilde{X}_t(x_o, w)$, such that $(t, x_o) \to \tilde{X}_t(x_o, w)$ is continuous over $[0, T_o] \times \{|x_o| \leq R + 1\}$ almost surely. But

$$X_t(x_o, w) = x_o \quad \text{if} \quad |x_o| > R.$$

We conclude that $(t, x_o) \to \tilde{X}_t(x_o, w)$ can be extended continuously to $[0, T_o] \times R^d$. Let $(\theta_{T_o} w)(t) = w(t + T_o) - w(T_o)$. Define for $0 < t \leq T_o$,

$$\tilde{X}_{T_o+t}(x_o, w) = \tilde{X}_t(\tilde{X}_{T_o}(x_o, w), \theta_{T_o} w).$$

Then $\tilde{X}_{T_o+\cdot}(x_o, w)$ satisfies the s.d.e (23) driven by the Brownian motion $\theta_{T_o} w$ with the initial condition $\tilde{X}_{T_o}(x_o, w)$. By pathwise uniqueness, we see that $\tilde{X}_{T_o+t}(x_o, w) = X_{T_o+t}(x_o, w)$ almost surely for all $t \in [0, T_o]$. This means that $\tilde{X}_t(x_o, w)$ is a continuous version of $X_t(x_o, w)$ over $|0, 2T_o] \times R^d$. Continuing in this way, we get a continuous version on the whole space $[0, +\infty[\times R^d$.

Step 2: general case. For $R > 0$, let $f_R(x)$ denote a smooth function with compact support satisfying

$$f_R(x) = 1 \quad \text{for } |x| \leq R \quad \text{and} \quad f_R(x) = 0 \quad \text{for } |x| > R + 1. \tag{61}$$

Define

$$\sigma_R(x) = \sigma(x) f_R(x) \quad \text{and} \quad b_R(x) = b(x) f_R(x). \tag{62}$$

Let $X^R_t(x,w)$ be the unique solution of the s.d.e. (0.1) with σ and b replaced by σ_R and b_R. Let $\tilde{X}^R_t(x,w)$ denote a continuous version of $X^R_t(x,w)$. Such a version exists according to step 1. For $K>0$, set

$$\tau^R_K(x) = \inf\{t>0;\ |\tilde{X}^R_t(x,w)| \geq K\} \tag{63}$$

$$\tau_K(x) = \inf\{t>0;\ |X_t(x,w)| \geq K\} \tag{64}$$

By the pathwise uniqueness, for $|x|\leq K$, we have

$$\tau_K(x) = \tau^{K+2}_K(x) \tag{65}$$

almost surely. For $|x|\leq R$, define

$$\tilde{X}_\cdot(x,w) = \tilde{X}^{R+2}_\cdot(x,w) \quad \text{on} \quad [0, \tau^{R+2}_R(x)) \tag{66}$$

Then it is clear that $\tilde{X}_\cdot(x,w)$ is a version of $X_\cdot(x,w)$. Let us prove that $\tilde{X}_t(x,w)$ is continuous in (t,x) for almost all w. Fix x_0 with $|x_0|\leq K$. Since the life time of the solution is infinity, there exists $R>0$ such that $\tau^{R+2}_R(x_0)>t$. This implies that $\sup_{0\leq s\leq t}|\tilde{X}^{R+2}_s(x_o,w)|<R$. By the continuity, we can find a neighbourhood $B_\delta(x_o)$ of x_o such that $\sup_{0\leq s\leq t}|\tilde{X}^{R+2}_s(x,w)|<R$ or $\tau^{R+2}_R(x)>t$ for all $x\in B_\delta(x_o)$. Hence, $\tilde{X}_s(x,w)=\tilde{X}^{R+2}_s(x,w)$ for all $x\in B_\delta(x_o)$ and $s\leq t$, which implies that $\tilde{X}_s(x_o,w)$ is continuous with respect to (s,x_o). ∎

Remark 6.4: Consider $r(s) = \log\frac{1}{s}\cdot\log\log\frac{1}{s}$ for $s\in]0,1/2e]$. Clearly $s\to r(s)$ is decreasing and $s\to sr(s)$ is concave over $]0,1/2e]$. Applying (59), we get

$$\varphi(t) \leq \exp\Big(-[\log\frac{1}{\varphi(0)}]^{e^{-C_pt}}\Big).$$

In order to apply the Kolmogorov's modification theorem, we have to find $\alpha>0$ such that

$$\exp\Big(-[\log\frac{1}{\varphi(0)}]^{e^{-C_pt}}\Big) \leq \varphi(0)^\alpha,$$

or

$$[\log\frac{1}{\varphi(0)}]^{e^{-C_pt}} \geq \alpha\log\frac{1}{\varphi(0)}$$

which is impossible when $|x_o-y_o|$ is small for any $t>0$. ∎

7 Flow of homeomorphisms

Theorem 7.1 *Assume (H1.b) with $r(\xi)=log\frac{1}{\xi}$ and that σ and b are compactly supported. Then the solution of the s.d.e (23) admits a version $X_t(x_o,w)$ such that $x_o\to X_t(x_o,w)$ is a homeomorphism of R^d almost surely for all $t>0$.*

Proof. Let $R>0$ such that

$$\sigma(x)=0 \quad \text{and} \quad b(x)=0 \quad \text{for} \quad |x|\geq R.$$

From section 6, we know that the solution of s.d.e. (23) admits a continuous version, still denoted by $X_t(x_o, w)$, in (t, x_o), such that $|X_t(x_o, w)| \leq R+1$ if $|x_o| \leq R+1$. So $x_o \rightarrow X_t(x_o, w)$ defines a continuous map from $B(R+1)$ to $B(R+1)$, where

$$B(r) = \{x \in R^d;\ |x| \leq r\}.$$

Lemma 7.2 *Let $x_o \neq y_o$ and $\alpha < 0$. Then there exists $C_\alpha, K_\alpha > 0$ such that for $x_o, y_o \in B(R+1)$,*

$$E\Big(|X_t(x_o) - X_t(y_o)|^{2\alpha}\Big) \leq C_\alpha\, |x_o - y_o|^{2\alpha e^{-K_\alpha t}}. \tag{67}$$

Proof. Let $0 < \varepsilon < \frac{|x_o - y_o|^2}{8(R+1)^2}$. Consider $\eta_t(w) = X_t(x_o) - X_t(y_o)$ and

$$\xi_t(w) = \frac{|\eta_t(w)|^2}{8(R+1)^2}.$$

By theorem 5.1, we know that $\xi_t(w) \neq 0$ almost surely for all $t > 0$. Define

$$\tau_\varepsilon = \inf\{t > 0,\ \xi_t \leq \varepsilon\}.$$

By Itô formula, for $s < t$,

$$\xi^\alpha_{t\wedge\tau_\varepsilon} - \xi^\alpha_{s\wedge\tau_\varepsilon} = \int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon} \alpha\xi_u^{\alpha-1}\, d\xi_u + \frac{1}{2}\alpha(\alpha-1)\int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon} \alpha\xi_u^{\alpha-2}\, d\xi_u \cdot d\xi_u.$$

Where

$$\begin{aligned} d\xi_t = \frac{1}{4(R+1)^2}\Big\{ &\ \langle \eta_t(w), (\sigma(X_t(x_o)) - \sigma(X_t(y_o)))dw_t\rangle \\ &+ \langle \eta_t(w), b(X_t(x_o)) - b(X_t(y_o))\rangle\, dt \\ &+ \tfrac{1}{2}||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2\, dt\Big\}, \end{aligned} \tag{68}$$

and

$$d\xi_t \cdot d\xi_t = \Big(\frac{1}{4(R+1)^2}\Big)^2 |(\sigma^*(X_t(x_o)) - \sigma^*(X_t(y_o)))\eta_t|^2\, dt.$$

Using lemma 6.1 for $p = 1$, there exists a constant $C > 0$ such that

$$|b(X_t(x_o)) - b(X_t(y_o))| \leq C\, |\eta_t| \log \frac{8(R+1)^2}{|\eta_t|^2} \tag{69}$$

and

$$||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2 \leq C\, |\eta_t|^2 \log \frac{8(R+1)^2}{|\eta_t|^2}. \tag{70}$$

Therefore by (69),

$$\begin{aligned} &\Big|\alpha\xi_u^{\alpha-1}\langle \eta_u, b(X_t(x_o)) - b(X_t(y_o))\rangle\Big| \\ &\leq -2C\alpha\, \xi_u^\alpha \log \tfrac{1}{\xi_u} = -2C\, \xi_u^\alpha \log \tfrac{1}{\xi_u^\alpha}. \end{aligned} \tag{71}$$

Similarly , we have

$$\tfrac{1}{2}\alpha(\alpha-1)\xi_u^{\alpha-2}\Big(\tfrac{1}{4(R+1)^2}\Big)^2|(\sigma^*(X_t(x_o))-\sigma^*(X_t(y_o)))\eta_t|^2 \\ \leq 2C|\alpha-1|\,\xi_u^\alpha\log\tfrac{1}{\xi_u^\alpha}. \tag{72}$$

Let dM_t be the martingale part of $d\xi_t$. Then we have

$$\xi^\alpha_{t\wedge\tau_\varepsilon}-\xi^\alpha_{s\wedge\tau_\varepsilon}\leq\int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon}\alpha\xi_u^{\alpha-1}\,dM_u+K(\alpha)\int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon}\xi_u^\alpha\log\frac{1}{\xi_u^\alpha}\,du.$$

Define $\varphi(t)=E(\xi^\alpha_{t\wedge\tau_\varepsilon})$. From the above inequality, we get

$$\varphi'(t)\leq K(\alpha)\varphi(t)\log\frac{1}{\varphi(t)},$$

which implies that

$$\varphi(t)\leq\varphi(0)^{e^{-K(\alpha)\,t}}.$$

Hence,

$$E(\xi^\alpha_{t\wedge\tau_\varepsilon})\leq\Big(\frac{|x_o-y_o|^2}{8(R+1)^2}\Big)^{\alpha\,e^{-K(\alpha)\,t}}\qquad\text{for } x_o,y_o\in B(R+1).$$

Since $\tau_\varepsilon\uparrow+\infty$ while $\varepsilon\downarrow0$, we get

$$E(\xi_t^\alpha)\leq\Big(\frac{|x_o-y_o|^2}{8(R+1)^2}\Big)^{\alpha\,e^{-K(\alpha)\,t}}\qquad\text{for } x_o,y_o\in B(R+1)$$

or (67) holds.

Lemma 7.3 *Let $\delta>0$ and $\Delta_\delta=\{(x_o,y_o)\in B(R+1)^2;\ |x_o-y_o|\geq\delta\}$. Take a continuous version $X_t(x_o)$ and set $\eta_t(x_o,y_o)=|X_t(x_o)-X_t(y_o)|^{-1}$. Then for any $p>1$, there exist constant $C_{p,T,\delta},K_1(p)>0$ such that*

$$E\Big(|\eta_t(x_o,y_o)-\eta_s(\tilde{x}_o,\tilde{y}_o)|^p\Big) \\ \leq C_{p,T,\delta}\Big(|x_o-\tilde{x}_o|^{pe^{-K_1(p)T}}+|y_o-\tilde{y}_o|^{pe^{-K_1(p)T}}+|t-s|^{p/2}\Big), \tag{73}$$

for $(x_o,y_o),(\tilde{x}_o,\tilde{y}_o)\in\Delta_\delta$ and $s,t\in[0,T]$.

Proof. We have

$$|\eta_t(x_o,y_o)-\eta_s(\tilde{x}_o,\tilde{y}_o)| \\ =\eta_t(x_o,y_o)\eta_s(\tilde{x}_o,\tilde{y}_o)\Big||X_t(x_o)-X_t(y_o)|-|X_s(\tilde{x}_o)-X_s(\tilde{y}_o)|\Big| \\ \leq\eta_t(x_o,y_o)\eta_s(\tilde{x}_o,\tilde{y}_o)\Big(|X_t(x_o)-X_s(\tilde{x}_o)|+|X_t(y_o)-X_s(\tilde{y}_o)|\Big).$$

By lemma 7.1, there exists a constant $C_{p,T,\delta}>0$ such that

$$E(\eta_t(x_o,y_o)^{4p})\leq C_{p,T,\delta}\quad\text{for }(x_o,y_o)\in\Delta_\delta,\ t\in[0,T].$$

Now combining with (60), we get

$$E\Big(|\eta_t(x_o,y_o)-\eta_s(\tilde{x}_o,\tilde{y}_o)|^p\Big)$$

$$\leq C_{p,T,\delta}\Big(|x_o-\tilde{x}_o|^{2pe^{-K_1(p)T}}+|y_o-\tilde{y}_o|^{2pe^{-K_1(p)T}}+2|t-s|^p\Big)^{1/2}$$

which is dominated by the right hand side of (73). So we get the result.

Proof of theorem 7.1 By (73), for $p > 2(2d+1)$ and $T_o > 0$ small enough, we can apply the Kolmogorov's modification theorem to get that $\eta_t(x_o, y_o)$ has a continuous version $\tilde{\eta}_t(x_o, y_o)$ on $[0, T_o] \times \Delta_o$. This means that $(t, x_o, y_o) \to \tilde{\eta}_t(x_o, y_o)$ is continuous on $[0, T_o] \times \Delta_o$ almost surely. Let D be a countable dense subset of $[0, T_o] \times \Delta_o$. Then almost surely, for all $(t, x_o, y_o) \in D$,

$$\tilde{\eta}_t(x_o, y_o) = |X_t(x_o) - X_t(y_o)|^{-1},$$

or

$$|X_t(x_o) - X_t(y_o)| = \tilde{\eta}_t(x_o, y_o)^{-1}. \tag{74}$$

Now by continuity, the relation (74) holds for all $(t, x_o, y_o) \in [0, T_o] \times \Delta_o$. In particular, this relation shows that almost surely for all $t > 0$, $x_o \to X_t(x_o)$ is injective on $B(R+1)$. Since $X_t(x_o) = x_o$ for $|x_o| > R$, $x_o \to X_t(x_o)$ is injective on the whole R^d. Let $\overline{R^d} = R^d \cup \{\infty\}$ which is homeomorphe to the sphere S^d. Extend the map X_t to $\overline{R^d}$ by setting $X_t(\infty) = \infty$. It is clear that X_t is continuous near ∞. So X_t can be seen as a continuous map from S^d into S^d. Since X_t is homotope to the identity, we conclude that X_t is a surjective map on S^d. Therefore X_t is a homeomorphism of S^d and its restriction X_t on R^d is a homeomorphism of R^d. Now using the relation

$$X_{T_o+t}(x_o, w) = X_t(X_{T_o}(x_o, w), \theta_{T_o} w),$$

we conclude that X_t is a homeomorphism of R^d for all $t > 0$. ∎

8 Statement of large deviation principle

Let $\sigma: R^d \to R^d \otimes R^m$ and $b: R^d \to R^d$ be continuous functions. Consider the following Itô s.d.e:

$$dX^\varepsilon(t) = \varepsilon^{\frac{1}{2}} \sigma(X^\varepsilon(t))\, dW_t + b(X^\varepsilon(t))\, dt, \quad X^\varepsilon_o(w) = x_o \tag{75}$$

where $t \to W_t$ is a R^m-valued standard Brownian motion. In this section we will state a large deviation principle for the law of X^ε. The proofs will be given in section 9 and section 10.

Let $C_x([0,1], R^m)$ be the space of continuous functions from $[0,1]$ into R^m with initial value x. If $g \in C_0([0,1], R^m)$ is absolutely continuous, set $e(g) = \int_0^1 |\dot{g}(t)|^2 dt$. Set $e(g) = \infty$ if g is not is absolutely continuous. Let $F(g)$ be the solution to the differential equation

$$\begin{aligned} F(g)(t) &= x_0 + \int_0^t b(F(g)(s))ds \\ &\quad + \int_0^t \sigma(F(g)(s))\dot{g}(s)ds, \quad 0 < t < \infty, \end{aligned} \tag{76}$$

the uniqueness and non explosion being obtained in a similar way as as in section 3 for the differential equation (13). See also lemma 9.3 below.

Theorem 8.1 *Let* μ_ε *be the law of* $X^\varepsilon(\cdot)$ *on* $C_{x_0}([0,1], R^d)$. *Assume* $(H1.b)$ *with* $r(\xi) = log\frac{1}{\xi}$ *and* $(H2.b)$ *with* $\rho(\xi) = log\xi$. *Then* $\{\mu_\varepsilon, \varepsilon > 0\}$ *satisfies a large deviation principle with the following good rate function*

$$I(f) = \inf\left\{\frac{1}{2}e(g)^2; F(g) = f\right\}, \qquad f \in C_{x_0}([0,1], R^d) \tag{77}$$

i.e.,

(i) for any closed subset $C \subset C_{x_0}([0,1], R^d)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \le -\inf_{f\in C} I(f) \tag{78}$$

(ii) for any open subset $G \subset C_{x_0}([0,1], R^d)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \ge -\inf_{f\in G} I(f) \tag{79}$$

9 Large deviations when σ, b are bounded

The theory of large deviations for diffusion processes under Lipschitzian coefficients is well established (see [A], [S]). Some new developments in infinite dimensional situations are discussed in [FZ1,2], [Z1,2]. The main task of this work is to handle the non Lipschitzian feature. For $n \ge 1$, let $X_n^\varepsilon(\cdot)$ to be the solution to

$$\begin{aligned} X_n^\varepsilon(t) &= x_0 + \int_0^t b(X_n^\varepsilon(\frac{[ns]}{n}))ds \\ &\quad + \varepsilon^{\frac{1}{2}} \int_0^t \sigma\left(X_n^\varepsilon\left(\frac{[ns]}{n}\right)\right) dW_s. \end{aligned} \tag{80}$$

We need the following lemma from Stroock [S,p.81].

Lemma 9.1. *Let* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *be* $(\mathcal{F}_t)_{t\ge 0}$ *-progressively measurable with values in* $R^d \otimes R^m$ *and* R^d *respectively. Assume* $||\alpha(\cdot)|| \le A$ *and* $|\beta(\cdot)| \le B$, *and set* $\xi(t) = \int_0^t \alpha(s)dW_s + \int_0^t \beta(s)ds$. *Then for* $T > 0$ *and* $R > 0$ *satisfying* $d^{\frac{1}{2}}BT < R$:

$$P(\sup_{0\le t\le T} |\xi(t)| \ge R) \le 2d \exp\big(-(R - d^{\frac{1}{2}}BT)^2/2A^2dT\big). \tag{81}$$

Proposition 9.2. *Assume* $(H1.b)$ *holds with* $r(\xi) = log\frac{1}{\xi}$ *and* b, σ *are bounded. Then for any* $\delta > 0$,

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\le t\le 1} |X^\varepsilon(t) - X_n^\varepsilon(t)| > \delta) = -\infty. \tag{82}$$

Proof. Assume $\delta < e^{-1} \wedge \delta_0 < 1$. Let $Y_n^\varepsilon(t) := X^\varepsilon(t) - X_n^\varepsilon(t)$ and $\xi_n^\varepsilon(t) = |Y_n^\varepsilon(t)|^2$. We have

$$\begin{aligned} Y_n^\varepsilon(t) &= \int_0^t \left[b(X^\varepsilon(s)) - b(X_n^\varepsilon(\frac{[ns]}{n})) \right] ds \\ &\quad + \varepsilon^{\frac{1}{2}} \int_0^t \left[\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\frac{[ns]}{n})) \right] dW_s, \end{aligned} \tag{83}$$

and

$$\begin{aligned} d\xi_n^\varepsilon(t) &= 2\varepsilon^{\frac{1}{2}} \langle Y_n^\varepsilon(t), \Big(\sigma(X^\varepsilon(t)) - \sigma(X_n^\varepsilon(\frac{[nt]}{n})) \Big) dW_t \rangle \\ &\quad + 2 \langle Y_n^\varepsilon(t), \Big(b(X^\varepsilon(t)) - b(X_n^\varepsilon(\frac{[nt]}{n})) \Big) \rangle dt \\ &\quad + \varepsilon || \sigma(X^\varepsilon(t)) - \sigma(X_n^\varepsilon(\frac{[nt]}{n})) ||^2 dt. \end{aligned} \tag{84}$$

The stochastic contraction $d\xi_n^\varepsilon \cdot d\xi_n^\varepsilon$ is given by

$$d\xi_n^\varepsilon \cdot d\xi_n^\varepsilon = 4\varepsilon | \Big(\sigma(X^\varepsilon(t)) - \sigma(X_n^\varepsilon(\frac{[nt]}{n})) \Big)^* Y_n^\varepsilon(t)|^2 \, dt, \tag{85}$$

where σ^* denotes the transpose of σ. Let $\rho > 0$. Define the function $\psi_\rho : [0, \delta_o] \to R$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s \log \frac{1}{s} + \rho}. \tag{86}$$

It is clear that for any $0 < \xi < 1$,

$$\psi_\rho(\xi) \uparrow \psi_0(\xi) = \int_0^\xi \frac{ds}{s \log \frac{1}{s}} = +\infty, \quad \text{as } \rho \downarrow 0.$$

Define for $\lambda > 0$

$$\Phi_{\rho,\lambda}(\xi) = e^{\lambda \psi_\rho(\xi)}. \tag{87}$$

Then we have

$$\Phi'_{\rho,\lambda}(\xi) \, (\xi \log \frac{1}{\xi} + \rho) = \lambda \Phi_{\rho,\lambda}(\xi), \tag{88}$$

and

$$\begin{aligned} \Phi''_{\rho,\lambda}(\xi) &= \lambda^2 \Phi_{\rho,\lambda}(\xi) \frac{1}{(\xi \log \frac{1}{\xi} + \rho)^2} + \lambda \Phi_{\rho,\lambda}(\xi) \frac{(1 - \log \frac{1}{\xi})}{(\xi \log \frac{1}{\xi} + \rho)^2} \\ &\le \lambda^2 \Phi_{\rho,\lambda}(\xi) \frac{1}{(\xi \log \frac{1}{\xi} + \rho)^2} \quad \text{if } \xi \le e^{-1}. \end{aligned} \tag{89}$$

Now, choose a positive constant $\delta_1 < e^{-1} \wedge \delta_0$ satisfying $\delta_1 \log \frac{1}{\delta_1} < \rho$. Define $\tau_n^\varepsilon = \inf\{t \ge 0; |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{[nt]}{n})| \ge \delta_1\}$, and set $\xi_{n,\delta_1}^\varepsilon(t) = \xi_n^\varepsilon(t \wedge \tau_n^\varepsilon), t \ge 0$. Putting $T_n^\varepsilon = \inf\{t \ge 0, |\xi_{n,\delta_1}^\varepsilon(t)| \ge \delta^2\}$, we have,

$$\begin{aligned} P(\sup_{0 \le t \le 1} |Y_n^\varepsilon(t)| > \delta) &= P(\sup_{0 \le t \le 1} |Y_n^\varepsilon(t)| > \delta, \tau_n^\varepsilon \le 1) \\ &\quad + P(\sup_{0 \le t \le 1} |Y_n^\varepsilon(t)| > \delta, \tau_n^\varepsilon > 1) \\ &\le P(\tau_n^\varepsilon \le 1) + P(T_n^\varepsilon \le 1). \end{aligned} \tag{90}$$

Observe,

$$P(\tau_n^\varepsilon \le 1) \le \sum_{k=1}^{n} P\Big(\sup_{\frac{k-1}{n}\le t\le \frac{k}{n}} |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{k-1}{n})| \ge \delta_1\Big). \tag{91}$$

Using Lemma 9.1 and the boundness of σ, b, there exists a constant $c_{\delta_1} > 0$ such that

$$P(\tau_n^\varepsilon \le 1) \le n\exp(-nc_{\delta_1}/\varepsilon). \tag{92}$$

Hence,

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0} \varepsilon\log P(\tau_{n,\rho}^\varepsilon \le 1) = -\infty. \tag{93}$$

For notational simplicity, write T for T_n^ε and τ for τ_n^ε. By Ito's formula,

$$\begin{aligned}
& \Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(t\wedge T)) \\
= \;& 1 + 2\varepsilon^{\frac{1}{2}}\int_0^{t\wedge T\wedge\tau} \Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))\Big\langle Y_n^\varepsilon(s), \Big(\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\tfrac{[ns]}{n}))\Big)\, dW_s\Big\rangle \\
+ \;& 2\int_0^{t\wedge T\wedge\tau} \Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))\Big\langle Y_n^\varepsilon(s), b(X^\varepsilon(s)) - b(X_n^\varepsilon(\tfrac{[ns]}{n}))\Big\rangle ds \\
+ \;& \varepsilon\int_0^{t\wedge T\wedge\tau} \Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))||\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\tfrac{[ns]}{n}))||^2 ds \\
+ \;& 2\varepsilon\int_0^{t\wedge T\wedge\tau} \Phi''_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))\,|\Big(\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\tfrac{[ns]}{n}))\Big)^* Y_n^\varepsilon(s)|^2\, ds.
\end{aligned} \tag{94}$$

Note that for $s \le T\wedge\tau$, $|Y_n^\varepsilon(s)| \le \delta_1 \le e^{-1} < 1$ and

$$|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})| \le \delta_1 < e^{-1}.$$

Therefore, for $s \le T\wedge\tau$,

$$\begin{aligned}
& |\Big\langle Y_n^\varepsilon(s), b(X^\varepsilon(s)) - b(X_n^\varepsilon(\frac{[ns]}{n}))\Big\rangle| \\
&= |\Big\langle Y_n^\varepsilon(s), b(X^\varepsilon(s)) - b(X_n^\varepsilon(s))\Big\rangle| + |\Big\langle Y_n^\varepsilon(s), b(X_n^\varepsilon(s)) - b(X_n^\varepsilon(\frac{[ns]}{n}))\Big\rangle| \\
&\le C|Y_n^\varepsilon(s)|^2\log(\frac{1}{|Y_n^\varepsilon(s)|}) + C|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})|\log(\frac{1}{|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})|}) \\
&\le \frac{1}{2}C\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)}) + \delta_1\log(\frac{1}{\delta_1}) \le C(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)}) + \rho),
\end{aligned} \tag{95}$$

where we have used the fact that the function $x\log(\frac{1}{x})$ is increasing on $[0, e^{-1}]$. Furthermore, for $s \le T\wedge\tau$,

$$\begin{aligned}
& ||\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(\frac{[ns]}{n}))||^2 \\
&\le 2||\sigma(X^\varepsilon(s)) - \sigma(X_n^\varepsilon(s)))||^2 + 2||\sigma(X_n^\varepsilon(s)) - \sigma(X_n^\varepsilon(\frac{[ns]}{n}))||^2 \\
&\le 2C\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)}) + 2C|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})|^2\log(\frac{1}{|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})|})
\end{aligned}$$

$$\leq C(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho). \tag{96}$$

Similarly,

$$|\left(\sigma(X^\varepsilon(s))-\sigma(X_n^\varepsilon(\frac{[ns]}{n}))\right)^* Y_n^\varepsilon(s)|^2$$
$$\leq |Y_n^\varepsilon(s)|^2||\left(\sigma(X^\varepsilon(s))-\sigma(X_n^\varepsilon(\frac{[ns]}{n}))\right)^*||^2$$
$$\leq C\xi_n^\varepsilon(s)(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho)\leq C(\xi_n^\varepsilon(s)\log(\frac{1}{\xi_n^\varepsilon(s)})+\rho)^2. \tag{97}$$

Taking these inequalities into account, it follows from (94) that

$$\begin{aligned} &E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(t\wedge T))] \\ \leq\quad & CE\Big[\int_0^{t\wedge T}\Phi'_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))(\xi_{n,\delta_1}^\varepsilon(s)\log(\tfrac{1}{\xi_{n,\delta_1}^\varepsilon(s)})+\rho)ds\Big] \\ +\quad & C\varepsilon E\Big[\int_0^{t\wedge T}\Phi''_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s))(\xi_{n,\delta_1}^\varepsilon(s)\log(\tfrac{1}{\xi_{n,\delta_1}^\varepsilon(s)})+\rho)^2 ds\Big], \end{aligned} \tag{98}$$

which is smaller by (88) and (89) than

$$1+C\varepsilon(\lambda^2+\lambda)\int_0^t E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(s\wedge T))]ds.$$

By Gronwall's inequality, we obtain that

$$E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(1\wedge T))]\leq e^{C(\varepsilon\lambda^2+\lambda)}. \tag{99}$$

On the other hand,

$$E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(1\wedge T))]$$
$$\geq E[\Phi_{\rho,\lambda}(\xi_{n,\delta_1}^\varepsilon(T)),T\leq 1]=e^{\lambda\psi_\rho(\delta^2)}P(T\leq 1). \tag{100}$$

Combining (90) with (100), we have

$$P(T\leq 1)\leq e^{-\lambda\psi_\rho(\delta^2)}e^{C(\varepsilon\lambda^2+\lambda)}.$$

Taking $\lambda=\frac{1}{\varepsilon}$ it follows that

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon log P(T\leq 1)\leq -\psi_\rho(\delta^2)+2C.$$

This together with (93) implies that

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon log P(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)|>\delta)\leq -\psi_\rho(\delta^2)+2C \tag{101}$$

Sending ρ to 0 completes the proof. ∎

For $n\geq 1$, define a map $F_n(\cdot):C_0([0,1],R^m)\to C_{x_0}([0,1],R^d)$ by

$$\begin{aligned} F_n(\omega)(0) &= x_0, \\ F_n(\omega)(t) &= F_n(\omega)\Big(\frac{k}{n}\Big)+b\Big(F_n(\omega)\Big(\frac{k}{n}\Big)\Big)\Big(t-\frac{k}{n}\Big) \\ &\quad +\sigma\Big(F_n(\omega)\Big(\frac{k}{n}\Big)\Big)\Big(\omega(t)-\omega\Big(\frac{k}{n}\Big)\Big), \end{aligned} \tag{102}$$

for $\frac{k}{n} \leq t \leq \frac{k+1}{n}$. It is easy to see that F_n is a continuous map from $C_0([0,1], R^m)$ to $C_{x_0}([0,1], R^d)$.

Lemma 9.3. $\lim_{n\to\infty} \sup_{\{g; e(g)\leq\alpha\}} \sup_{0\leq t\leq 1} |F_n(g)(t) - F(g)(t)| = 0.$

Proof . Note that for g with $e(g) \leq \alpha$,

$$\begin{aligned} F_n(g)(t) &= x_0 + \int_0^t b(F_n(g)(\frac{[ns]}{n}))ds \\ &+ \int_0^t \sigma\Big(F_n(g)\Big(\frac{[ns]}{n}\Big)\Big)\dot{g}(s)ds. \end{aligned} \tag{103}$$

Thus,

$$\begin{aligned} &F_n(g)(t) - F(g)(t) \\ &= \int_0^t [b(F_n(g)\Big(\frac{[ns]}{n}\Big)) - b(F(g)(s))]ds \\ &+ \int_0^t [\sigma(F_n(g)\Big(\frac{[ns]}{n}\Big)) - \sigma(F(g)(s))]\dot{g}(s)ds. \end{aligned} \tag{104}$$

Since b, σ are bounded, we have for $t \leq 1$

$$\begin{aligned} |F_n(g)(t) - F_n(g)\Big(\frac{[nt]}{n}\Big)| &\leq \int_{\frac{[nt]}{n}}^t |b(F_n(g)(\frac{[ns]}{n}))|ds \\ &+ \int_{\frac{[nt]}{n}}^t ||\sigma(F_n(g)\Big(\frac{[ns]}{n}\Big))|||\dot{g}|(s)ds \\ &\leq C_\alpha\Big(\frac{1}{n}\Big)^{\frac{1}{2}}, \end{aligned} \tag{105}$$

where C_α is a constant depending only on α and the uniform norms of b and σ. Let $Y_n^g(t) = F_n(g)(t) - F(g)(t)$ and $Z_n^g(t) = |Y_n^g(t)|^2$. For any $0 < \delta < e^{-1}$, define $\tau_n(g) = \inf\{t \geq 0, |Y_n^g(t)| > \delta\}$. Given $\rho > 0$, define

$$\Phi_\rho(\xi) = e^{\psi_\rho(\xi)},$$

where

$$\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s\log\frac{1}{s} + \rho}.$$

Then

$$\Phi'_\rho(\xi)(\xi\log\frac{1}{\xi} + \rho) = \Phi_\rho(\xi). \tag{106}$$

By the chain rule,

$$\begin{aligned} &\Phi_\rho(Z_n^g(t\wedge\tau_n(g))) \\ &= 1 + 2\int_0^{t\wedge\tau_n(g)} \Phi'_\rho(Z_n^g(s))\Big\langle Y_n^g(s), b(F_n(g)(\frac{[ns]}{n})) - b(F(g)(s))\Big\rangle ds \\ &+2\int_0^{t\wedge\tau_n(g)} \Phi'_\rho(Z_n^g(s))\Big\langle Y_n^g(s), (\sigma(F_n(g)(\frac{[ns]}{n}))-\sigma(F(g)(s)))\dot{g}(s)\Big\rangle ds. \end{aligned} \tag{107}$$

Using $(H1.b)$ and (105), for $s \le \tau_n(g)$,

$$
\begin{aligned}
&|\left\langle Y_n^g(s), b(F_n(g)\left(\frac{[ns]}{n}\right)) - b(F(g)(s))\right\rangle| \\
&\le |\left\langle Y_n^g(s), b(F_n(g)\left(\frac{[ns]}{n}\right)) - b(F_n(g)(s))\right\rangle| \\
&\quad +|\left\langle Y_n^g(s), b(F_n(g)(s)) - b(F(g)(s))\right\rangle| \\
&\le C|Y_n^g(s)||F_n(g)\left(\frac{[ns]}{n}\right) - (F_n(g)(s)| \log(\frac{1}{|F_n(g)(\frac{[ns]}{n}) - (F_n(g)(s)|}) \\
&\quad +\frac{1}{2}CZ_n^g(s)\log(\frac{1}{Z_n^g(s)}) \\
&\le CC_\alpha(\frac{1}{n})^{\frac{1}{2}}\log(n^{\frac{1}{2}}) + CZ_n^g(s)\log(\frac{1}{Z_n^g(s)}) \\
&\le C(Z_n^g(s)\log(\frac{1}{Z_n^g(s)}) + \rho), \qquad (108)
\end{aligned}
$$

for $n \ge N_\alpha^1$, where N_α^1 depends on α and ρ. Similarly, for $s \le \tau_n(g)$ and $n \ge N_\alpha^1$,

$$
\begin{aligned}
&|Y_n^g(s)|||\sigma(F_n(g)\left(\frac{[ns]}{n}\right)) - \sigma(F(g)(s))|| \\
&\le C(Z_n^g(s)\log(\frac{1}{Z_n^g(s)}) + \rho). \qquad (109)
\end{aligned}
$$

It follows from (107) that for $n \ge N_\alpha^1$ and all $g \in \{g; e(g) \le \alpha\}$,

$$
\begin{aligned}
&\Phi_\rho(Z_n^g(t\wedge\tau_n(g))) \\
&\le 1 + C\int_0^{t\wedge\tau_n(g)} \Phi'_\rho(Z_n^g(s))(\frac{1}{Z_n^g(s)}) + \rho)(1+|\dot{g}(s)|)ds \\
&\le C\int_0^t \Phi_\rho(Z_n^g(s\wedge\tau_n(g)))(1+|\dot{g}(s)|)ds. \qquad (110)
\end{aligned}
$$

By Gronwall's lemma,

$$
\Phi_\rho(Z_n^g(1\wedge\tau_n(g))) \le Ce^{1+\int_0^1 |\dot{g}(s)|ds}.
$$

Since $\Phi_\rho(\xi)$ is incresing in ξ, it follows that for $n \ge N_\alpha^1$,

$$
\Phi_\rho\left(\sup_{g\in\{g;e(g)\le\alpha\}} Z_n^g(1\wedge\tau_n(g))\right) \le Ce^{1+\alpha}. \qquad (111)
$$

Consequently, for any $\rho > 0$,

$$
\limsup_{n\to\infty} \Phi_\rho\left(\sup_{g\in\{g;e(g)\le\alpha\}} Z_n^g(1\wedge\tau_n(g))\right) \le Ce^{1+\alpha}. \qquad (112)
$$

To complete the proof it suffices to show that for any $\delta > 0$ there exists an integer N such that if $n \ge N$, then $\tau_n(g) > 1$ for all $g \in \{g; e(g) \le \alpha\}$.

This is now a consequence of (112). In fact, otherwise, there exists $\delta > 0$, a subsequence $\{n_k, k \geq 1\}$ of positive integers and $g_{n_k} \in \{g; e(g) \leq \alpha\}$ such that $\tau_{n_k}(g_{n_k}) > 1$. This implies that

$$\Phi_\rho(\sup_{g\in\{g;e(g)\leq\alpha\}} Z^g_{n_k}(1 \wedge \tau_{n_k}(g))) \geq \Phi_\rho(Z^{g_{n_k}}_{n_k}(1 \wedge \tau_{n_k}(g_{n_k}))) \geq \Phi_\rho(\delta^2).$$

Combing this with (112), we get

$$\Phi_\rho(\delta^2) \leq Ce^{1+\alpha}, \tag{113}$$

for all ρ. This leads to a contradiction since the left side of (113) tends to infinity as ρ goes to 0. The proof is complete. ∎

Proof of Theorem 9.1 when b, σ are bounded Notice that $X^\varepsilon_n(s) = F_n(\varepsilon^{\frac{1}{2}}W)(s)$, where W is the Brownian motion. The theorem follows from Proposition 5.2 , Lemma 5.3 and Theorem 4.2.23 in [DZ]. ∎

10 Large deviations: general case

In this section we will remove the boundeness assumptions on b and σ. We begin with

Proposition 10.1 *Assume (H2.b) with $\rho(\xi) = log\xi$. Then*

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq 1} |X^\varepsilon(t) - x_0| > R) = -\infty \tag{114}$$

where $X^\varepsilon(\cdot)$ is the solution to equation (75).

Proof. Let δ_0 be a fixed small positive constant, say $\delta_0 < \frac{1}{2}$. Let $f \in C^1(R_+)$ be a strictly positive C^1 function on R_+ that satisfies

$$f(s) = \begin{cases} -s\log s & \text{if } 0 \leq s \leq 1-\delta_0, \\ s\log s & \text{if } s \geq 1+\delta_0. \end{cases}$$

From now on, we will use C to denote a generic constant which may change from line to line. It is easy to see that there exists a positive contant C such that

$$s|\log s| + 1 \leq C(f(s)+1), \quad f'(s) \geq -C \quad \text{for } s \geq 0. \tag{115}$$

Define

$$\psi(\xi) = \int_0^\xi \frac{ds}{f(s)+1}, \quad \xi \geq 0,$$

and put

$$\Phi_\lambda(\xi) = e^{\lambda\psi(\xi)}, \quad \lambda \geq 0.$$

It follows from (115) that

$$\Phi'_\lambda(\xi) = \lambda\Phi_\lambda(\xi)\frac{1}{f(\xi)+1} \leq C\lambda\Phi_\lambda(\xi)\frac{1}{(\xi|\log\xi|+1)}, \tag{116}$$

and

$$\Phi_\lambda''(\xi) = \lambda^2\Phi_\lambda(\xi)\frac{1}{(f(\xi)+1)^2} - \lambda\Phi_\lambda(\xi)\frac{f'(\xi)}{(f(\xi)+1)^2}$$
$$\le C(\lambda^2+\lambda)\Phi_\lambda(\xi)\frac{1}{(\xi|\log\xi|+1)^2}. \tag{117}$$

Let $\eta^\varepsilon(t) = X^\varepsilon(t) - x_o$ and $\xi^\varepsilon(t) = |\eta^\varepsilon(t)|^2$. Define

$$\tau_R = \inf\{t > 0, |\eta^\varepsilon(t)| \ge R\}, \quad R > 0.$$

Now by Itô formula, we have

$$\begin{aligned}
\Phi_\lambda(\xi^\varepsilon(t\wedge\tau_R)) &= 1 + 2\varepsilon^{\frac{1}{2}}\int_0^{t\wedge\tau_R}\Phi_\lambda'(\xi^\varepsilon(s))\langle\eta^\varepsilon(s),\sigma(X^\varepsilon(s))dW_s\rangle \\
&+ 2\int_0^{t\wedge\tau_R}\Phi_\lambda'(\xi^\varepsilon(s))\,\langle\eta^\varepsilon(s),b(X^\varepsilon(s))\rangle\,ds \\
&+ \varepsilon\int_0^{t\wedge\tau_R}\Phi_\lambda'(\xi^\varepsilon(s))||\sigma(X^\varepsilon(s))||^2\,ds \\
&+ 2\varepsilon\int_0^{t\wedge\tau_R}\Phi_\lambda''(\xi^\varepsilon(s))\,|\sigma^*(X^\varepsilon(s))\eta^\varepsilon(s)|^2\,ds.
\end{aligned} \tag{118}$$

By $(H2.b)$, there exists $C_1 > 0$ such that

$$\begin{cases} ||\sigma(x)||^2 \le C_1\,(|x-x_o|^2\,\log|x-x_o|+1), \\ |b(x)| \le C_1\,(|x-x_o|\,\log|x-x_o|+1). \end{cases}$$

It follows that

$$\frac{|\sigma^*(X^\varepsilon(s))\eta^\varepsilon(s)|^2}{(\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1)^2} \le C_1\,\frac{\xi^\varepsilon(s)\,(\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1)}{(\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1)^2}$$

which is dominated by a constant C. By (117), we get

$$\int_0^{t\wedge\tau_R}\Phi_\lambda''(\xi^\varepsilon(s))\,|\sigma^*(X^\varepsilon(s))\eta^\varepsilon(s)|^2\,ds \le C(\lambda^2+\lambda)\int_0^{t\wedge\tau_R}\Phi_\lambda(\xi^\varepsilon(s))\,ds. \tag{119}$$

In the same way, for some constant $C > 0$, we have

$$\frac{|\langle\eta^\varepsilon(s),b(X^\varepsilon(s))\rangle| + ||\sigma(X^\varepsilon(s))||^2}{\xi^\varepsilon(s)\,|\log\xi^\varepsilon(s)|+1} \le C, \quad s > 0. \tag{120}$$

Combining above inequalities together, we get

$$E(\Phi_\lambda(\xi^\varepsilon(t\wedge\tau_R))) \le 1 + C(\varepsilon\lambda^2+\lambda)\int_0^t E(\Phi_\lambda(\xi^\varepsilon(t\wedge\tau_R)))\,ds,$$

which implies that

$$E(\Phi_\lambda(\xi^\varepsilon(1\wedge\tau_R))) \le e^{C(\varepsilon\lambda^2+\lambda)}.$$

Let $\lambda = \frac{1}{\varepsilon}$. It follows that

$$P(\sup_{0\le t\le 1}|X^\varepsilon(t) - x_0| > R)\Phi_{\frac{1}{\varepsilon}}(R) \le E(\Phi_\lambda(\xi^\varepsilon(1\wedge\tau_R)))$$

$$\leq e^{2C\frac{1}{\varepsilon}}.$$

This gives that

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq 1} |X^\varepsilon(t) - x_0| > R) \leq C - \psi(R). \tag{121}$$

Note that $\lim_{R\to\infty} \psi(R) = +\infty$. Letting R tend to $+\infty$ in (121) proves the proposition. ∎

For $R > 0$, define $m_R = \sup\{|b(x)|, ||\sigma(x)||; |x| \leq R\}$ and $b_i^R = (-m_R - 1) \vee b_i \wedge (m_R + 1)$, $\sigma_{i,j}^R = (-m_R - 1) \vee \sigma_{i,j} \wedge (m_R + 1)$, $1 \leq i \leq d$, $0 \leq j \leq m$. Put $b_R = (b_1^R, b_2^R, ..., b_d^R)$ and $\sigma_R = (\sigma_{i,j}^R)_{1\leq i\leq d, 1\leq j\leq m}$. Then for $|x| \leq R$,

$$b_R(x) = b(x), \quad \sigma_R(x) = \sigma(x).$$

and b_R, σ_R satisfy (H1.b) and (H2.b) with the same constants.
Let $X_R^\varepsilon(\cdot)$ be the solution to

$$\begin{aligned} X_R^\varepsilon(t) &= x_0 + \int_0^t b_R(X_R^\varepsilon(s))ds \\ &+ \varepsilon^{\frac{1}{2}} \int_0^t \sigma_R(X_R^\varepsilon(s))dW_s, \ t > 0. \end{aligned} \tag{122}$$

For g with $e(g) < \infty$, let $F_R(g)$ be the solution to

$$\begin{aligned} F_R(g)(t) &= x_0 + \int_0^t b_R(F_R(g)(s))ds \\ &+ \int_0^t \sigma_R(F_R(g)(s))\dot{g}(s)ds. \end{aligned} \tag{123}$$

Define

$$I_R(f) = \inf\left\{\frac{1}{2}e(g)^2; F_R(g) = f\right\}, \quad f \in C_{x_0}([0,1] \to R^d). \tag{124}$$

If $\sup_{0\leq t\leq 1} |F(g)(t)| \leq R$, then $F(g) = F_R(g)$. Therefore,

$$I(f) = I_R(f), \quad \text{ for } f \text{ with } \sup_{0\leq t\leq 1} |f(t)| \leq R. \tag{125}$$

Lemma 10.2 *$I(\cdot)$ is a good rate function on $C_{x_0}([0,1], R^d)$,i.e., for any $\alpha \geq 0$, the level set $\{f; I(f) \leq \alpha\}$ is compact.*

Proof. Reasoning as in Lemma 9.3, it is easy to see that for $\alpha > 0$, $\sup_{\{g; e(g)\leq\alpha\}} ||F(g)||_\infty \leq R$ for some constant R. Thus $F_R(\cdot) = F(\cdot)$ on $\{g; e(g) \leq \alpha\}$. On the other hand, It is easy to see that $F_R(\cdot)$ is continuous on the level set $\{g; e(g) \leq \alpha\}$, so is $F(\cdot)$. This is sufficient to conclude that $I(\cdot)$ is a good rate functional since $e(\cdot)$ is. ∎

Proof of Theorem 8.1 in the Unbounded Case.

Recall that the process $X_R^\varepsilon(\cdot)$ is defined by (122). Let μ_ε^R denote the law of $X_R^\varepsilon(\cdot)$ on $C_{x_0}([0,1], R^d)$. According to previous section, $\{\mu_\varepsilon^R, \varepsilon > 0\}$ satisfies a large deviation principle with good rate function $I_R(\cdot)$. Note that μ_ε^R and μ_ε coincide on the ball $\{f; ||f||_\infty \leq R\}$. For $R > 0$ and a closed subset $C \subset C_{x_0}([0,1], R^d)$, set $C_R = C \cap \{f; ||f||_\infty \leq R\}$. Then,

$$\begin{aligned} \mu_\varepsilon(C) &\leq \mu_\varepsilon(C_R) + P(\sup_{0\leq t\leq 1} |X^\varepsilon(t)| > R) \\ &= \mu_\varepsilon^R(C_R) + P(\sup_{0\leq t\leq 1} |X^\varepsilon(t)| > R). \end{aligned} \tag{126}$$

By the large deviation principle for $\{\mu_\varepsilon^R, \varepsilon > 0\}$,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \\ &\leq (-\inf_{f\in C_R} I_R(f)) \vee (\limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq 1} |X^\varepsilon(t)| > R)) \\ &= (-\inf_{f\in C_R} I(f)) \vee (\limsup_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leq t\leq 1} |X^\varepsilon(t)| > R)). \end{aligned} \tag{127}$$

Applying Proposition 10.1 and Letting $R \to \infty$, we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(C) \leq (-\inf_{f\in C} I(f)), \tag{128}$$

which is the upper bound.
Let G be an open subset of $C_{x_0}([0,1] \to R^d)$. Fix any $\phi_0 \in G$. Choosing $\delta > 0$ such that $B(\phi_0, \delta) = \{f; ||f - \phi_0||_\infty \leq \delta\} \subset G$. Let $R = ||\phi_0||_\infty + \delta$. Since

$$B(\phi_0, \delta) \subset \{f; ||f||_\infty \leq R\},$$

we have

$$\begin{aligned} -I(\phi_0) = -I_R(\phi_0) &\leq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon^R\Big(B(\phi_0, \delta)\Big) \\ &= \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon\Big(B(\phi_0, \delta)\Big) \\ &\leq (\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G)). \end{aligned} \tag{129}$$

Since ϕ_0 is arbitrary, it follows that

$$-\inf_{f\in G} I(f) \leq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G), \tag{130}$$

which is the lower bound. ∎

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