Large deviations for stochastic nonlinear beam equations

Tusheng Zhang

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, U.K. Email: tzhang@maths.man.ac.uk

Abstract

We establish a large deviation principle for the solutions of stochastic partial differential equations for nonlinear vibration of elastic panels (also called stochastic nonlinear beam equations).

Key words: Stochastic partial differential equations, stochastic beam equations, large deviations, exponential martingales, exponential integrability.

AMS Subject Classification: Primary 60H15 Secondary 60F10, 35R60.

1 Introduction

Consider a bounded open interval on the real line, say, $(0,1)$. Let $L^2 = L^2(0,1)$. Denote by $H_0^1 = H_0^1(0,1)$ and $H_0^2 = H_0^2(0,1)$ the Sobolev spaces of order one and two satisfying the homogeneous boundary conditions. Denote by H_0^{-k} the dual space of H_0^k. $(\cdot,\cdot)$ will denote the L^2-inner product and $<\cdot,\cdot>$ denotes the dual pairing. The norms on L^2, H_0^k and H_0^{-k} will be denoted respectively by $||\cdot||$, $||\cdot||_k$ and $||\cdot||_{-k}$. Consider the linear operator

$$Au = \alpha\partial_x^2 u - \gamma\partial_x^4 u,$$

and the nonlinear operator

$$B(u) = \beta\left(\int_0^1 |\partial_x u|^2 dx\right)\partial_x^2 u.$$

The mathematical model for the nonlinear penal vibration is governed by the following partial differential equation:

$$\begin{aligned}\partial_t^2 u_t &= \left(\alpha + \int_0^1 |\partial_y u_t|^2 dy\right)\partial_x^2 u_t - \gamma\partial_x^4 u_t + F(\dot{u}_t, u_t) \\ &\quad u_t(0) = u_t(1) = 0, \quad \partial_x u_t(0) = \partial_x u_t(1) = 0, \\ &\quad u_0(x) = \phi_0(x), \quad \partial_t u_0(x) = \phi_1(x),\end{aligned} \tag{1}$$

where $\dot{u}_t$ denotes the derivative of u with respect to the variable t. A detailed study of the model can be found in the book by Dowell [D2]. The equation was also proposed by

Woinowsky-Krieger in [WK] as a model for the transversal deflection of an extensible beam of natural length 1. An equation in two space variables similar to (1) was suggested in [C2] as a model of nonlinear oscillations of a plate in a supersonic flow of gas. It has also been studied by many other people, see [B],[BMS],[F],[KP] and references therein.

Let $W_t, t \geq 0$ be a Wiener process taking values in a Hilbert space. Without loss of generality, we may assume that W_t is l^2-valued Wiener process which admits the following representation:

$$W_t = \sum_{k=1}^{\infty} \lambda_k \beta_t^k e_k,$$

where $\lambda_k, k \geq 1$ is a sequence of non-negative numbers such that $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$, $\beta_t^k, k \geq 1$ is a sequence of independent standard Brownian motions and $\{e_k, k \geq 1\}$ is the canonical orthonormal basis of l^2.

Taking into account the random fluctuations, Chow and Menaldi (1999) considered in [CM1] the stochastic nonlinear partial differential equation for vibration of elastic panels:

$$\begin{aligned}
\partial_t^2 u_t^\varepsilon &= \left(\alpha + \int_0^1 |\partial_y u_t^\varepsilon|^2 dy\right) \partial_x^2 u_t^\varepsilon - \gamma \partial_x^4 u_t^\varepsilon + \varepsilon \sigma(u_t^\varepsilon) \dot{W}_t \\
&+ F(\dot{u}_t^\varepsilon, u_t^\varepsilon) \\
& u_t^\varepsilon(0) = u_t^\varepsilon(1) = 0, \quad \partial_x u_t^\varepsilon(0) = \partial_x u_t^\varepsilon(1) = 0, \\
& u_0^\varepsilon(x) = \phi_0(x) \in H_0^2, \quad \partial_t u_0^\varepsilon(x) = \phi_1(x) \in L^2,
\end{aligned} \tag{2}$$

where for every $u \in H_0^2$, $\sigma(u)$ stands for a map from l^2 into H_0^2 which will be specified later, and $F(\cdot, \cdot)$ denotes a map from $L^2 \times H_0^2$ into L^2. It is proved in [CM1] that under reasonable conditions on σ, (2) has a unique solution with the property:

$$u^\varepsilon \in C([0,T]; H_0^2) \quad \text{and} \quad \dot{u}^\varepsilon \in C([0,T]; L^2).$$

A general formulation of the equation in an abstract Hilbert space was later studied by Brzeźniak, Maslowski and Seidler (2005) in [BMS], where existence, uniqueness and asymptotic stability of the solution were discussed.

The aim of this paper is to establish a large deviation principle (LDP) for the the vector process $v_t^\varepsilon = (u_t^\varepsilon, \dot{u}_t^\varepsilon)$ on the product space $C([0,T]; H_0^2) \times C([0,T]; L^2)$ as $\varepsilon \to 0$.

The large deviation problem for stochastic partial differential equations (SPDEs) has been studied by many people, but mainly for stochastic parabolic equations. For example, an LDP for stochastic reaction equations with nonlinear reaction term was established by Cerrai and Röckner (2004) in [CR]. An LDP for stochastic a Burgers'-type SPDEs was considered by Cardon-Weber (1999) in [CW]. A uniform LDP for parabolic SPDEs was proved by Chenal and Millet (1997) in [CM2]. In [RS], Rovira and Sanz-Sole (1996) proved an LDP for a class of nonlinear hyperbolic SPDEs. An LDP was obtained by P.L. Chow (1992) for some parabolic SPDEs in [C1]. An LDP for stochastic reaction equations was established by R. Sowers (1992) in [S]. A small time large deviation principle for stochastic parabolic equations was obtained by the author (2000) in [Z]. For the general theory of large deviations, readers are referred to the monograph [DZ].

Because of the different nature of nonlinearity for different types of equations, the large deviations for SPDEs has to be dealt with on individual bases. There are two main issues which distinguish the current work from the previous ones. The first is the cubic nonlinear term $B(u)$ in the equation (2) and the second is the second order differentiation in t (not like the parabolic cases). Note that even the existence and uniqueness of the solution of this kind of equation was newly established. Although the second order (in t) equation (2) can also be written as a system of parabolic equations as it was done in [BMZ], but by doing so the operator (differential) becomes degenerate. The properties of the corresponding semigroups are therefore not good enough for the large deviation estimates, not like the parabolic cases in the existing literature. To tackle the first issue, our idea is to prove that the probability that the energy of the solution is big is exponentially small. To this end, a remarkable result of Davis [D1], Barlow and Yor in [BY] on the moment estimates of martingales plays a key role. To treat the second order differentiation in t, we fully exploit the energy equality proved by Chow, Menaldi and Pardoux, and establish some exponential integrability of Hilbert space-valued martingales. To achieve this, some exponential martingales are specially constructed.

The rest of the paper is organized as follows: In Section 2, the precise result is stated. In Section 3, the skeleton equation is studied. It is proved that the solution is a continuous map from the level set into the space $C([0,T];H_0^2) \times C([0,T];L^2)$. Section 4 is devoted to the proof of the large deviation principle. The long proof is split into several lemmas for clarity.

We end this introduction with a remark.

Remark 1.1. The main result in this paper is stated in the setting of one space dimension. This is just for simplicity. Our approach works equally well in high space dimensions and also in the general setting formulated in [BMS].

Throughout the paper, the generic constants may be different from line to line. If it is essential, the dependence of a constant on the parameters will be written explicitly.

2 Statement of the main result

We now state the precise conditions on σ. Let $\sigma_k(\cdot), k \geq 1$ be a sequence of mappings from H_0^2 into H_0^2 and $F(\cdot,\cdot)$ a mapping from from $L^2 \times H_0^2$ into L^2. Introduce

(A.1) .

$$|trace\sigma(u)|^2 = \sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u)||_2^2 \leq c(1 + ||u||_2^2)$$

(A.2) . $\sum_{k=1}^{\infty} ||\sigma_k(u)||^2$ is bounded on bounded subsets of H_0^2

(A.3) .

$$|trace(\sigma(u) - \sigma(v))|^2 = \sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u) - \sigma_k(v)||^2 \leq c(||u - v||^2)$$

(A.4) .

$$||F(v,u)|| \leq c(1 + ||v|| + ||u||_2)$$

(A.5) .

$$||F(v_1,u_1) - F(v_2,u_2)|| \leq c(||v_1 - v_2|| + ||u_1 - u_2||_2)$$

Throughout this paper, we assume (A.1)-(A.5) are in place. The time interval we consider is fixed as $[0, T]$. We notice that the Cameron-Martin space $\mathcal{H}$ corresponding to the Wiener process W_t is given by

$$\mathcal{H} = \{h_t = \sum_{k=1}^{\infty} \lambda_k h_t^k e_k; \quad \sum_{k=1}^{\infty} \int_0^T (\dot{h}_s^k)^2 ds < \infty\}.$$

For $h \in \mathcal{H}$, let u_t^h denote the solution of the following deterministic PDE, the so called skeleton equation:

$$\begin{aligned} d\dot{u}_t^h &= Au_t^h dt + B(u_t^h)dt + \sum_{k=1}^{\infty} \lambda_k \sigma_k(u_t^h)\dot{h}_t^k dt \\ &+ F(\dot{u}_t^h, u_t^h)dt \\ &\quad u_0^h = \phi_0 \in H_0^2, \qquad \dot{u}_0^h = \phi_1 \in L^2. \end{aligned} \tag{3}$$

For $h_t = \sum_{k=1}^{\infty} \lambda_k h_t^k e_k \in \mathcal{H} \subset C([0,T]; l^2)$, define

$$I(h) = \frac{1}{2} \sum_{k=1}^{\infty} \int_0^T (\dot{h}_t^k)^2 dt.$$

Set $I(h)=\infty$ if $h\in C([0,T];l^2)\setminus\mathcal{H}$. Notice that $I(\cdot)$ is the rate function for the large deviations of the l^2-valued Brownian motion

$$W_t=\sum_{k=1}^{\infty}\lambda_k\beta_t^k e_k.$$

This is clear by considering the finite dimensional version: $W_t^d=\sum_{k=1}^{d}\lambda_k\beta_t^k e_k$.

For $f\in C([0,T];H_0^2)\times C([0,T];L^2)$, introduce

$$\mathcal{L}_f=\{h\in\mathcal{H};\quad f(t)=(u_t^h,\dot{u}_t^h)\}.$$

Define

$$R(f)=\begin{cases}\inf_{h\in\mathcal{L}_f}I(h) & \text{if}\quad \mathcal{L}_f\neq\emptyset,\\ +\infty & \text{if}\quad \mathcal{L}_f=\emptyset.\end{cases}$$

Theorem 1 *Assume (A.1)-(A.5). Let μ_ε be the law of $(u^\varepsilon,\dot{u}^\varepsilon)$ on the product space $C([0,T];H_0^2)\times C([0,T];L^2)$. Then $\{\mu_\varepsilon,\varepsilon>0\}$ satisfies a large deviation principle with rate function $R(f)$, i.e.,*

(1) for every closed subset $C\subset C([0,T];H_0^2)\times C([0,T];L^2)$,

$$\limsup_{\varepsilon\to0}\varepsilon^2 log\mu_\varepsilon(C)\leq -inf_{f\in C}R(f), \tag{4}$$

(2) for every open subset $G\subset C([0,T];H_0^2)\times C([0,T];L^2)$,

$$\liminf_{\varepsilon\to0}\varepsilon^2 log\mu_\varepsilon(G)\geq -inf_{f\in G}R(f). \tag{5}$$

3 The skeleton equation

The purpose of this section is to study the skeleton equation. For $h\in\mathcal{H}$, recall that u_t^h denote the solution of the following deterministic PDE, the so called skeleton equation:

$$\begin{aligned} d\dot{u}_t^h &= Au_t^h dt+B(u_t^h)dt+\sum_{k=1}^{\infty}\lambda_k\sigma_k(u_t^h)\dot{h}_t^k dt\\ &+ F(\dot{u}_t^h,u_t^h)dt \\ & \quad u_0^h=\phi_0\in H_0^2,\qquad \dot{u}_0^h=\phi_1\in L^2. \end{aligned}\tag{6}$$

For $a>0$, we aim to show that the mapping $v^h=(u^h,\dot{u}^h)$ from $(\{h;I(h)\leq a\},||\cdot||_\infty)$ into $C([0,T];H_0^2)\times C([0,T];H)$ is continuous, where $||\cdot||_\infty$ denotes the uniform norm on $C([0,T];l^2)$.

Proposition 2 *The map: $v^h=(u_t^h,\dot{u}_t^h)$ from $(\{h;I(h)\leq a\},||\cdot||_\infty)$ into $C([0,T];H_0^2)\times C([0,T];H)$ is continuous.*

Proof. Let $h^n \in \{h; I(h) \le a\}$ with $\lim_{n\to\infty} \sup_{0\le t\le T} ||h^n_t - h_t||_{l^2} = 0$. Define

$$e(t,u) = \frac{1}{2}\{||\dot{u}_t||^2 + \alpha||\partial_x u_t||^2 + \frac{\beta}{2}||\partial_x u_t||^4 + \gamma||\partial_x^2 u_t||^2\}. \tag{7}$$

By the energy equality proved in Theorem 3.1 in [CM1] and (A.1), we have

$$\begin{aligned}
e(t,u^h) &= e(0,u^h) + \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{u}^h_s, \sigma_k(u^h_s))\dot{h}^k_s ds + \int_0^t (F(\dot{u}^h_s, u^h_s), \dot{u}^h_s) ds \\
&\le e(0,u^h) + \int_0^t ||\dot{u}^h_s|| \left(\sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u^h_s)||^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (\dot{h}^k_s)^2\right)^{\frac{1}{2}} ds \\
&+ c\int_0^t ||\dot{u}^h_s||(1 + ||\dot{u}^h_s|| + ||u^h_s||_2) ds \\
&\le e(0,u^h) + c\int_0^t ||\dot{u}^h_s||(1 + ||u^h_s||_2) \left(\sum_{k=1}^{\infty} (\dot{h}^k_s)^2\right)^{\frac{1}{2}} ds \\
&+ c_T + c\int_0^t e(s,u^h) ds \\
&\le c_T + e(0,u^h) + c\int_0^t e(s,u^h)\left(\left(\sum_{k=1}^{\infty} (\dot{h}^k_s)^2\right)^{\frac{1}{2}} + 1\right) ds.
\end{aligned} \tag{8}$$

When h is fixed, it is easy to check that $\sup_{0\le t\le T} e(t,u^h) < \infty$. Thus, applying Grownwall's inequality, we get

$$\sup_{h\in\{h;I(h)\le a\}} \sup_{0\le t\le T} e(t,u^h) = M < \infty. \tag{9}$$

Observe that

$$||A\phi||_{-2} \le (\gamma + \alpha)||\phi||_2, \quad \phi \in H_0^2, \tag{10}$$

$$\begin{aligned}
||B(\phi) - B(\psi)|| &\le \beta||\partial_x^2\phi|| ||\partial_x\phi + \partial_x\psi|| ||\partial_x\phi - \partial_x\psi|| \\
&\quad + \beta||\partial_x\psi||^2 ||\partial_x^2\phi - \partial_x^2\psi||, \quad \phi,\psi \in H_0^2.
\end{aligned} \tag{11}$$

Thus, (9) implies that there exists constants C_1 and C_2 such that

$$\sup_{h\in\{h;I(h)\le a\}} \sup_{0\le t\le T} ||Au^h_t||_{-2} \le C_1, \tag{12}$$

and

$$||B(u^{h_1}_t) - B(u^{h_2}_t)|| \le C_2||u^{h_1}_t - u^{h_2}_t||_2, \quad h_1, h_2 \in \{h; I(h) \le a\}. \tag{13}$$

Regarded as an equation in H_0^{-2}, one has

$$\begin{aligned}\dot{u}_t^h &= \phi_1 + \int_0^t A u_s^h ds + \int_0^t B(u_s^h) ds + \sum_{k=1}^{\infty} \lambda_k \int_0^t \sigma_k(u_s^h) \dot{h}_s^k ds \\ &+ \int_0^t F(\dot{u}_s^h, u_s^h) ds. \end{aligned} \tag{14}$$

By (9), (A.1) and (A.4), we have

$$\begin{aligned}&\sum_{k=1}^{\infty} \lambda_k \int_s^t ||\sigma_k(u_l^h)|| |\dot{h}_l^k| dl \\ &\leq \int_s^t \left(\sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u_l^h)||^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (\dot{h}_l^k)^2 \right)^{\frac{1}{2}} dl \\ &\leq c \int_s^t (1 + ||u_l^h||_2) \left(\sum_{k=1}^{\infty} (\dot{h}_l^k)^2 \right)^{\frac{1}{2}} dl \leq c \int_s^t e(l, u^h) \left(\sum_{k=1}^{\infty} (\dot{h}_l^k)^2 \right)^{\frac{1}{2}} dl \\ &\leq C_{M,a} (t-s)^{\frac{1}{2}}, \end{aligned} \tag{15}$$

and

$$\int_s^t ||F(\dot{u}_l^h, u_l^h)|| dl \leq c \int_s^t (1 + e(l, u^h)) dl \leq C_{M,a}(t-s), \tag{16}$$

for some constant $C_{M,a}$. Combing this with (12) and (13), we see that there exists a constant C_3 so that

$$\sup_{h \in \{h; I_d(h) \leq a\}} ||u_t^h - u_s^h||_{-2} \leq C_3 |t-s|^{\frac{1}{2}}. \tag{17}$$

Introduce

$$e_L(t, v) = \frac{1}{2} \{ ||\dot{v}_t||^2 + \alpha ||\partial_x v_t||^2 + \gamma ||\partial_x^2 v_t||^2 \}.$$

Set $v_t^n = u_t^{h_n} - u_t^h$. Write

$$h_t^n = \sum_{k=1}^{\infty} \lambda_k h_t^{k,n} e_k.$$

Note that $\sup_{0 \leq t \leq T} e(t, v)$ dominates the norm of $(v_t, \dot{v}_t)$ in the space of $C([0,T]; H_0^2) \times C([0,T]; H)$. Applying the energy inequality in Lemma 3.1 [CM1], we have

$$\begin{aligned} e_L(t, v^n) &= \int_0^t (B(u_s^{h_n}) - B(u_s^h), \dot{v}_s^n) ds + \int_0^t (F(\dot{u}_s^{h_n}, u_s^{h_n}) - F(\dot{u}_s^h, u_s^h), \dot{v}_s^n) ds \\ &+ \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_s^n, \sigma_k(u_s^{h_n}) \dot{h}_s^{k,n} - \sigma_k(u_s^h) \dot{h}_s^k) ds. \end{aligned} \tag{18}$$

In virtue of (13),

$$
\begin{aligned}
|(B(u_s^{h_n}) - B(u_s^h), \dot{v}_s^n)| &\le ||B(u_s^{h_n}) - B(u_s^h)|| ||\dot{v}_s^n|| \\
&\le c e_L(s, v^n),
\end{aligned}
$$

for some constant c. By the Lipschitz condition and the Sobolev imbedding,

$$
\begin{aligned}
&|(F(\dot{u}_s^{h_n}, u_s^{h_n}) - F(\dot{u}_s^h, u_s^h), \dot{v}_s^n)| \\
&\le c(||\dot{v}_s^n||^2 + ||u_s^n) - u_s^h||_2^2) \le c e_L(s, v^n). \qquad (19)
\end{aligned}
$$

Let $s_m = \frac{[ms]}{m}$. Write

$$
\sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_s^n, \sigma_k(u_s^{h_n})\dot{h}_s^{k,n} - \sigma_k(u_s^h)\dot{h}_s^k) ds = C_t^1 + C_t^2 + C_t^3 + C_t^4, \qquad (20)
$$

where

$$
\begin{aligned}
C_t^1 &= \sum_{k=1}^{\infty} \lambda_k \int_0^t \left(\dot{v}_s^n, (\sigma_k(u_s^{h_n}) - \sigma_k(u_s^h))\dot{h}_s^{k,n} \right) ds \\
C_t^2 &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_s^n - \dot{v}_{s_m}^n, \sigma_k(u_s^h))(\dot{h}_s^{k,n} - \dot{h}_s^k) ds \\
C_t^3 &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_{s_m}^n, \sigma_k(u_s^h) - \sigma_k(u_{s_m}^h))(\dot{h}_s^{k,n} - \dot{h}_s^k) ds \\
C_t^4 &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_{s_m}^n, \sigma_k(u_{s_m}^h))(\dot{h}_s^{k,n} - \dot{h}_s^k) ds
\end{aligned}
$$

We now estimate each of the terms. Keeping (A.3) in mind, we have

$$
\begin{aligned}
|C_t^1| &\le \int_0^t ||\dot{v}_s^n|| \left(\sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u_s^{h_n}) - \sigma_k(u_s^h)||^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (\dot{h}_s^{k,n})^2 \right)^{\frac{1}{2}} ds \\
&\le c \int_0^t ||\dot{v}_s^n|| ||u_s^{h_n} - u_s^h||_2 \left(\sum_{k=1}^{\infty} (\dot{h}_s^{k,n})^2 \right)^{\frac{1}{2}} ds \\
&\le c \int_0^t e_L(s, v^n) \left(\sum_{k=1}^{\infty} (\dot{h}_s^{k,n})^2 \right)^{\frac{1}{2}} ds. \qquad (21)
\end{aligned}
$$

In view of (17) and (A.1),

$$\begin{aligned}|C_t^2| &\le \int_0^t ||\dot v_s^n - \dot v_{s_m}^n||_{-2}\left(\sum_{k=1}^\infty \lambda_k^2 ||\sigma_k(u_s^h)||_2^2\right)^{\frac12}\left(\sum_{k=1}^\infty (\dot h_s^{k,n} - \dot h_s^k)^2\right)^{\frac12} ds \\ &\le c\frac{1}{\sqrt m}\int_0^t (1+||u_s^h||_2)\left(\sum_{k=1}^\infty (\dot h_s^{k,n} - \dot h_s^k)^2\right)^{\frac12} ds \\ &\le c_{a,M}\frac{1}{\sqrt m}, \qquad (22)\end{aligned}$$

where M is defined as in (9). Since $||\dot u_s^h||$ is dominated by $\sqrt{e(s,u^h)}$, (9) implies that

$$\sup_{h\in\{h; I(h)\le a\}} ||u_t^h - u_s^h|| \le C|t-s|. \qquad (23)$$

This together with (A.3) implies that

$$\begin{aligned}|C_t^3| &\le \int_0^t ||\dot v_{s_m}^n||\left(\sum_{k=1}^\infty \lambda_k^2 ||\sigma_k(u_s^h) - \sigma_k(u_{s_m}^h)||^2\right)^{\frac12}\left(\sum_{k=1}^\infty (\dot h_s^{k,n} - \dot h_s^k)^2\right)^{\frac12} ds \\ &\le c\int_0^t (||u_s^h - u_{s_m}^h||)\left(\sum_{k=1}^\infty (\dot h_s^{k,n} - \dot h_s^k)^2\right)^{\frac12} ds \\ &\le c_{a,M}\frac{1}{m}. \qquad (24)\end{aligned}$$

Now

$$\begin{aligned}|C_t^4| \le{}& \Big|\sum_{k=1}^\infty \lambda_k \sum_{l=1}^{[mt]-1} (\dot v^n_{\frac{l}{m}}, \sigma_k(u^h_{\frac{l}{m}}))\Big((h^{k,n}_{\frac{(l+1)}{m}} - h^k_{\frac{(l+1)}{m}}) - (h^{k,n}_{\frac{l}{m}} - h^k_{\frac{l}{m}})\Big)\Big| \\ &+\Big|\sum_{k=1}^\infty \lambda_k (\dot v^n_{\frac{[mt]}{m}}, \sigma_k(u^h_{\frac{[mt]}{m}}))\Big((h_t^{k,n} - h_t^k) - (h^{k,n}_{\frac{[mt]}{m}} - h^k_{\frac{[mt]}{m}})\Big)\Big| \\ \le{}& \sum_{l=1}^{[mt]-1} ||\dot v^n_{\frac{l}{m}}||\left(\sum_{k=1}^\infty ||\sigma_k(u^h_{\frac{l}{m}})||^2\right)^{\frac12}\left(\sum_{k=1}^\infty \lambda_k^2\Big((h^{k,n}_{\frac{(l+1)}{m}} - h^k_{\frac{(l+1)}{m}}) - (h^{k,n}_{\frac{l}{m}} - h^k_{\frac{l}{m}})\Big)^2\right)^{\frac12} \\ &+||\dot v^n_{\frac{[mt]}{m}}||\left(\sum_{k=1}^\infty ||\sigma_k(u^h_{\frac{[mt]}{m}})||^2\right)^{\frac12}\left(\sum_{k=1}^\infty \lambda_k^2\Big((h_t^{k,n} - h_t^k) - (h^{k,n}_{\frac{[mt]}{m}} - h^k_{\frac{[mt]}{m}})\Big)^2\right)^{\frac12} \\ \le{}& c_{m,M}\sup_{0\le t\le T} ||h_t^n - h_t||_{l^2}, \qquad (25)\end{aligned}$$

where we have used (9) and the assumption (A.2). Putting together (18)–(25) we arrive at

$$\begin{aligned}e_L(t,v^n) &\le c\left(\frac{1}{\sqrt m} + c_{m,M}\sup_{0\le t\le T}||h_t^n - h_t||_{l^2}\right) + c\int_0^t e_L(s,v^n)ds \\ &+ c\int_0^t e_L(s,v^n)\left(\sum_{k=1}^\infty (\dot h_s^{k,n})^2\right)^{\frac12} ds. \qquad (26)\end{aligned}$$

Applying the Gronwall's inequality, we get

$$\begin{aligned}
&\sup_{0\le t\le T} e_L(t,v^n)\\
\le\ & c\left(\frac{1}{\sqrt{m}}+c_{m,M}\sup_{0\le t\le T}||h_t^n-h_t||_{l^2}\right)exp\left(cT+c\int_0^T\left(\sum_{k=1}^{\infty}(\dot{h}_s^{k,n})^2\right)^{\frac{1}{2}}ds\right)\\
\le\ & c_a\left(\frac{1}{\sqrt{m}}+c_{m,M}\sup_{0\le t\le T}||h_t^n-h_t||_{l^2}\right). \qquad (27)
\end{aligned}$$

Given $\varepsilon>0$. We first choose m such that $c_a\frac{1}{m}\le\frac{\varepsilon}{2}$. Then for such a m, there exists N so that for $n\ge N$,

$$c_a c_{m,M}\sup_{0\le t\le T}||h_t^n-h_t||_{l^2}\le\frac{\varepsilon}{2}. \qquad (28)$$

Therefore, for $n\ge N$,

$$\sup_{0\le t\le T} e_L(t,u^{h^n}-u^h)\le\varepsilon,$$

which finishes the proof of the Theorem.

Corollary 3 *The rate function $R(\cdot)$ defined in Section 2 is a good rate function, i.e., for every $a>0$, $\{g;\quad R(g)\le a\}$ is compact.*

Proof. Notice that

$$\{g;\quad R(g)\le a\}=\{(u^h,\dot{u}^h);\quad I(h)\le a\}$$

So the Corollary is a consequence of Proposition 2 and the fact that $\{h;I(h)\le a\}$ is compact in $C([0,T];l^2)$.

4 Large deviations

Consider

$$\begin{aligned}
d\dot{u}_t^{\varepsilon}\ =\ & \left(\alpha+\beta\int_0^1|\partial_y u_t^{\varepsilon}|^2dy\right)\partial_x^2u_t^{\varepsilon}dt-\gamma\partial_x^4u_t^{\varepsilon}dt+\varepsilon\sum_{k=1}^{\infty}\lambda_k\sigma_k(u_t^{\varepsilon})d\beta_t^k\\
+\ & \int_0^t F(\dot{u}_s^{\varepsilon},u_s^{\varepsilon})ds\\
& u_t^{\varepsilon}(0)=u_t^{\varepsilon}(1)=0,\quad \partial_x u_t^{\varepsilon}(0)=\partial_x u_t^{\varepsilon}(1)=0, \qquad (29)\\
& u_0^{\varepsilon}(x)=\phi_0(x)\in H_0^2,\quad \partial_t u_0^{\varepsilon}(x)=\phi_1(x)\in L^2.
\end{aligned}$$

In this section, we will establish the large deviation principle. We first prepare a number of preliminary results. Let $e(t,u^{\varepsilon})$ be defined as in (7) in section 3.

Lemma 4 *It holds that*

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 log P(\sup_{0\le t\le T} e(t,u^\varepsilon) > M) = -\infty. \tag{30}$$

Proof. By the energy equality (3.14) in [CM1], we have

$$\begin{aligned} e(t,u^\varepsilon) &= e(0,u^\varepsilon) + \varepsilon \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{u}_s^\varepsilon, \sigma_k(u_s^\varepsilon)) d\beta_s^k \\ &+ \frac{1}{2}\varepsilon^2 \int_0^t \sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u_s^\varepsilon)||^2 ds \\ &+ \int_0^t (\dot{u}_s^\varepsilon, F(\dot{u}_s^\varepsilon, u_s^\varepsilon)) ds. \end{aligned} \tag{31}$$

Recall that it is proved in [D]and [BY] that there exists a universal constant c such that, for any $p \ge 2$ and any continuous martingale (M_t) with $M_0 = 0$, one has

$$||M_t^*||_p \le c p^{\frac{1}{2}} || < M >_t^{\frac{1}{2}} ||_p, \tag{32}$$

where $M_t^* = \sup_{0\le s\le t}|M_s|$, and $||\cdot||_p$ stands for the $L^p(\Omega)$-norm. Using (32) and (A.1), we have for $p\ge 2$,

$$
\begin{aligned}
&\Big(E[\sup_{0\le t\le l} e(t,u^\varepsilon)^p]\Big)^{\frac{1}{p}}\\
\le\ & e(0,u^\varepsilon)+\frac{1}{2}\varepsilon^2\Big(E[|\int_0^l\sum_{k=1}^\infty \lambda_k^2||\sigma_k(u_s^\varepsilon)||^2ds|^p]\Big)^{\frac{1}{p}}\\
+\ & \varepsilon\Big(E[\sup_{0\le t\le l}|\sum_{k=1}^\infty\lambda_k\int_0^t(\dot u_s^\varepsilon,\sigma_k(u_s^\varepsilon))d\beta_s^k|^p]\Big)^{\frac{1}{p}}+\Big(E[|\int_0^l|(\dot u_s^\varepsilon,F(\dot u_s^\varepsilon,u_s^\varepsilon))|ds|^p]\Big)^{\frac{1}{p}}\\
\le\ & e(0,u^\varepsilon)+c\frac{1}{2}\varepsilon^2\Big(E[|\int_0^l(1+||u_s^\varepsilon||_2^2)ds|^p]\Big)^{\frac{1}{p}}+c\Big(E[|\int_0^l(1+||\dot u_s^\varepsilon||^2+||u_s^\varepsilon||_2^2)ds|^p]\Big)^{\frac{1}{p}}\\
+\ & \varepsilon cp^{\frac{1}{2}}\Big(E[|\int_0^l\sum_{k=1}^\infty\lambda_k^2(\dot u_s^\varepsilon,\sigma_k(u_s^\varepsilon))^2ds|^{\frac{p}{2}}]\Big)^{\frac{1}{p}}\\
\le\ & c_T+c\frac{1}{2}\varepsilon^2\int_0^l(1+(E[||u_s^\varepsilon||_2^{2p}])^{\frac{1}{p}})ds+\varepsilon cp^{\frac{1}{2}}\Big(E[|\int_0^l||\dot u_s^\varepsilon||^2(1+||u_s^\varepsilon||_2^2)ds|^{\frac{p}{2}}]\Big)^{\frac{1}{p}}\\
+\ & c\int_0^l\Big(E[e(s,u^\varepsilon)^p]\Big)^{\frac{1}{p}}ds\\
\le\ & c_T+(c\frac{1}{2}\varepsilon^2+c)\int_0^l(E[e(s,u^\varepsilon)^p])^{\frac{1}{p}}ds+\varepsilon cp^{\frac{1}{2}}\Big(\int_0^l(E[||\dot u_s^\varepsilon||^p(1+||u_s^\varepsilon||_2^2)^{\frac{p}{2}}])^{\frac{2}{p}}ds\Big)^{\frac{1}{2}}\\
\le\ & c_T+(c\frac{1}{2}\varepsilon^2+c)\int_0^l(E[e(s,u^\varepsilon)^p])^{\frac{1}{p}}ds+\varepsilon cp^{\frac{1}{2}}\Big(\int_0^l(E[\frac{1}{2}||\dot u_s^\varepsilon||^{2p}+\frac{1}{2}(1+||u_s^\varepsilon||_2^2)^p])^{\frac{2}{p}}ds\Big)^{\frac{1}{2}}\\
\le\ & c_T+(c\frac{1}{2}\varepsilon^2+c)\int_0^l(E[e(s,u^\varepsilon)^p])^{\frac{1}{p}}ds+\varepsilon cp^{\frac{1}{2}}\Big(\int_0^l(E[e(s,u^\varepsilon)^p])^{\frac{2}{p}}ds\Big)^{\frac{1}{2}},
\end{aligned}
\tag{33}
$$

where we have used the inequality

$$\Big(E[|\int_0^l f_s ds|^m]\Big)^{\frac{1}{m}}\le\int_0^l(E[|f_s|^m])^{\frac{1}{m}}ds$$

in several places for an appropriate m. Therefore,

$$\Big(E[\sup_{0\le t\le l}e(t,u^\varepsilon)^p]\Big)^{\frac{2}{p}}\le c_T+(\varepsilon^2cp+c\varepsilon^4+c^2)\int_0^l(E[e(s,u^\varepsilon)^p])^{\frac{2}{p}}ds.$$

By Gronwall's inequality, there exist constants c_1 and c_2 so that

$$E[\sup_{0\le t\le T}e(t,u^\varepsilon)^p]\le c_1^pe^{c_2\varepsilon^2p^2}. \tag{34}$$

This implies, by Chebyshev inequality,

$$P(\sup_{0\le t\le T} e(t,u^\varepsilon) > M) \le M^{-p} c_1^p e^{c_2 \varepsilon^2 p^2}.$$

Letting $p = \frac{1}{\varepsilon^2}$, we get

$$\varepsilon^2 log P(\sup_{0\le t\le T} e(t,u^\varepsilon) > M) \le log(\frac{1}{M}) + log(c_1) + c_2$$

which yields

$$\lim_{M\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 log P(\sup_{0\le t\le T} e(t,u^\varepsilon) > M) = -\infty. \tag{35}$$

We need a result of exponential integrability for a Hilbert space-valued martingale.

Proposition 5 *Let $f_k(s), k \ge 1$ be a sequence of adapted L^2-valued stochastic processes. Assume that there exists a constant K such that*

$$\sum_{k=1}^\infty ||f_k(s)||^2 \le K \qquad \textit{almost surely for all} \quad s \ge 0.$$

Define

$$M_t = \sum_{k=1}^\infty \int_0^t f_k(s) d\beta_s^k.$$

Then there exists a constant $\delta_0 > 0$ such that

$$\sup_{t\ne s} E\left[exp\left(\delta_0 \frac{||M_t - M_s||^2}{|t-s|}\right)\right] < \infty. \tag{36}$$

Proof. For simplicity, denote L^2 by H. Without loss of generality, we may assume $s = 0$. Otherwise consider $Y_u = M_{s+u} - M_s$. For $g \in C^2(H)$, by Ito's formula,

$$\begin{aligned} & exp(g(M_t)) = exp(g(M_0)) \\ + & \sum_{k=1}^\infty \int_0^t exp(g(M_s))(g'(M_s), f_k(s)) d\beta_s^k \\ + & \frac{1}{2}\sum_{k=1}^\infty \int_0^t exp(g(M_s))\Big((g'(M_s)\otimes g'(M_s) + g''(M_s))f_k(s), f_k(s)\Big) ds. \end{aligned} \tag{37}$$

Put

$$h_s = \frac{1}{2}\sum_{k=1}^\infty \Big((g'(M_s)\otimes g'(M_s) + g''(M_s))f_k(s), f_k(s)\Big). \tag{38}$$

By integration by parts formula and (37), it is easy to verify that

$$N_t^g = exp(g(M_t) - g(0) - \int_0^t h_s ds)$$

is a non-negative local martingale. Now, for $\lambda > 0$ (which will be specified later), let $g_\lambda(x) = (1+\lambda|x|_H^2)^{\frac{1}{2}}$. Then

$$g'_\lambda(x) = \lambda(1+\lambda|x|_H^2)^{-\frac{1}{2}}x,$$

$$g''_\lambda(x) = -\lambda^2(1+\lambda|x|_H^2)^{-\frac{3}{2}}x\otimes x + \lambda(1+\lambda|x|_H^2)^{-\frac{1}{2}}I_H,$$

where I_H stands for the identity operator. It is easy to see that

$$\sup_x |g'_\lambda(x)| \le \lambda^{\frac{1}{2}}, \quad \sup_x ||g''_\lambda(x)|| \le \lambda^{\frac{1}{2}},$$

where $||\cdot||$ stands for the operator norm. Define h_s^λ as in (38) replacing g by g_λ. Then,

$$|h_s^\lambda| \le \frac{1}{2}\lambda\sum_{k=1}^\infty ||f_k(s)||^2 \le \frac{1}{2}\lambda K. \tag{39}$$

For any $r > 0$ and every $\lambda > 0$, we have

$$\begin{aligned}
& P\Big(\frac{||M_t||}{\sqrt{t}} > r\Big) = P\Big(g_\lambda(M_t) \ge (1+\lambda tr^2)^{\frac{1}{2}}\Big) \\
= \ & P\Big(g_\lambda(M_t) - g_\lambda(0) - \int_0^t h_s^\lambda ds + g_\lambda(0) + \int_0^t h_s^\lambda ds \ge (1+\lambda tr^2)^{\frac{1}{2}}\Big) \\
\le \ & P\Big(g_\lambda(M_t) - g_\lambda(0) - \int_0^t h_s^\lambda ds \ge (1+\lambda tr^2)^{\frac{1}{2}} - g_\lambda(0) - \frac{1}{2}\lambda Kt\Big) \\
\le \ & E[N_t^{g_\lambda}]exp\Big(- (1+\lambda tr^2)^{\frac{1}{2}} + g_\lambda(0) + \frac{1}{2}\lambda Kt\Big) \\
\le \ & exp\Big(- (1+\lambda tr^2)^{\frac{1}{2}} + g_\lambda(0) + \frac{1}{2}\lambda Kt\Big),
\end{aligned} \tag{40}$$

where the fact $E[N_t^{g_\lambda}] \le 1$ has been used. Choosing $\lambda = t^{-1}\delta r^2$, we get that

$$P\Big(\frac{||M_t||}{\sqrt{t}} > r\Big) \le exp\Big(- \delta^{\frac{1}{2}}r^2 + 1 + \frac{1}{2}K\delta r^2\Big)$$

$$= exp\Big(- \delta(\frac{1}{\delta^{\frac{1}{2}}} - \frac{1}{2}K)r^2\Big).$$

Take $\delta > 0$ small enough so that $\delta^* := \delta(\frac{1}{\delta^{\frac{1}{2}}} - \frac{1}{2}K) > 0$. We arrive at

$$P\Big(\frac{||M_t||}{\sqrt{t}} > r\Big) \le exp(-\delta^* r^2 + 1), \tag{41}$$

where δ^* is independent of t. We can now easily deduce (36). Fix $\delta_0 < \delta^*$ and let $\xi = \frac{||M_t||}{\sqrt{t}}$. We have

$$E\left[exp\left(\delta_0 \frac{||M_t||^2}{|t|}\right)\right] = E[exp(\delta_0 \xi^2)] = -\int_0^\infty exp(\delta_0 r^2) dP(\xi > r)$$

$$\leq 1 + 2\delta_0 \int_0^\infty exp(\delta_0 r^2) P(\xi > r) r dr$$

$$\leq 1 + 2\delta_0 \int_0^\infty exp(\delta_0 r^2) exp(-\delta^* r^2 + 1) r dr < +\infty \tag{42}$$

which completes the proof of the Proposition.

Lemma 6 *For every $\delta_1 > 0$, $M > 0$,*

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 log P\left(\sup_{|s-t|\leq \frac{1}{m}, s,t\leq T} ||\dot{u}_s^\varepsilon - \dot{u}_t^\varepsilon||_{-2} \geq \delta_1, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\right) = -\infty \tag{43}$$

Proof. In view of (10), (11) and (A.4), $||Au_s^\varepsilon||_{-2}$, $||B(u_s^\varepsilon)||$ and $||F(\dot{u}_s^\varepsilon, u_s^\varepsilon)||$ are uniformly bounded by a constant (depending only on M) on the set $\{\omega, t \leq T; \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\}$. Regarded as an equation in H_0^{-2},

$$\begin{aligned} \dot{u}_t^\varepsilon &= \phi_1 + \int_0^t Au_s^\varepsilon ds + \int_0^t B(u_s^\varepsilon) ds + \varepsilon \sum_{k=1}^\infty \lambda_k \int_0^t \sigma_k(u_s^\varepsilon) d\beta_s^k \\ &+ \int_0^t F(\dot{u}_s^\varepsilon, u_s^\varepsilon) ds. \end{aligned} \tag{44}$$

Therefore, on $\{\omega; \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\}$,

$$\sup_{|s-t|\leq \frac{1}{m}, s,t\leq T} ||\dot{u}_t^\varepsilon - \dot{u}_s^\varepsilon||_{-2} \leq c_M \frac{1}{m} + \varepsilon \sup_{|s-t|\leq \frac{1}{m}, s,t\leq T} ||N_t - N_s||, \tag{45}$$

where

$$N_t = \sum_{k=1}^\infty \lambda_k \int_0^t \sigma_k(u_s^\varepsilon) d\beta_s^k.$$

Thus for sufficiently big m,

$$P\left(\sup_{|s-t|\leq \frac{1}{m}, s,t\leq T} ||\dot{u}_s^\varepsilon - \dot{u}_t^\varepsilon||_{-2} \geq \delta_1, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\right)$$

$$\leq P\left(\varepsilon \sup_{|s-t|\leq \frac{1}{m}, s,t\leq T} ||N_t - N_s|| \geq \frac{1}{2}\delta_1, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\right).$$

So it remains to show

$$\lim_{m\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2 log P\left(\varepsilon \sup_{|s-t|\le\frac{1}{m},s,t\le T}||N_t-N_s||\ge\frac{1}{2}\delta_1,\ \sup_{0\le t\le T} e(t,u^\varepsilon)\le M\right)=-\infty \quad (46)$$

Notice that by (A.1) there exists a constant K_M such that

$$\{\omega;\ \sup_{0\le t\le T} e(t,u^\varepsilon)\le M\}\subset\{\omega;\ \sup_{0\le t\le T}\sum_{k=1}^{\infty}\lambda_k^2||\sigma_k(u_t^\varepsilon)||^2\le K_M\} \quad (47)$$

Define

$$\tau=\inf\{s\ge 0;\sum_{k=1}^{\infty}\lambda_k^2||\sigma_k(u_s^\varepsilon)||^2>K_M\}$$

It follows from (47) that

$$\{\omega;\ \sup_{0\le t\le T} e(t,u^\varepsilon)\le M\}\subset\{\tau\ge T\}$$

Therefore,

$$P\left(\varepsilon \sup_{|s-t|\le\frac{1}{m},s,t\le T}||N_t-N_s||\ge\frac{1}{2}\delta_1,\ \sup_{0\le t\le T} e(t,u^\varepsilon)\le M\right)$$
$$\le P\left(\varepsilon \sup_{|s-t|\le\frac{1}{m},s,t\le T}||N_{t\wedge\tau}-N_{s\wedge\tau}||\ge\frac{1}{2}\delta_1\right).$$

So, to prove (46), we may drop the event $\{\sup_{0\le t\le T} e(t,u^\varepsilon)\le M\}$ and assume in the rest of the proof that

$$\sup_{0\le t\le T}\sum_{k=1}^{\infty}\lambda_k^2||\sigma_k(u_t^\varepsilon)||^2\le K_M. \quad (48)$$

Applying Proposition 5, there exists a constant $\lambda_M>0$ such that

$$\sup_{t\ne s,s,t\le T} E\left[exp\left(\lambda_M\frac{||N_t-N_s||^2}{|t-s|}\right)\right]<\infty. \quad (49)$$

Introduce

$$D=\int_0^T\int_0^T exp\left(\lambda_M\frac{||N_t-N_s||^2}{|t-s|}\right)dsdt.$$

Then we have $E[D]<\infty$. Now by Garsia lemma (see [W]) we have

$$||N_t-N_s||\le\frac{8}{\sqrt{\lambda_M}}\int_0^{|t-s|}\left(log\frac{D}{u^2}\right)^{\frac{1}{2}}dp(u), \quad (50)$$

where $p(u) = u^{\frac{1}{2}}$. For any $\delta < \frac{1}{2}$, say $\delta = \frac{1}{4}$, (50) implies that there exists a constant c such that

$$||N_t - N_s|| \leq \frac{8c}{\sqrt{\lambda_M}}\left(\sqrt{logD}+1\right)|t-s|^{\frac{1}{4}}.$$

Consequently,

$$\sup_{|s-t|\leq\frac{1}{m},s,t\leq T}||N_t - N_s|| \leq \frac{8c}{\sqrt{\lambda_M}}\left(\sqrt{logD}+1\right)(\frac{1}{m})^{\frac{1}{4}}.$$

Therefore,

$$\begin{aligned}
&P\left(\varepsilon \sup_{|s-t|\leq\frac{1}{m},s,t\leq T}||N_t - N_s|| \geq \frac{1}{2}\delta_1\right)\\
&\leq P\left(\sqrt{logD} > \frac{1}{2}\frac{\delta_1}{\varepsilon}\frac{\sqrt{\lambda_M}}{8c}\frac{1}{(\frac{1}{m})^{\frac{1}{4}}} - 1\right)\\
&\leq P\left(D > exp\left(\frac{1}{2}\frac{\delta_1}{\varepsilon}\frac{\sqrt{\lambda_M}}{8c}\frac{1}{(\frac{1}{m})^{\frac{1}{4}}} - 1\right)^2\right)\\
&\leq E[D]exp\left\{-(\frac{1}{2}\frac{\delta_1}{\varepsilon}\frac{\sqrt{\lambda_M}}{8c}\frac{1}{(\frac{1}{m})^{\frac{1}{4}}} - 1)^2\right\}.
\end{aligned}$$

This yields

$$\lim_{m\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2 P\left(\varepsilon \sup_{|s-t|\leq\frac{1}{m},s,t\leq T}||N_t - N_s|| \geq \frac{1}{2}\delta_1\right) = -\infty,$$

which completes the proof.

Lemma 7 *Let $f_s, s \geq 0$ be a H_0^{-2}-valued adapted stochastic process such that* $\sup_{0\leq s\leq T}||f_s||_{-2} \leq \delta_1$. *Set*

$$M_t = \sum_{k=1}^{\infty}\lambda_k\int_0^t < f_s, \sigma_k(u_s^\varepsilon) > d\beta_s^k. \tag{51}$$

Then there exist positive constants $c_1 > 0$, $c_2 > 0$ and c_M such that for $\eta_1 > 0$,

$$P\left(\sup_{0\leq t\leq T}|M_t| > \eta_1, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\right) \leq c_1 exp\left(-\frac{c_2\eta_1^2}{c_M T\delta_1^2}\right). \tag{52}$$

Proof. Notice that $M_t, t \geq 0$ is a martingale whose bracket satisfies

$$\begin{aligned} < M >_t &= \sum_{k=1}^{\infty} \lambda_k^2 \int_0^t < f_s, \sigma_k(u_s^\varepsilon) >^2 ds \\ &\leq \int_0^t ||f_s||_{-2}^2 \sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u_s^\varepsilon)||_2^2 ds \\ &\leq c \int_0^t ||f_s||_{-2}^2 (1 + ||u_s^\varepsilon||_2^2) ds \\ &\leq c\delta_1^2 \int_0^t e(s, u^\varepsilon) ds. \end{aligned} \tag{53}$$

Thus, on $\{\omega, \sup_{0\leq t\leq T} e(t, u^\varepsilon) \leq M\}$, $< M >_t \leq c_M \delta_1^2 =: b$. By the martingale representation theorem, there exists a standard Brownian motion $B_s, s \geq 0$ such that $M_t = B_{<M>_t}$. We have

$$\{\omega, \sup_{0\leq t\leq T} |M_t| > \eta_1, \sup_{0\leq t\leq T} e(t, u^\varepsilon) \leq M\} = \{\omega, \sup_{0\leq t\leq T} |B_{<M>_t}| > \eta_1, \sup_{0\leq t\leq T} e(t, u^\varepsilon) \leq M\}$$

$$\subset \{\omega; \sup_{0\leq u\leq b} |B_u| > \eta_1, \sup_{0\leq t\leq T} e(t, u^\varepsilon) \leq M\} = \{\omega; \sqrt{b} \sup_{0\leq u\leq 1} |\tilde{B}_u| > \eta_1, \sup_{0\leq t\leq T} e(t, u^\varepsilon) \leq M\}$$

where $\tilde{B}$ is another Brownian motion by the scaling invariance property. It is well known that there exists a constant $c_2 > 0$ such that $c_1 := E[exp(c_2 \sup_{0\leq u\leq 1} |\tilde{B}_u|^2)] < \infty$. Thus,

$$P(\sup_{0\leq t\leq T} |M_t| > \eta_1, \sup_{0\leq t\leq T} e(t, u^\varepsilon) \leq M) \leq P(\sup_{0\leq u\leq 1} |\tilde{B}_u| > \frac{\eta_1}{\sqrt{b}})$$

$$\leq E[exp(c_2 \sup_{0\leq u\leq 1} |\tilde{B}_u|^2)] exp(-\frac{c_2\eta_1^2}{b})$$

$$= c_1 exp\Big(-\frac{c_2\eta_1^2}{c_M\delta_1^2}\Big),$$

proving the Lemma.

Recall

$$e_L(t, v) = \frac{1}{2}\{||\dot{v}_t||^2 + \alpha||\partial_x v_t||^2 + \gamma||\partial_x^2 v_t||^2\}. \tag{54}$$

Theorem 8 *For every $\eta > 0$, $R > 0$, $h \in \mathcal{H}$, there exists $\delta > 0$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon^2 log P\Big(\sup_{0\leq t\leq T} e_L(t, u^\varepsilon - u^h) \geq \eta, \sup_{0\leq t\leq T} ||\varepsilon W - h||_\infty < \delta\Big) \leq -R \tag{55}$$

Proof. As an equation in H_0^{-2}, we have

$$\begin{aligned}\dot{u}_t^\varepsilon - \dot{u}_t^h &= \int_0^t A(u_s^\varepsilon - u_s^h)ds + \int_0^t (B(u_s^\varepsilon) - B(u_s^h))ds \\ &+ \varepsilon \sum_{k=1}^\infty \lambda_k \int_0^t \sigma_k(u_s^\varepsilon)d\beta_s^k - \sum_{k=1}^\infty \lambda_k \int_0^t \sigma_k(u_s^h)\dot{h}_s^k ds \\ &+ \int_0^t (F(\dot{u}_s^\varepsilon, u_s^\varepsilon) - F(\dot{u}_s^h, u_s^h))ds. \end{aligned} \quad (56)$$

For simplicity, denote $v_t := u_t^\varepsilon - u_t^h$. By the energy inequality (3.10) in [CM1], we have

$$\begin{aligned} e_L(t,v) &= \int_0^t (B(u_s^\varepsilon) - B(u_s^h), \dot{v}_s)ds + \frac{1}{2}\varepsilon^2 \int_0^t \sum_{k=1}^\infty \lambda_k^2 ||\sigma_k(u_s^\varepsilon)||^2 ds \\ &+ \sum_{k=1}^\infty \lambda_k \int_0^t (\dot{v}_s, \varepsilon\sigma_k(u_s^\varepsilon))d\beta_s^k - \sum_{k=1}^\infty \lambda_k \int_0^t (\dot{v}_s, \sigma_k(u_s^h))\dot{h}_s^k ds \\ &+ \int_0^t (F(\dot{u}_s^\varepsilon, u_s^\varepsilon) - F(\dot{u}_s^h, u_s^h), \dot{v}_s)ds \end{aligned} \quad (57)$$

Let $M > \sup_{0\le t\le T} e(t, u^h)$. In view of (11) and (A.1), on the event $\{\omega, \sup_{0\le t\le T} e(t,u^\varepsilon) \le M\}$,

$$||B(u_s^\varepsilon) - B(u_s^h)|| \le C_M \Big(||u_s^\varepsilon - u_s^h||_2 \Big)$$

$$\sum_{k=1}^\infty \lambda_k^2 ||\sigma_k(u_s^\varepsilon)||^2 \le C_M.$$

By the Lipschitz condition, (A.5) and the Sobolev imbedding,

$$|(F(\dot{u}_s^\varepsilon, u_s^\varepsilon) - F(\dot{u}_s^h, u_s^h), \dot{v}_s)| \le c e_L(s,v).$$

So it follows from (57) that on $\{\omega, \sup_{0\le t\le T} e(t,u^\varepsilon) \le M\}$,

$$\begin{aligned} e_L(t,v) &\le C_M \int_0^t e_L(s,v)ds + \frac{1}{2}\varepsilon^2 c_M T \\ &+ |M_t^\varepsilon|, \end{aligned} \quad (58)$$

where

$$M_t^\varepsilon = \sum_{k=1}^\infty \lambda_k \int_0^t (\dot{v}_s, \varepsilon\sigma_k(u_s^\varepsilon))d\beta_s^k - \sum_{k=1}^\infty \lambda_k \int_0^t (\dot{v}_s, \sigma_k(u_s^h))\dot{h}_s^k ds. \quad (59)$$

Let $s_m = \frac{[ms]}{m}$ and write

$$M_t^\varepsilon = N_t^{1,m} + N_t^{2,m} + N_t^{3,m} + N_t^{4,m},$$

where

$$
\begin{aligned}
N_t^{1,m} &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_s, \sigma_k(u_s^\varepsilon) - \sigma_k(u_s^h)) \dot{h}_s^k ds \\
N_t^{2,m} &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_s - \dot{v}_{s_m}, \sigma_k(u_s^\varepsilon))(\varepsilon d\beta_s^k - \dot{h}_s^k ds) \\
N_t^{3,m} &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_{s_m}, \sigma_k(u_s^\varepsilon) - \sigma_k(u_{s_m}^\varepsilon))(\varepsilon d\beta_s^k - \dot{h}_s^k ds) \\
N_t^{4,m} &= \sum_{k=1}^{\infty} \lambda_k \int_0^t (\dot{v}_{s_m}, \sigma_k(u_{s_m}^\varepsilon))(\varepsilon d\beta_s^k - \dot{h}_s^k ds).
\end{aligned}
$$

Now,

$$
\begin{aligned}
|N_t^{1,m}| &\leq \int_0^t ||\dot{v}_s|| \left(\sum_{k=1}^{\infty} \lambda_k^2 ||\sigma_k(u_s^\varepsilon) - \sigma_k(u_s^h)||^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (\dot{h}_s^k)^2 \right)^{\frac{1}{2}} ds \\
&\leq c \int_0^t ||\dot{v}_s|| (||u_s^\varepsilon - u_s^h||_2) \left(\sum_{k=1}^{\infty} (\dot{h}_s^k)^2 \right)^{\frac{1}{2}} ds \\
&\leq c \int_0^t e_L(s,v) \left(\sum_{k=1}^{\infty} (\dot{h}_s^k)^2 \right)^{\frac{1}{2}} ds. \qquad (60)
\end{aligned}
$$

So we deduce from (58) that on $\{\omega, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\}$,

$$
\begin{aligned}
e_L(t,v) &\leq C_M \int_0^t e_L(s,v)(1 + (\sum_{k=1}^{\infty} (\dot{h}_s^k)^2 \Big)^{\frac{1}{2}}) ds + \frac{1}{2}\varepsilon^2 c_M T \\
&+ \sum_{i=2}^{4} |N_t^{i,m}|. \qquad (61)
\end{aligned}
$$

By Gronwall's inequality we obtain that on $\{\omega, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\}$,

$$
e_L(t,v) \leq e^{C_M T} \left(\frac{1}{2}\varepsilon^2 c_M T + \sum_{i=2}^{4} \sup_{0\leq t\leq T} |N_t^{i,m}| \right)
$$

which implies that there exists a constant $\varepsilon_1 > 0$ such that for $\varepsilon \leq \varepsilon_1$,

$$
P\left(\sup_{0\leq t\leq T} e_L(t, u^\varepsilon - u^h) \geq \eta, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M, ||\varepsilon W - h||_\infty < \delta \right)
$$

$$
\leq P\left(\sum_{i=2}^{4} \sup_{0\leq t\leq T} |N_t^{i,m}| > \frac{1}{2} e^{-C_M T}\eta, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M, ||\varepsilon W - h||_\infty < \delta \right) \qquad (62)
$$

Now,

$$P\bigg(\sum_{i=2}^{4}\sup_{0\le t\le T}|N_t^{i,m}>\frac{1}{2}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M,\ \sup_{0\le t\le T}||\varepsilon W-h||_\infty<\delta\bigg)$$

$$\le\ P\bigg(\sup_{0\le t\le T}|N_t^{4,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M,\ \sup_{0\le t\le T}||\varepsilon W-h||_\infty<\delta\bigg) \quad (63)$$

$$+\ P\bigg(\sup_{0\le t\le T}|N_t^{2,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\bigg)$$

$$+\ P\bigg(\sup_{0\le t\le T}|N_t^{3,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\bigg). \quad (64)$$

Furthermore, for $\delta_1>0$,

$$P\bigg(\sup_{0\le t\le T}|N_t^{2,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\bigg)$$

$$\le\ P\bigg(\sup_{0\le t\le T}|N_t^{2,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M,\ \sup_{|t-s|\le\frac{1}{m}}||\dot v_t-\dot v_s||_{-2}\le\delta_1\bigg)$$

$$+\ P\bigg(\sup_{|t-s|\le\frac{1}{m}}||\dot v_t-\dot v_s||_{-2}>\delta_1,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\bigg). \quad (65)$$

By the Girsanov theorem, we know that $W_t-\frac{1}{\varepsilon}h_t, t\ge 0$ is a Wiener process under the probability measure P^* given by

$$\frac{dP^*}{dP}\bigg|_{\mathcal{F}_t}=exp\bigg(\frac{1}{\varepsilon}\int_0^t<\dot h_s,dW_s>-\frac{1}{2}\frac{1}{\varepsilon^2}\sum_{k=1}^\infty\lambda_k^4\int_0^t(\dot h_s^k)^2ds\bigg).$$

Through a change of measure and applying Lemma 7, we can show that there exist constants c_1, c_2 such that

$$P\bigg(\sup_{0\le t\le T}|N_t^{2,m}|>\frac{1}{4}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M,\ \sup_{|t-s|\le\frac{1}{m}}||\dot v_t-\dot v_s||_{-2}\le\delta_1\bigg)$$

$$\le c_1exp\bigg(-\frac{c_2\eta^2}{cMT\varepsilon^2\delta_1^2}\bigg). \quad (66)$$

Notice that on $\{\omega;\sup_{0\le t\le T}e(t,u^\varepsilon)\le M\}$,

$$\begin{aligned}&\sum_{k=1}^\infty\lambda_k^2(\dot v_{s_m},\sigma_k(u_s^\varepsilon)-\sigma_k(u_{s_m}^\varepsilon))^2\\ \le\ &||\dot v_{s_m}||^2\sum_{k=1}^\infty\lambda_k^2||\sigma_k(u_s^\varepsilon)-\sigma_k(u_{s_m}^\varepsilon)||^2\\ \le\ &c||\dot v_{s_m}||^2(||u_s^\varepsilon-u_{s_m}^\varepsilon||^2)\\ \le\ &||\dot v_{s_m}||^2(||\int_{s_m}^s\dot u_l^\varepsilon dl||^2)\le C_M(\frac{1}{m})^2. \quad (67)\end{aligned}$$

Using this and following the same proof of Lemma 7, we can show that

$$P\left(\sup_{0\le t\le T}|N_t^{3,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\right)$$

$$\le c_1 exp\left(-\frac{c_2\eta^2}{cMT\varepsilon^2(\frac{1}{m})^2}\right). \tag{68}$$

Now given $R>0$, $\eta>0$. According to Lemma 4, we can choose M large enough and $\varepsilon_2>0$ such that for $\varepsilon\le\varepsilon_2$,

$$P\left(\sup_{0\le t\le T}e(t,u^\varepsilon)>M\right)\le exp(-\frac{R}{\varepsilon^2}). \tag{69}$$

Next, we choose δ_1 , according to (66), so that for $\varepsilon\le\varepsilon_3$

$$P\left(\sup_{0\le t\le T}|N_t^{2,m}|>\frac{1}{4}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M,\ \sup_{|t-s|\le\frac{1}{m}}||\dot{v}_t-\dot{v}_s||_{-2}\le\delta_1\right)$$

$$\le exp(-\frac{R}{\varepsilon^2}). \tag{70}$$

where ε_3 is a positive number . For such a $\delta_1>0$, by Lemma 6 and (68) there exist an integer m and $\varepsilon_4>0$ so that for $\varepsilon\le\varepsilon_4$,

$$P\left(\sup_{|t-s|\le\frac{1}{m}}||\dot{v}_t-\dot{v}_s||_{-2}>\delta_1,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\right)\le exp(-\frac{R}{\varepsilon^2}), \tag{71}$$

$$P\left(\sup_{0\le t\le T}|N_t^{3,m}|>\frac{1}{6}e^{-C_MT}\eta,\ \sup_{0\le t\le T}e(t,u^\varepsilon)\le M\right)\le exp(-\frac{R}{\varepsilon^2}). \tag{72}$$

When such a m is fixed,

$$\begin{aligned}N_t^{4,m} &= \sum_{k=1}^{\infty}\lambda_k\sum_{l=0}^{[mt]-1}(\dot{v}_{\frac{l}{m}},\sigma_k(u^\varepsilon_{\frac{l}{m}}))[(\varepsilon\beta^k_{\frac{l+1}{m}}-h^k_{\frac{l+1}{m}})-(\varepsilon\beta^k_{\frac{l}{m}}-h^k_{\frac{l}{m}})] \\ &+ \sum_{k=1}^{\infty}\lambda_k(\dot{v}_{\frac{[mt]}{m}},\sigma_k(u^\varepsilon_{\frac{[mt]}{m}}))[(\varepsilon\beta^k_t-h^k_t)-(\varepsilon\beta^k_{\frac{[mt]}{m}}-h^k_{\frac{[mt]}{m}})].\end{aligned} \tag{73}$$

Therefore,

$$\begin{aligned}
&|N_t^{4,m}| \\
\leq\ & \sum_{l=0}^{[mt]-1} ||\dot{v}_{\frac{l}{m}}|| \Big(\sum_{k=1}^{\infty} ||\sigma_k(u^\varepsilon_{\frac{l}{m}})||^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} \lambda_k^2 [(\varepsilon \beta^k_{\frac{l+1}{m}} - h^k_{\frac{l+1}{m}}) - (\varepsilon \beta^k_{\frac{l}{m}} - h^k_{\frac{l}{m}})]^2 \Big)^{\frac{1}{2}} \\
+\ & ||\dot{v}_{\frac{[mt]}{m}}|| \Big(\sum_{k=1}^{\infty} ||\sigma_k(u^\varepsilon_{\frac{[mt]}{m}})||^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} \lambda_k^2 [(\varepsilon \beta^k_t - h^k_t) - (\varepsilon \beta^k_{\frac{[mt]}{m}} - h^k_{\frac{[mt]}{m}})]^2 \Big)^{\frac{1}{2}} \\
\leq\ & \sum_{l=0}^{[mt]-1} ||\dot{v}_{\frac{l}{m}}|| \Big(\sum_{k=1}^{\infty} ||\sigma_k(u^\varepsilon_{\frac{l}{m}})||^2 \Big)^{\frac{1}{2}} ||\varepsilon W - h||_\infty \\
+\ & ||\dot{v}_{\frac{[mt]}{m}}|| \Big(\sum_{k=1}^{\infty} ||\sigma_k(u^\varepsilon_{\frac{[mt]}{m}})||^2 \Big)^{\frac{1}{2}} ||\varepsilon W - h||_\infty. \qquad (74)
\end{aligned}$$

By the assumption (A.2), we know that on the event $\{\omega, \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M\}$,

$$\sup_{0\leq s\leq T} \sum_{k=1}^{\infty} ||\sigma_k(u^\varepsilon_s)||^2 \quad \text{and} \quad \sup_{0\leq s\leq T} ||\dot{v}_s||$$

are uniformly bounded by some constant c_M. Thus, we see from (74) that there exists $\delta > 0$ such that the event

$$\Big\{ \sup_{0\leq t\leq T} |N_t^{4,m}| > \frac{1}{6} e^{-C_M T}\eta,\ \sup_{0\leq t\leq T} e(t,u^\varepsilon) \leq M,\ \sup_{0\leq t\leq T} ||\varepsilon W - h||_\infty < \delta \Big\}$$

is empty. Combing this with (62), (69), (70), (71) and (72) we obtain that for sufficiently small ε,

$$P\Big(\sup_{0\leq t\leq T} e_L(t, u^\varepsilon - u^h) \geq \eta,\ \sup_{0\leq t\leq T} ||\varepsilon W - h|| < \delta \Big) \leq 5exp(-\frac{R}{\varepsilon^2}).$$

This completes the proof.

After we established the key results: Proposition 2 and Theorem 8, there exists now a well-known method (see [A],[DP]) to deduce the large deviation principle. For completeness, we include a proof.

Theorem 9 $\{\mu_\varepsilon, \varepsilon > 0\}$ *satisfies a large deviation principle with rate function* $R(f)$.

Proof. Denote by $d(\cdot,\cdot)$ the distance in the metric space $C([0,T]; H_0^2) \times C([0,T]; L^2)$. First, fix any closed subset $C \subset C([0,T]; H_0^2) \times C([0,T]; L^2)$ and choose $a < \inf_{g\in C} R(g)$. Define

$$\begin{aligned}
K_a &= \{g \in C([0,T]; H_0^2) \times C([0,T]; L^2) \mid R(g) \leq a\} \\
C_a &= \{f \in \mathcal{H} \mid I(f) \leq a\},
\end{aligned}$$

then

$$K_a = \{(u^f, \dot{u}^f); f \in C_a\}, \ K_a \cap C = \emptyset,$$

where $(u^f, \dot{u}^f)$ denotes the solution of the skeleton equation in Section 2 with h replaced by f. For any $g \in K_a$, there exists an open neighborhood of g, V_g such that

$$V_g \cap C = \emptyset.$$

One can choose $\rho_g > 0$, such that

$$G_g = \{h \in C([0,T]; H_0^2) \times C([0,T]; L^2), d(h,g) \le \rho_g\} \subseteq V_g.$$

For any $f_g \in C_a$ such that $g = (u^{f_g}, \dot{u}^{f_g})$ and $R > a$, by Theorem 8 one can find two constants $\varepsilon_g > 0, \alpha_g > 0$ such that for any $\varepsilon < \varepsilon_g$

$$P(\varepsilon W \in F_g, (u^\varepsilon, \dot{u}^\varepsilon) \in G_g^C) \le \exp(-\frac{R}{\varepsilon^2}).$$

Where $F_g = \{f \in C([0,T], l^2); ||f - f_g||_\infty < \alpha_g\}$. Therefore,

$$P(\varepsilon W \in F_g, (u^\varepsilon, \dot{u}^\varepsilon) \in V_g^C) \le P(\varepsilon W \in F_g, (u^\varepsilon, \dot{u}^\varepsilon) \in G_g^C) \le \exp(-\frac{R}{\varepsilon^2}). \tag{75}$$

Since $(F_g)_{g \in K_a}$ forms a cover for the compact set C_a of $C([0,T], l^2)$, there exist $g_1, \dots, g_l \in K_a$ such that

$$F = \cup_{i=1}^l F_{g_i} \supset C_a.$$

Therefore, we have

$$\begin{aligned} P(\varepsilon W \in F, (u^\varepsilon, \dot{u}^\varepsilon) \in C) &= P(\cup_{i=1}^l [\{\varepsilon W \in F_{g_i}\} \cup \{(u^\varepsilon, \dot{u}^\varepsilon) \in C\}]) \\ &\le P(\cup_{i=1}^l [\{\varepsilon W \in F_{g_i}\} \cup \{(u^\varepsilon, \dot{u}^\varepsilon) \in V_{g_i}^C\}]) \\ &\le \sum_{i=1}^l P(\varepsilon W \in F_{g_i}, (u^\varepsilon, \dot{u}^\varepsilon) \in V_{g_i}^C) \\ &\le l \exp(-\frac{R}{\varepsilon^2}). \end{aligned}$$

It follows that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_\varepsilon(C) &\le \limsup_{\varepsilon \to 0} \varepsilon^2 \log[P(\varepsilon W \in F^C) + P(\varepsilon W \in F, (u^\varepsilon, \dot{u}^\varepsilon) \in C)] \\ &\le (-\inf_{f \in F^c} I(f)) \vee (-R) \\ &= -a, \end{aligned}$$

where we have used the fact that εW satisfies a large deviation principle with rate function $I(\cdot)$. Let $a \to \inf_{g \in C} R(g)$ to get

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu_\varepsilon(C) \le -\inf_{g \in C} R(g).$$

Let $G \subset C([0,T];H_0^2) \times C([0,T];L^2)$ be an open set, take $g \in G$ with $R(g) < \infty$. Then there exists $f \in \mathcal{H}$ such that

$$g = (u^f, \dot{u}^f), \quad R(g) = I(f)$$

Choose $\rho > 0$, such that

$$\{h \in C([0,T];H_0^2) \times C([0,T];L^2) | d(h,g) \leq \rho\} \subset G.$$

For any $R > R(g)$, by Theorem 8, $\exists \alpha > 0, \varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$P(d((u^\varepsilon, \dot{u}^\varepsilon), (u^f, \dot{u}^f)) > \rho, \|\varepsilon W - f\|_\infty < \alpha) \leq \exp(-\frac{R}{\varepsilon^2}).$$

Therefore,

$$\begin{aligned} P((u^\varepsilon, \dot{u}^\varepsilon) \in G) &\geq P(d((u^\varepsilon, \dot{u}^\varepsilon), g) \leq \rho) \\ &\geq P(d((u^\varepsilon, \dot{u}^\varepsilon), (u^f, \dot{u}^f)) \leq \rho, \|\varepsilon W - f\|_\infty < \alpha) \\ &\geq P(\|\varepsilon W - f\|_\infty < \alpha) - P(d((u^\varepsilon, \dot{u}^\varepsilon), (u^f, \dot{u}^f)) > \rho, \|\varepsilon W - f\|_\infty < \alpha) \\ &\geq P(\|\varepsilon W - f\|_\infty < \alpha) - \exp(-\frac{R}{\varepsilon^2}). \end{aligned}$$

But

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^2 \log P(\|\varepsilon W - f\|_\infty < \alpha) &\geq -\inf\{I(\varphi), \|\varphi - f\|_\infty \leq \alpha\} \\ &\geq -I(f). \end{aligned}$$

and since $R > R(g)$,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P((u^\varepsilon, \dot{u}^\varepsilon) \in G) \geq -I(f) = -R(g).$$

Since g is the arbitrary,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P((u^\varepsilon, \dot{u}^\varepsilon) \in G) \geq -\inf_{g \in G} I(g)$$

This completes the proof of Theorem.

Acknowledgement

This work was initiated when the author visited the University of Oslo. He would like to thank Bernt Øksendal for his hospitality and for the stimulating discussions.

References

[A] R. Azencott: Grandes déviations et applications.Ecole d'été de Probabilité de Saint-Flour VIII 1978. Lecture Notes in Mathematics, vol. **774** (1980) 1-176.

[B] J.M. Ball: Initial-boundary value problems for an extensible beam. J. Math. Anal. Appl. **42** (1973), 61-90.

[BMS] Z. Brzeźniak, B. Maslowski and J. Seidler: Stochastic beam equations. Probab. Theory Relat. Fields **132** (2005),119-149.

[BY] M.T. Barlow, M. Yor: Semi-martingale inequalities via the Garsia-Rodemich-Rumsey Lemma, and applications to local times. J. Funt. Analysis **49** (1982),198-229.

[C1] P.L. Chow: Large deviation problem for some parabolic Ito equations. Comm. Pure. Appl. Math **45:1** (1992), 197-120.

[C2] I. D. Chueshov: Introduction to the theory of infinite-dimensional dissipative systems. Acta, Kharkiv, 2002.

[CM1] P.L.Chow and J.L. Menaldi: Stochastic PDE for nonlinear vibration of elastic panels. Differential and Integral Equation **12:3** (1999) 419-432.

[CM2] F. Chenal and A. Millet: Uniform large deviations for parabolic SPDEs and applications. Stochastic Process. Appl. **72** (1997), 161-186.

[CR] S. Cerrai and M. Röckner: Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschtiz reaction term. Ann. Probab. **32:1** (2004), 1100-1139.

[CW] C. Cardon-Weber: Large deviations for a Burgers'-type SPDE. Stochastic processes and their applications **84** (1999), 53-70.

[D1] B. Davis: On the L^p-norms of stochastic integrals and other martingales. Duke Math. J. **43**(1976) 697-704.

[D2] E.H. Dowell: Aeroelasticity of plates and shells. Leyden, Noordhoff, The netherlands, 1975.

[DP] H. Doss and P. Priouret: Petites perturbations de systémes dynamiques avec réflexion. Séminaire de Probabilités XVII (1982) 351-370.

[DZ] A. Dembo and A. Zeitouni: Large deviations techniques and applications. Springer-Verlag, Berlin Heidelberg, 1998.

[F] W.E. Fitzgibbon: Global existence and boundedness of solutions to the extensible beam equation. SIAM J. Math. Anal. **13** (1982), 739-745.

[KP] S. Kouémou Patcheu: On a global solution and asymptotic behaviour for the generalized damped extensible beam equation. J. Differential Equations **135** (1997),299-314.

[P] E. Pardoux: Equations aux dérivées partielles stochastiques non lineaires montones, Thesis Détat, Université Paris sud, 1975.

[PZ] G.D. Prato and J. Zabczyk: Stochastic equations in infinite dimensions. Cambridge University Press, 1992.

[RS] C. Rovira and M. Sanz-Solé: The law of the solution to a nonlinear hyperbolic SPDE. J. Theoret. Probab. **4:9** (1996) 863-901.

[S] R.B. Sowers: Large deviations for a reaction-diffusion equation with non Gaussian perturbations. Ann. Probab. **20:1** (1992), 504-537.

[W] J. B. Walsh: An intriduction to stochastic partial differential equations. École d'Été de Probabilités de Saint-Flour XIV-1984. Lecture Notes in Mathematics 1180 (1986) 265-437.

[WK] S. Woinowsky-Krieger: The effect of an axial force on the vibration of hinged bars. J. Appl. Mech. **17** (1950) 35-36.

[Z] T.S.Zhang: On small time asymptotics of diffusions on Hilbert spaces. Ann. Probab. **28:2** (2000), 537-557.