Optimal control of SPDEs with delay and time-advanced backward stochastic partial differential equations

Bernt Øksendal[1),2)] Agnès Sulem [3)]

Tusheng Zhang[4),1)]

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Abstract

We study optimal control problems for (time-) delayed stochastic differential equations with jumps. We establish sufficient and necessary (Pontryagin type) maximum principles for an optimal control of such systems. The associated adjoint processes are shown to satisfy a (time-) advanced backward stochastic differential equation (ABSDE). Several results on existence and uniqueness of such ABSDEs

[1]Center of Mathematics for Applications (CMA), University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway. Email: oksendal@math.uio.no

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[2]Norwegian School of Economics and Business Administration (NHH), Helleveien 30, N-5045 Bergen, Norway.

[3]INRIA Paris-Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, Le Chesnay Cedex, 78153, France. Email: agnes.sulem@inria.fr

[4]School of Mathematics, University of Manchester, Oxford Road, Manchester M139PL, United Kingdom. Email: tusheng.zhang@manchester.ac.uk

are shown. The results are illustrated by an application to optimal consumption from a cash flow with delay.

1 Time-advanced BSPDEs with jumps

We now study time-advanced backward stochastic differential equations driven both by Brownian motion $B(t)$ and compensated Poisson random measures $\tilde{N}(dt, dz)$.

1.1 Framework

Let $B(t) = B(t, \omega)$ be a Brownian motion and $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$, where ν is the Lévy measure of the jump measure $N(\cdot, \cdot)$, be an independent compensated Poisson random measure on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$. Let D be a smooth domain in R^n. Consider the following general second order differential operator:

$$Au = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u(x)$$

where $a = (a_{ij}(x)) : D \to R^{n\times n}$ $(n > 2)$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$\lambda_1|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda_2|\xi|^2, \ \forall\, \xi \in R^n \ \text{ and } \ x \in D \tag{1.1}$$

for some positive constants $\lambda_1,\ \lambda_2,\ \ b = (b_1, ...b_n),:\ D \to R^n$ and $c:\ D \to R$ are bounded measurable functions. Set $H = L^2(D)$. Let $H_0^{1,2}(D)$ denote the Sobolev space of order one with zero boundary condition. In view of (1.1) and the boundedness of b and $c(x)$, for $u \in H_0^{1,2}(D)$ we have

$$
\begin{aligned}
-<Au,u> &= \frac{1}{2}\int_D (\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})dx \\
&\quad - \sum_{i=1}^n \int_D b_i(x)\frac{\partial u}{\partial x_i}u(x)dx - \int_D c(x)u(x)^2 dx \\
&\geq \frac{1}{2}\lambda_1 \int_D |\nabla u|^2(x)dx - \frac{1}{4}\lambda_1 \int_D |\nabla u|^2(x)dx - C_{\lambda_1}\int_D u^2(x)dx \\
&= \frac{1}{4}\lambda_1 \int_D |\nabla u|^2(x)dx - C_{\lambda_1}\int_D u^2(x)dx \qquad (1.2)
\end{aligned}
$$

Given a positive constant δ, denote by $D([0,\delta],H)$ the space of all càdlàg paths from $[0,\delta]$ into H. For a path $X(\cdot):\mathbb{R}_+\to H$, X_t will denote the function defined by $X_t(s)=X(t+s)$ for $s\in[0,\delta]$. Put $\mathcal{H}=L^2(R\to H;\nu)$. Consider the L^2 spaces $V_1:=L^2([0,\delta]\to H, ds)$ and $V_2:=L^2([0,\delta]\to \mathcal{H}, ds)$. Let

$$
F:\mathbb{R}_+\times H\times H\times V_1\times H\times H\times V_1\times\mathcal{H}\times\mathcal{H}\times V_2\times\Omega\to H
$$

be a predictable function. Introduce the following Lipschitz condition: There exists a constant C such that

$$
\begin{aligned}
&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega)-F(t,\bar{p}_1,\bar{p}_2,\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r},\omega)|_H \\
&\quad \leq C(|p_1-\bar{p}_1|_H+|p_2-\bar{p}_2|_H+|p-\bar{p}|_{V_1}+|q_1-\bar{q}_1|_H+|q_2-\bar{q}_2|_H \\
&\quad +|q-\bar{q}|_{V_1}+|r_1-\bar{r}_1|_{\mathcal{H}}+|r_2-\bar{r}_2|_{\mathcal{H}}+|r-\bar{r}|_{V_2}. \qquad (1.3)
\end{aligned}
$$

1.2 First existence and uniqueness theorem

We first consider the following time-advanced backward stochastic partial differential equation (BSPDE) in the unknown $\mathcal{F}_t$ adapted processes $(p(t),q(t),r(t,z))\int H\times H\times\mathcal{H}$:

$$
\begin{aligned}
dp(t) &= -Ap(t)dt \\
&\quad + E[F\left(t,p(t),p(t+\delta)\chi_{[0,T-\delta]}(t),p_t\chi_{[0,T-\delta]}(t),q(t),q(t+\delta)\chi_{[0,T-\delta]}(t),\right. \\
&\quad \left. q_t\chi_{[0,T-\delta]}(t),r(t),r(t+\delta)\chi_{[0,T-\delta]}(t),r_t\chi_{[0,T-\delta]}(t)\right)|\mathcal{F}_t]dt \\
&\quad + q(t)dB(t)+\int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz)\ ;\ t\in[0,T] \qquad (1.4)
\end{aligned}
$$

$$p(T) = G, \tag{1.5}$$

where G is a given H-valued $\mathcal{F}_T$-measurable random variable.

Note that the time-advanced BSPDE (**??**)-(**??**) for the adjoint processes of the Hamiltonian is of this form.

For this type of time-advanced BSPDEs we have the following result:

Theorem 1.1 *Assume that $E[G^2] < \infty$ and that condition (1.3) is satisfied. Then the BSPDE (1.4)-(1.5) has a unique solution $p(t), q(t), r(t,z))$ such that*

$$E\left[\int_0^T \left\{ |p(t)|_H^2 + |q(t)|_H^2 + \int_{\mathbb{R}} |r(t,z)|_H^2 \nu(dz) \right\} dt\right] < \infty. \tag{1.6}$$

Moreover, the solution can be found by inductively solving a sequence of BSPDEs backwards as follows:

Step 0: In the interval $[T-\delta, T]$ we let $p(t), q(t)$ and $r(t,z)$ be defined as the solution of the classical BSPDE (see [**?**])

$$\begin{aligned} dp(t) &= -Ap(t)dt + F\left(t, p(t), 0, 0, q(t), 0, 0, r(t,z), 0, 0\right) dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz)\ ;\ t \in [T-\delta, T] \end{aligned} \tag{1.7}$$

$$p(T) = G. \tag{1.8}$$

Step k ; $k \geq 1$**:** If the values of $(p(t), q(t), r(t,z))$ have been found for $t \in [T-k\delta, T-(k-1)\delta]$, then if $t \in [T-(k+1)\delta, T-k\delta]$ the values of $p(t+\delta), p_t, q(t+\delta), q_t, r(t+\delta, z)$ and r_t are known and hence the BSPDE

$$\begin{aligned} dp(t) &= -Ap(t)dt \\ &\quad + E[F\left(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t\right) |\mathcal{F}_t] dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz)\ ;\ t \in [T-(k+1)\delta, T-k\delta] \end{aligned} \tag{1.9}$$

$$p(T-k\delta) = \text{ the value found in Step } k-1 \tag{1.10}$$

has a unique solution in $[T-(k+1)\delta, T-k\delta]$.

We proceed like this until k is such that $T-(k+1)\delta \leq 0 < T-k\delta$ and then we solve the corresponding BSPDE on the interval $[0, T-k\delta]$.

Proof. The proof follows directly from the above inductive procedure. The estimate (1.6) is a consequence of known estimates for classical BSPDEs. □

1.3 Second existence and uniqueness theorem

Next, we consider the following backward stochastic partial differential equation in the unknown $\mathcal{F}_t$-adapted processes $(p(t), q(t), r(t, z)) \int H \times H \times \mathcal{H}$:

$$\begin{aligned} dp(t) &= -Ap(t)dt \\ &\quad + E[F(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t)|\mathcal{F}_t]dt \\ &\quad + q(t)dB_t + \int_{\mathbb{R}} r(t, z)\tilde{N}(dt, dz), \ ; \ t \in [0, T] \end{aligned} \tag{1.11}$$

$$p(t) = G(t), \qquad t \in [T, T+\delta]. \tag{1.12}$$

where G is a H-valued given continuous $\mathcal{F}_t$-adapted stochastic process.

Theorem 1.2 *Assume* $E[\sup_{T \leq t \leq T+\delta} |G(t)|_H^2] < \infty$ *and that the condition* (1.3) *is satisfied. Then the backward stochastic differential equation* (1.11) *admits a unique solution* $(p(t), q(t), r(t, z))$ *such that*

$$E[\int_0^T \{|p(t)|_H^2 + |q(t)|_H^2 + \int_{\mathbb{R}} |r(t, z)|_H^2 \nu(dz)\}dt] < \infty.$$

Proof.

Step 1 Assume F is independent of p_1, p_2 and p. Set $q^0(t) := 0, r^0(t, x) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t, z))$ to be the unique solution to the following backward stochastic partial differential equation equation:

$$\begin{aligned} dp^n(t) &= -Ap^n(t)dt \\ &\quad + E[F(t, q^{n-1}(t), q^{n-1}(t+\delta), q_t^{n-1}, r^{n-1}(t, \cdot), r^{n-1}(t+\delta, \cdot), r_t^{n-1}(\cdot))|\mathcal{F}_t]dt \\ &\quad + q^n(t)dB_t + r^n(t, z)\tilde{N}(dt, dz), \quad t \in [0, T] \end{aligned} \tag{1.13}$$

$$p^n(t) = G(t) \qquad t \in [T, T+\delta].$$

It is a consequence of the martingale representation theorem that the above equation admits a unique solution, see, e.g. [**?**]. We extend q^n, r^n to $[0, T+\delta]$ by setting $q^n(s) = 0$, $r^n(s, z) = 0$ for $T \leq s \leq T+\delta$. We are going to show that $(p^n(t), q^n(t), r^n(t, z))$ forms a Cauchy sequence. By Itô's formula, we have

$$
\begin{aligned}
0 &= |p^{n+1}(T) - p^n(T)|_H^2 \\
&= |p^{n+1}(t) - p^n(t)|_H^2 - 2\int_t^T < A(p^{n+1}(s) - p^n(s)), p^{n+1}(s) - p^n(s) > ds \\
&+ 2\int_t^T < p^{n+1}(s) - p^n(s), (E[F(s, q^n(s), q^n(s+\delta), q_s^n, r^n(s,\cdot), r^n(s+\delta,\cdot), r_s^n(\cdot))|\mathcal{F}_s] \\
&\quad - E[F(s, q^{n-1}(s), q^{n-1}(s+\delta), q_s^{n-1}, r^{n-1}(s,\cdot), r^{n-1}(s+\delta,\cdot), r_s^{n-1}(\cdot)))|\mathcal{F}_s] >_H ds \\
&+ \int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|_H^2 ds\, \nu(dz) + \int_t^T |q^{n+1}(s) - q^n(s)|_H^2 ds \\
&+ 2\int_t^T < p^{n+1}(s) - p^n(s), q^{n+1}(s) - q^n(s) >_H dB_s \\
&+ \int_t^T \int_{\mathbb{R}} \{|r^{n+1}(s,z) - r^n(s,z)|_H^2 \\
&\qquad + 2 < p^{n+1}(s-) - p^n(s-), r^{n+1}(s,z) - r^n(s,z) >_H\} \tilde{N}(ds,dz) \quad (1.14)
\end{aligned}
$$

Rearranging terms, in view of (1.3) and (1.2) we get

$$
\begin{aligned}
& E[|p^{n+1}(t) - p^n(t)|_H^2] + \frac{1}{2}\int_t^T |\nabla(p^{n+1}(s) - p^n(s))|_H^2 ds \\
&\quad + E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|_H^2 ds \nu(dz)\right] + E\left[\int_t^T |q^{n+1}(s) - q^n(s)|_H^2 ds\right] \\
&\leq 2E\left[\int_t^T | < p^{n+1}(s) - p^n(s), E[F(s, q^n(s), q^n(s+\delta), r^n(s,\cdot), r^n(s+\delta,\cdot)) \right. \\
&\quad -F(s, q^{n-1}(s), q^{n-1}(s+\delta), r^{n-1}(s,\cdot), r^{n-1}(s+\delta,\cdot))|\mathcal{F}_s] >_H |ds] \\
&\quad + C\int_t^T |p^{n+1}(s) - p^n(s)|_H^2 ds \qquad \leq \\
&\quad + \varepsilon E\left[\int_t^T |q^n(s+\delta) - q^{n-1}(s+\delta)|_H^2 ds\right] + \varepsilon E\left[\int_t^T (\int_s^{s+\delta} |q^n(u) - q^{n-1}(u)|_H^2 du) ds\right] \\
&\quad + \varepsilon E\left[\int_t^T |r^n(s) - r^{n-1}(s)|_{\mathcal{H}}^2 ds\right] \\
&\quad + \varepsilon E\left[\int_t^T |r^n(s+\delta) - r^{n-1}(s+\delta)|_{\mathcal{H}}^2 ds\right] + \varepsilon E\left[\int_t^T \left(\int_s^{s+\delta} |r^n(u) - r^{n-1}(u)|_{\mathcal{H}}^2 du\right) ds\right]
\end{aligned}
$$

(1.15)

Note that

$$E\left[\int_t^T |q^n(s+\delta)-q^{n-1}(s+\delta)|_H^2 ds\right] \le E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right]. \tag{1.16}$$

Interchanging the order of integration,

$$E\left[\int_t^T \left(\int_s^{s+\delta} |q^n(u)-q^{n-1}(u)|_H^2 du\right) ds\right] = E\left[\int_t^{T+\delta} |q^n(u)-q^{n-1}(u)|_H^2 du(\int_{u-\delta}^u ds\right]$$
$$\le \delta E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right]. \tag{1.17}$$

Similar inequalities hold also for $r^n - r^{n-1}$. It follows from (1.15) that

$$E[|p^{n+1}(t)-p^n(t)|_H^2] + \frac{1}{2}\int_t^T |\nabla(p^{n+1}(s)-p^n(s))|_H^2 ds$$
$$+ E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\nu(dz)\right] + E\left[\int_t^T |q^{n+1}(s)-q^n(s)|_H^2 ds\right]$$
$$\le C_\varepsilon E\left[\int_t^T |p^{n+1}(s)-p^n(s)|_H^2 ds\right] + (2+M)\varepsilon E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right]$$
$$+ 3\varepsilon E\left[\int_t^T |r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2 ds\right]. \tag{1.18}$$

Choose $\varepsilon > 0$ sufficiently small so that

$$E[|p^{n+1}(t)-p^n(t)|_H^2] + \frac{1}{2}\int_t^T |\nabla(p^{n+1}(s)-p^n(s))|_H^2 ds$$
$$+ E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\, \nu(dz)\right] + E\left[\int_t^T |q^{n+1}(s)-q^n(s)|_H^2 ds\right]$$
$$\le C_\varepsilon E\left[\int_t^T |p^{n+1}(s)-p^n(s)|_H^2 ds\right] + \frac{1}{2}E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right]$$
$$+ \frac{1}{2}E\left[\int_t^T |r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2 ds\right]. \tag{1.19}$$

This implies that

$$
-\frac{d}{dt}\left(e^{C_\varepsilon t}E\left[\int_t^T |p^{n+1}(s)-p^n(s)|_H^2 ds\right]\right) + \frac{1}{2}e^{C_\varepsilon t}\int_t^T |\nabla(p^{n+1}(s)-p^n(s))|_H^2 ds
$$

$$
+ e^{C_\varepsilon t}E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\ \nu(dz)\right] + e^{C_\varepsilon t}E\left[\int_t^T |q^{n+1}(s)-q^n(s)|_H^2 ds\right]
$$

$$
\leq \frac{1}{2}e^{C_\varepsilon t}E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right] + \frac{1}{2}e^{C_\varepsilon t}E\left[\int_t^T |r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2 ds\right]. \tag{1.20}
$$

Integrating the last inequality we get

$$
E\left[\int_0^T |p^{n+1}(s)-p^n(s)|_H^2 ds\right] + \int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |q^{n+1}(s)-q^n(s)|_H^2 ds\right]
$$

$$
+ \int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\nu(dz)\right]
$$

$$
+ \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t}\int_t^T |\nabla(p^{n+1}(s)-p^n(s))|_H^2 ds
$$

$$
\leq \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right] + \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2 ds\right] \tag{1.21}
$$

In particular,

$$
\int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\nu(dz)\right]
$$

$$
+ \int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |q^{n+1}(s)-q^n(s)|_H^2 ds\right]
$$

$$
\leq \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |q^n(s)-q^{n-1}(s)|_H^2 ds\right] + \frac{1}{2}\int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |r^n(s)-r^{n-1}(s)|_{\mathcal{H}}^2 ds\right] \tag{1.22}
$$

This yields

$$
\int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\nu(dz)\right] + \int_0^T dt\ e^{C_\varepsilon t}E\left[\int_t^T |q^{n+1}(s)-q^n(s)|_H^2 ds\right]
$$

$$
\leq \left(\frac{1}{2}\right)^n C \tag{1.23}
$$

for some constant C. It follows from (1.21) that

$$E\left[\int_0^T |p^{n+1}(s) - p^n(s)|_H^2 ds\right] \le \left(\frac{1}{2}\right)^n C. \tag{1.24}$$

(1.18) and ((1.21) further gives

$$E\left[\int_0^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|_H^2 ds\nu(dz)\right] + E\left[\int_0^T |q^{n+1}(s) - q^n(s)|_H^2 ds\right] \le \left(\frac{1}{2}\right)^n CnC_\varepsilon. \tag{1.25}$$

In view of (1.18), (1.21) and (1.22), we conclude that there exist progressively measurable processes $(p(t), q(t), r(t,z))$ such that

$$\begin{aligned}
&\lim_{n\to\infty} E[|p^n(t) - p(t)|_H^2] = 0, \\
&\lim_{n\to\infty} \int_0^T E[|p^n(t) - p(t)|_H^2]dt = 0, \\
&\lim_{n\to\infty} \int_0^T E[|\nabla(p^n(t) - p(t))|_H^2]dt = 0, \\
&\lim_{n\to\infty} \int_0^T E[|q^n(t) - q(t)|_H^2]dt = 0, \\
&\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}} E[|r^n(t,z) - r(t,z)|_H^2]\nu(dz)dt = 0.
\end{aligned}$$

Letting $n \to \infty$ in (1.13) we see that $(p(t), q(t), r(t,z))$ satisfies

$$\begin{aligned}
p(t) &- \int_t^T < Ap(s), p(s) > ds \\
&+ \int_t^T E[F(s, q(s), q(s+\delta), q_s, r(s,\cdot), r(s+\delta,\cdot), r_s(\cdot))|\mathcal{F}_s]ds \\
&+ \int_t^T q(s)dB_s + \int_t^T \int_{\mathbb{R}} r(s,z)\tilde{N}(ds,dz) = g(T)
\end{aligned} \tag{1.26}$$

i.e., $(p(t), q(t), r(t,z))$ is a solution. Uniqueness follows easily from Ito's formula, a similar calculation of deducing (1.14) and (1.15), and Gronwall's Lemma.

Step 2. General case. Let $p^0(t)=0$. For $n\geq 1$, define $(p^n(t),q^n(t),r^n(t,z))$ to be the unique solution to the following BSDE:

$$\begin{aligned} dp^n(t) &= -Ap^n(t)dt + q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz) \\ &\quad + E[F(t,p^{n-1}(t),p^{n-1}(t+\delta),p_t^{n-1},q^n(t),q^n(t+\delta),q_t^n,r^n(t,\cdot),r^n(t+\delta,\cdot),r_t^n(\cdot))|\mathcal{F}_t]dt, \end{aligned} \tag{1.27}$$

$$p^n(t)=G(t); \qquad t\in[T,T+\delta].$$

The existence of $(p^n(t),q^n(t),r^n(t,z))$ is proved in Step 1. By the same arguments leading to (1.18), we deduce that

$$\begin{aligned} &E[|p^{n+1}(t)-p^n(t)|_H^2] + \frac{1}{2}E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|_H^2 ds\nu(dz)\right] \\ &+\frac{1}{2}E\left[\int_t^T|q^{n+1}(s)-q^n(s)|_H^2ds\right] + \frac{1}{2}\lambda_1 E\left[\int_t^T|\nabla(p^{n+1}(s)-p^n(s))|_H^2ds\right] \\ &\leq CE\left[\int_t^T|p^{n+1}(s)-p^n(s)|_H^2ds\right] + \frac{1}{2}E\left[\int_t^T|p^n(s)-p^{n-1}(s)|_H^2ds\right] \end{aligned} \tag{1.28}$$

This implies that

$$-\frac{d}{dt}\left(e^{Ct}E\left[\int_t^T|p^{n+1}(s)-p^n(s)|_H^2ds\right]\right) \leq \frac{1}{2}e^{Ct}E\left[\int_t^T|p^n(s)-p^{n-1}(s)|_H^2ds\right] \tag{1.29}$$

Integrating (1.29) from u to T we get

$$\begin{aligned} &E\left[\int_u^T|p^{n+1}(s)-p^n(s)|_H^2ds\right] \leq \frac{1}{2}\int_u^T dte^{C(t-u)}E\left[\int_t^T|p^n(s)-p^{n-1}(s)|_H^2ds\right] \\ &\leq e^{CT}\int_u^T dtE[\int_t^T|p^n(s)-p^{n-1}(s)|_H^2ds]. \end{aligned} \tag{1.30}$$

Iterating the above inequality we obtain that

$$E[\int_0^T|p^{n+1}(s)-p^n(s)|_H^2ds] \leq \frac{e^{CnT}T^n}{n!}$$

Using above inequality and a similar argument as in Step 1, it can be shown that $(p^n(t),q^n(t),r^n(t,z))$ converges to some limit $(p(t),q(t),r(t,z))$, which is the unique solution of equation (1.11). □

Theorem 1.3 *Assume* $E\left[\sup_{T\le t\le T+\delta}|G(t)|^{2\alpha}\right]<\infty$ *for some* $\alpha>1$ *and that the following condition hold:*

$$
\begin{aligned}
&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r)-F(t,\bar p_1,\bar p_2,\bar p,\bar q_1,\bar q_2,\bar q,\bar r_1,\bar r_2,\bar r)|\\
&\quad\le C(|p_1-\bar p_1|_H+|p_2-\bar p_2|_H+\sup_{0\le s\le\delta}|p(s)-\bar p(s)|_H+|q_1-\bar q_1|_H+|q_2-\bar q_2|_H+|q-\bar q|_{V_1}\\
&\qquad+|r_1-\bar r_1|_{\mathcal H}+|r_2-\bar r_2|_{\mathcal H}+|r-\bar r|_{V_2}). \qquad (1.31)
\end{aligned}
$$

Then the BSPDE (1.11) *admits a unique solution* $(p(t),q(t),r(t,z))$ *such that*

$$E\left[\sup_{0\le t\le T}|p(t)|_H^{2\alpha}+\int_0^T\{|q|_H^2(t)+\int_{\mathbb R}|r|_H^2(t,z)\nu(dz)\}dt\right]<\infty.$$

Proof.

Step 1 . Assume F is independent of p_1,p_2 and p. In this case condition (1.31) reduces to assumption (1.3). By the Step 1 in the proof of Theorem 1.2, there is a unique solution $(p(t),q(t),r(t,z))$ to equation (1.11).

Step 2. General case. Let $p^0(t)=0$. For $n\ge1$, define $(p^n(t),q^n(t),r^n(t,z))$ to be the unique solution to the following BSDE:

$$
\begin{aligned}
dp^n(t)=&-Ap^n(t)dt+q^n(t)dB_t+r^n(t,z)\tilde N(dt,dz)\\
&+E[F(t,p^{n-1}(t),p^{n-1}(t+\delta),p_t^{n-1},q^n(t),q^n(t+\delta),q_t^n,r^n(t,\cdot),r^n(t+\delta,\cdot),r_t^n(\cdot))|\mathcal F_t]dt
\end{aligned} \qquad (1.32)
$$

$$p^n(t)=G(t),\qquad t\in[T,T+\delta].$$

By Step 1, $(p^n(t), q^n(t), r^n(t,z))$ exists. We are going to show that $(p^n(t), q^n(t), r^n(t,z))$ forms a Cauchy sequence. Using Itô's formula, we have

$$
\begin{aligned}
&|p^{n+1}(t) - p^n(t)|_H^2 + \int_t^T \int_{\mathbb{R}} |r^{n+1}(s,z) - r^n(s,z)|_H^2 ds\nu(dz) + \int_t^T |q^{n+1}(s) - q^n(s)|_H^2 ds \\
& - 2\int_t^T < Ap^n(s), p^n(s) > ds \\
& \quad = -2\int_t^T < p^{n+1}(s) - p^n(s), \\
& [E[F(s, p^n(s), p^n(s+\delta), p_s^n, q^{n+1}(s), q^{n+1}(s+\delta), q_s^{n+1}, r^{n+1}(s,\cdot), r^{n+1}(s+\delta,\cdot), r_s^{n+1}(\cdot)) \\
& - F(s, p^{n-1}(s), p^{n-1}(s+\delta), p_s^{n-1}, q^n(s), q^n(s+\delta), q_s^n, r^n(s,\cdot), r^n(s+\delta,\cdot), r_s^n(\cdot))|\mathcal{F}_s]] >_H ds \\
& - 2\int_t^T < p^{n+1}(s) - p^n(s), q^{n+1}(s) - q^n(s) > dB_s \\
& - \int_t^T \int_{\mathbb{R}} [|r^{n+1}(s,z) - r^n(s,z)|_H^2 + 2 < p^{n+1}(s-) - p^n(s-), (r^{n+1}(s,z) - r^n(s,z) >_H] \tilde{N}(ds,dz)
\end{aligned}
\tag{1.33}
$$

Take conditional expectation with respect to $\mathcal{F}_t$, take the supremum over the interval $[u, T]$ and use the condition (1.31) to get

$$\begin{aligned}
&\sup_{u\le t\le T}|p^{n+1}(t)-p^n(t)|_H^2+\sup_{u\le t\le T}E\left[\int_t^T|q^{n+1}(s)-q^n(s)|_H^2ds|\mathcal{F}_t\right]\\
&\quad+\sup_{u\le t\le T}E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|_H^2ds\nu(dz)|\mathcal{F}_t\right]\\
&\quad+\sup_{u\le t\le T}E\left[\int_t^T|\nabla(p^{n+1}(s)-p^n(s))|_H^2ds\nu(dz)|\mathcal{F}_t\right]\\
&\le C_\varepsilon\sup_{u\le t\le T}E\left[\int_u^T|p^{n+1}(s)-p^n(s)|_H^2ds|\mathcal{F}_t\right]\\
&\quad+C_1\varepsilon\sup_{u\le t\le T}E\left[\int_u^T|p^n(s)-p^{n-1}(s)|_H^2ds|\mathcal{F}_t\right]\\
&\quad+C_2\varepsilon\sup_{u\le t\le T}E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|_H^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]\\
&\quad+C_3\varepsilon\sup_{u\le t\le T}E\left[\int_t^T|q^{n+1}(s)-q^n(s)|_H^2ds|\mathcal{F}_t\right]\\
&\quad+C_4\varepsilon\sup_{u\le t\le T}E\left[\int_t^T\int_{\mathbb{R}}|r^{n+1}(s,z)-r^n(s,z)|_H^2ds\ \nu(dz)|\mathcal{F}_t\right]
\end{aligned}\tag{1.34}$$

Choosing $\varepsilon > 0$ such that $C_3\varepsilon < 1$ and $C_4\varepsilon < 1$ it follows from (1.34) that

$$\begin{aligned}
&\sup_{u\le t\le T}|p^{n+1}(t)-p^n(t)|_H^2\le C_\varepsilon\sup_{u\le t\le T}E\left[\int_u^T|p^{n+1}(s)-p^n(s)|_H^2ds|\mathcal{F}_t\right]\\
&\quad+(C_1+C_2)\varepsilon\sup_{u\le t\le T}E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|_H^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]
\end{aligned}\tag{1.35}$$

Note that $E\left[\int_u^T|p^{n+1}(s)-p^n(s)|_H^2ds|\mathcal{F}_t\right]$ and $E\left[\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|_H^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]$ are right-continuous martingales on $[0, T]$ with terminal random variables $\int_u^T|p^{n+1}(s)-p^n(s)|_H^2ds$ and $\int_u^T E[\sup_{s\le v\le T}|p^n(v)-p^{n-1}(v)|_H^2|\mathcal{F}_s]ds$. Thus for

$\alpha > 1$, we have

$$E\left[\left(\sup_{u\le t\le T} E\left[\int_u^T |p^{n+1}(s)-p^n(s)|_H^2 ds|\mathcal{F}_t\right]\right)^\alpha\right] \le c_\alpha E\left[\left(\int_u^T |p^{n+1}(s)-p^n(s)|_H^2 ds\right)^\alpha\right]$$
$$\le c_{T,\alpha} E\left[\int_u^T \sup_{s\le v\le T} |p^{n+1}(v)-p^n(v)|_H^{2\alpha} ds\right], \tag{1.36}$$

and

$$E\left[\left(\sup_{u\le t\le T} E\left[\int_u^T E[\sup_{s\le v\le T} |p^n(v)-p^{n-1}(v)|_H^2|\mathcal{F}_s]ds|\mathcal{F}_t\right]\right)^\alpha\right]$$
$$\le c_{T,\alpha} E\left[\int_u^T E[\sup_{s\le v\le T} |p^n(v)-p^{n-1}(v)|_H^{2\alpha}|\mathcal{F}_s]ds\right]$$
$$\le c_{T,\alpha} E\left[\int_u^T \sup_{s\le v\le T} |p^n(v)-p^{n-1}(v)|_H^{2\alpha} ds\right], \tag{1.37}$$

(1.35), (1.36) and (1.37) yield that for $\alpha > 1$,

$$E\left[\sup_{u\le t\le T} |p^{n+1}(t)-p^n(t)|_H^{2\alpha}\right] \le C_{1,\alpha} E\left[\int_u^T \sup_{s\le v\le T} |p^{n+1}(v)-p^n(v)|_H^{2\alpha} ds\right]$$
$$+ C_{2,\alpha} E\left[\int_u^T \sup_{s\le v\le T} |p^n(v)-p^{n-1}(v)|_H^{2\alpha} ds\right] \tag{1.38}$$

Put

$$g_n(u) = E\left[\int_u^T \sup_{t\le s\le T} |p^n(s)-p^{n-1}(s)|_H^{2\alpha}\right]$$

(1.38) implies that

$$-\frac{d}{dt}(e^{C_{1,\alpha}u} g_{n+1}(u)) \le e^{C_{1,\alpha}u} C_{2,\alpha} g_n(u) \tag{1.39}$$

Integrating (1.39) from t to T we get

$$g_{n+1}(t) \le c_{2,\alpha} \int_t^T e^{C_{1,\alpha}(s-t)} g_n(s)ds \le C_{2,\alpha} e^{C_{1,\alpha}T} \int_t^T g_n(s)ds. \tag{1.40}$$

Iterating the above inequality we obtain that

$$E\left[\int_0^T \sup_{t\le s\le T} |p^{n+1}(s)-p^n(s)|_H^{2\alpha} dt\right] \le \frac{e^{CnT}T^n}{n!}$$

Using above inequality and a similar argument as in step 1, we can show that $(p^n(t), q^n(t), r^n(t,z))$ converges to some limit $(p(t), q(t), r(t,z))$, which is the unique solution of equation (1.11). □