Stochastic Calculus for Dirichlet Processes

(Dedicated to S. Nakao on the occation of his 60th birthday)

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Abstract

Using time-reversal, we introduce the stochastic integration for zero-energy additive functionals of symmetric Markov processes, which extends an early work of S. Nakao. Various properties of such stochastic integrals are discussed and an Itô formula for Dirichlet processes is obtained.

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1 Introduction and Framework

It is well-known that stochastic integrals and Itô's formula for semimartingales play a central role in modern probability theory. However there are many important classes of Markov processes are not semimartingales. For example, symmetric diffusions on $\mathbb{R}^d$ whose infinitesimal operator are elliptic operators of divergence form $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with meaurable coefficients are not semimartingales. Even when X is a Brownian motion in $\mathbb{R}^d$, for $u \in W^{1,2}(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d;dx) \mid |\nabla u| \in L^2(\mathbb{R}^d;dx)\}$, $u(X_t)$ is not a semimartingale in general. To study such kind of processes, Fukushima obtained the following substitute for Ito's formula (cf [7]): for $u \in W^{1,2}(\mathbb{R}^d)$,

$$u(X_t) = u(X_0) + M_t^u + N_t^u, \tag{1.1}$$

where M^u is a square-integrable martingale and N^u is a continuous additive functional of zero energy. The above decomposition is called Fukushima's decomposition and holds for general symmetric Markov process X and for $u \in \mathcal{F}$, where $(\mathcal{E},\mathcal{F})$ is the Dirichlet space for X. Process $u(X)$ is a Dirichlet process as it has finite quadratic variations on compact time intervals. Nakao introduced stochastic integral $\int_0^t f(X_s)dN_s^u$ in [16] by using a Riesz representation theorem in a suitably constructed Hilbert space. Nakao's stochastic integral played an important role in the study of lower order perturbation of diffusion processes by Lunt, Lyons and Zhang [14] and by Fitzsimmons and Kuwae [5]. However Nakao's definition of stochastic integral is too restrictive to the study of lower order perturbation for symmetric Markov processes with discontinuous sample paths such as stable processes. The purpose of this paper is to present a new way of defining stochastic integral for Dirichlet processes associated with a symmetric Markov process. Our new approach uses only the time-reversal operator for the process X_t, and is thereby more direct and provides additional insight into stochastic integration for Dirichlet processes. This approach enables us to define $\Lambda(M)$ (see (1.5))for any locally square-integrable MAF M, subject to some mild conditions. Thus it not only recovers Nakao's results in [16] but also extends them significantly. The new stochastic integral allows us to study various transforms for symmetric Markov processes, which is carried out in a subsequent paper [2]. Below is a more detailed description of this paper.

Let $X = \{\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \theta_t, \zeta, \mathbf{P}_x, x \in E\}$ be an m-symmetric right Markov process on a Lusin space E, where m is a σ-finite measure with full support on E. Its associated Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$ is known to be quasi-regular (see [15]). By [1], $(\mathcal{E},\mathcal{F})$ is quasi-homeomorphic to a regular Dirichlet space on a locally compact separable metric space. Thus using this quasi-homeomorphism, without loss of generality, we may and do assume that X is an m-symmetric Hunt process on a locally compact metric space E such that its associated Dirichlet space $(\mathcal{E},\mathcal{F})$ is regular on $L^2(E;m)$ and that m is a positive Radon measure with full topological support on E.

Without loss of generality, we can take Ω to be the canonical path space $D([0,\infty[\to E_\partial)$ of right-continuous, left-limited (*rcll*, for short) functions from $[0,\infty[$ to E_∂. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \partial\}$ be the life time of X. As usual, $\mathcal{F}_\infty$ and $\mathcal{F}_t$ are the minimal completed σ-algebras obtained from $\mathcal{F}_\infty^0 := \sigma\{X_s \mid 0 \leq s < \infty\}$ and $\mathcal{F}_t^0 := \sigma\{X_s \mid 0 \leq s \leq t\}$, respectively, under $\mathbf{P}_x$; see the next section for more details. We sometimes use a filtration denoted by $(\mathcal{M}_t)$ on $(\Omega, \mathcal{M})$ in order to represent several filtrations, for example, $(\mathcal{F}_t^0)$, $(\mathcal{F}_{t+}^0)$ on $(\Omega, \mathcal{F}_\infty^0)$, $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}_\infty)$ and others introduced later. We set $X_t(\omega) := \partial$ for $t \geq \zeta(\omega)$ and use θ_t to denote the shift operator defined by $\theta_t(\omega)(s) := \omega(t+s)$, $t, s \geq 0$. Let ω_∂ be the path starting from ∂. Then $\omega_\partial(s) \equiv \partial$ for all $s \in [0, \infty[$. Note that $\theta_{\zeta(\omega)}(\omega) = \omega_\partial$ for all $\omega \in \Omega$, $\{\omega_\partial\} \in \mathcal{F}_0^0 \subset \mathcal{F}_t^0$ for all $t > 0$ and $\mathbf{P}_x(\{\omega_\partial\}) \leq \mathbf{P}_x(X_0 = \partial) = 0$ for $x \in E$. For a Borel subset B of E, $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ (the *exit time* of B) is an $(\mathcal{F}_t)$-stopping time. If B is closed, then τ_B is an $(\mathcal{F}_{t+}^0)$-stopping time. Also, ζ is an $(\mathcal{F}_t^0)$-stopping time because $\{\zeta \leq t\} = \{X_t = \partial\} \in \mathcal{F}_t^0$, $t \geq 0$. The transition semigroup of X, $\{P_t, t \geq 0\}$, is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta], \qquad t \geq 0.$$

Each P_t may be viewed as an operator on $L^2(E; m)$; collectively these operators form a strongly continuous semigroup of self-adjoint contractions. The Dirichlet form associated with X is the bilinear form

$$\mathcal{E}(u, v) := \lim_{t \downarrow 0} \frac{1}{t}(u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \left\{ u \in L^2(E; m) \,\middle|\, \sup_{t > 0} \ t^{-1}(u - P_t u, u)_m < \infty \right\}.$$

Here we use the notation $(f, g)_m := \int_E f(x)g(x)\, m(dx)$. It is well known that for $u \in \mathcal{F}$, u has a quasi-continuous m-version $\widetilde{u}$. As a rule we take u to be represented by its quasi-continuous version (when such exists), and drop the tilde from the notation. We refer the readers to [7] and [15] for notions such as quasi-continuous, quasi-everywhere (abbreviated as q.e. or $\mathcal{E}$-q.e.), $\mathcal{E}$-nest, martingale additive functional, continuous additive functionals, etc.

Let $\overset{\circ}{\mathcal{M}}$ and $\mathcal{N}$ denote, respectively, the space of MAFs of finite energy and the space of continuous additive functionals of zero energy. For $u \in \mathcal{F}$, the following Fukushima decomposition holds:

$$u(X_t) - u(X_0) = M_t^u + N_t^u, \qquad \text{for every } t \in [0, \infty[, \tag{1.2}$$

$\mathbf{P}_x$-a.s. for q.e. $x \in E$, where $M^u \in \overset{\circ}{\mathcal{M}}$ and $N^u \in \mathcal{N}$.

A positive continuous additive functional (PCAF) of X (call it A) determines a measure $\nu = \nu_A$ on the Borel subsets of E via the formula

$$\nu(f) = \uparrow \lim_{t \to 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_s)\, dA_s \right], \tag{1.3}$$

in which $f : E \to [0, \infty]$ is Borel measurable. The measure ν is necessarily *smooth*, in the sense that ν charges no exceptional set of X and there is an $\mathcal{E}$-nest $\{F_n\}$ of closed subsets of E such that $\nu(F_n) < \infty$ for each $n \in \mathbb{N}$. (Here an increasing sequence of closed sets $\{F_n\}$ is called an $\mathcal{E}$-nest if $\cup_{n=1}^\infty \mathcal{F}_{F_n}$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$, where $\mathcal{F}_{F_n} := \{u \in \mathcal{F} \mid u = 0 \ m\text{-a.e. on } E \setminus F_n\}$ and a family $\{F_n\}$ of closed sets is an $\mathcal{E}$-nest if and only if it is a nest, that is, $\mathbf{P}_x(\lim_{n\to\infty}\tau_{F_n} = \zeta) = 1$ q.e. $x \in E$.) Conversely, given a smooth measure ν, there is a unique PCAF A^ν such that (1.3) holds with $A = A^\nu$. In the sequel we refer to this bijection between smooth measures and PCAFs as the *Revuz correspondence*, and to ν as the Revuz measure of A^ν.

If M is a locally square-integrable martingale additive functional (MAF) of X on random time interval $[\![0,\zeta[\![$, then the process $\langle M\rangle$ (the dual predictable projection of $[M]$) is a PCAF (Proposition 2.8), and the associated Revuz measure (as in (1.3)) is denoted by $\mu_{\langle M\rangle}$. More generally, if M^u is the martingale part in the Fukushima decomposition of $u \in \mathcal{F}$, then $\langle M^u, M\rangle$ is a CAF locally of bounded variation, and we have the associated Revuz measure $\mu_{\langle M^u,M\rangle}$, which is locally the difference of smooth (positive) measures. For $u \in \mathcal{F}$, the Revuz measure $\mu_{\langle M^u\rangle}$ of $\langle M^u\rangle$ will usually be denoted by $\mu_{\langle u\rangle}$.

Let $(N(x,dy), H_t)$ be a Lévy system for X; that is, $N(x,dy)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ and H_t is a PCAF with bounded 1-potential such that for any nonnegative Borel function ϕ on $E_\partial \times E_\partial$ vanishing on the diagonal and any $x \in E_\partial$,

$$\mathbf{E}_x\left(\sum_{s\le t}\phi(X_{s-},X_s)\right) = \mathbf{E}_x\left(\int_0^t\int_{E_\partial}\phi(X_s,y)N(X_s,dy)dH_s\right).$$

To simplify notation, we will write

$$N\phi(x) := \int_{E_\partial}\phi(x,y)N(x,dy)$$

and

$$(N\phi * H)_t := \int_0^t N\phi(X_s)dH_s.$$

Let μ_H be the Revuz measure of the PCAF H. Then the jumping measure J and the killing measure κ of X are given by

$$J(dx,dy) = \frac{1}{2}N(x,dy)\mu_H(dx), \quad \text{and} \quad \kappa(dx) = N(x,\{\partial\})\mu_H(dx).$$

These measures feature in the Beurling-Deny decomposition of $\mathcal{E}$: for $f, g \in \mathcal{F}$,

$$\mathcal{E}(f,g) = \mathcal{E}^{(c)}(f,g) + \int_{E\times E}(f(x)-f(y))(g(x)-g(y))J(dx,dy) + \int_E f(x)g(x)\kappa(dx),$$

where $\mathcal{E}^{(c)}$ is the strongly local part of $\mathcal{E}$.

For $u \in \mathcal{F}$, the martingale part M_t^u in (1.2) can be decomposed as

$$M_t^u = M_t^{u,c} + M_t^{u,j} + M_t^{u,\kappa} \qquad \text{for every } t \in [0,\infty[,$$

$\mathbf{P}_x$-a.s. for q.e. $x \in E$, where $M_t^{u,c}$ is the continuous part of the martingale M^u, and

$$\begin{aligned} M_t^{u,j} &= \lim_{\varepsilon \downarrow 0} \left\{ \sum_{0<s\le t} (u(X_s) - u(X_{s-})) 1_{\{|u(X_s)-u(X_{s-})|>\varepsilon\}} 1_{\{s<\zeta\}} \right. \\ &\quad \left. - \int_0^t \left(\int_{\{y\in E:\, |u(y)-u(X_s)|>\varepsilon\}} (u(y) - u(X_s)) N(X_s, dy) \right) dH_s \right\}, \\ M_t^{u,\kappa} &= \int_0^t u(X_s) N(X_s, \{\partial\}) dH_s - u(X_{\zeta-}) 1_{\{t\ge\zeta\}}, \end{aligned}$$

are the jump and killing parts of M^u in $\overset{\circ}{\mathcal{M}}$, respectively. See Theorem A.3.9 of [7]. The limit in the expression for $M^{u,j}$ is in the sense of convergence in the norm of the space of MAF of finite energy and of convergence in probability under $\mathbf{P}_x$ for q.e. $x \in E$ (see [7]).

Let $\mathcal{N}^* \subset \mathcal{N}$ denote the class of continuous additive functionals of the form $N^u + \int_0^\cdot g(X_s)ds$ for some $u \in \mathcal{F}$ and $g \in L^2(E;m)$. Nakao [16] constructed a linear map Γ from $\overset{\circ}{\mathcal{M}}$ into $\mathcal{N}^*$ in the following way. It is shown in [16] that, for every $Z \in \overset{\circ}{\mathcal{M}}$, there is a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}_1(w, f) = \frac{1}{2}\mu_{\langle M^f + M^{f,\kappa},\, Z\rangle}(E) \quad \text{for every } f \in \mathcal{F}. \tag{1.4}$$

This unique w is denoted by $\gamma(Z)$. The operator Γ is defined by

$$\Gamma(Z)_t = N_t^{\gamma(Z)} - \int_0^t \gamma(Z)(X_s)ds \qquad \text{for } Z \in \mathcal{M}. \tag{1.5}$$

It is shown in Nakao [16] that $\Gamma(Z)$ can be characterized by the following equation

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbf{E}_{g\cdot m}\left[\Gamma(Z)_t\right] = -\frac{1}{2}\mu_{\langle M^g + M^{g,\kappa},\, Z\rangle}(E) \qquad \text{for every } g \in \mathcal{F}_b. \tag{1.6}$$

Here $\mathcal{F}_b := \mathcal{F} \cap L^\infty(E;m)$. So in particular we have $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Nakao [16] used the operator Γ to define a stochastic integral

$$\int_0^t f(X_s)dN_s^u := \Gamma(f * M^u)_t - \frac{1}{2}\langle M^{f,c} + M^{f,j},\, M^{u,c} + M^{u,j}\rangle_t, \tag{1.7}$$

where $u \in \mathcal{F}$, $f \in \mathcal{F} \cap L^2(E; \mu_{\langle u\rangle})$ and $(f * M^u)_t := \int_0^t f(X_{s-})dM_s^u$. If we define

$$\widetilde{\mathcal{N}} := \{N \in \mathcal{N} \mid N = N^u + A^\mu \text{ for some } u \in \mathcal{F} \text{ and some signed smooth measure } \mu\},$$

then we see by (1.5) that $\int_0^\cdot f(X_s)dN_s^u \in \widetilde{\mathcal{N}}$ if $u \in \mathcal{F}$ and $f \in \mathcal{F} \cap L^2(E;\mu_{\langle u\rangle})$. However, the conditions on the integrand $f(X_t)$ and on the integrator N^u in Nakao's definition of stochastic integral are too restrictive for applications, particularly when we study the perturbation theory of general symmetric Markov processes (see [2]).

The purpose of this paper is to give a new way of defining $\Gamma(M)$ and Nakao's stochastic integral for zero energy AFs N^u.

For a finite rcll AF M_t, it is known (see [3, Lemma 3.2]) that there is a Borel function φ on $E_\partial \times E_\partial$ with $\varphi(x,x) = \varphi(\partial, x) = 0$ for all $x \in E_\partial$ so that

$$M_t - M_{t-} = \varphi(X_{t-}, X_t) \qquad \text{for every } t \in]0, \zeta_p[, \ \mathbf{P}_m\text{-a.s.}$$

where ζ_p is the predictable part of the lifetime ζ. We will call φ the jump function of M. (In [3, Lemma 3.2], it is stated that φ is only defined on $E \times E_\partial$, but its construction remains valid by setting $\varphi(\partial, x) = 0$ for $x \in E_\partial$.) We have a similar result for locally square-integrable MAFs on $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$ (see Definition 2.5(iii) for the definition of locally square-integrable MAF on $I(\zeta)$), where ζ_i is the totally inaccessible part of the lifetime ζ: Let M be a locally square-integrable MAF on $I(\zeta)$. Then there exists a jump function φ on $E_\partial \times E_\partial$ for M satisfying the same property as stated so that $M_t - M_{t-} = \varphi(X_{t-}, X_t)$ for every $t \in]0, \zeta_p[$, $\mathbf{P}_m$-a.s. (see Corollary 2.9). Assume

$$\int_0^t \int_E \left(\widehat{\varphi}^2 1_{\{|\widehat{\varphi}| \le 1\}} + |\widehat{\varphi}| 1_{\{|\widehat{\varphi}| > 1\}} \right) (X_s, y) N(X_s, dy) dH_s < \infty \qquad \text{for every } t < \zeta, \tag{1.8}$$

$\mathbf{P}_x$-a.s. for q.e. $x \in E$, where $\widehat{\varphi}(x,y) := \varphi(x,y) + \varphi(y,x)$. Define, $\mathbf{P}_m$-a.s. on $[0, \zeta[$,

$$\Lambda(M)_t := -\frac{1}{2} \left(M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t \right) \qquad \text{for } t \in [0, \zeta[,$$

where K_t is the purely discontinuous local MAF on $I(\zeta)$ with

$$K_t - K_{t-} = -\widehat{\varphi}(X_{t-}, X_t), \quad t < \zeta, \mathbf{P}_x\text{-a.s. for q.e. } x \in E,$$

and r_t is the time-reversal operator.

A function f is said to be locally in $\mathcal{F}$ (denoted as $f \in \mathcal{F}_{\text{loc}}$) if there is an increasing sequence of finely open Borel set $\{D_k, k \ge 1\}$ with $\cup_{k=1}^\infty D_k = E$ q.e. and for every $k \ge 1$, there is $f_k \in \mathcal{F}$ such that $f = f_k$ m-a.e. on D_k. For two subsets A, B of E, we denote $A = B$ q.e. if $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is exceptional. By definition, every $f \in \mathcal{F}_{\text{loc}}$ admits a quasi-continuous m-version, so we may assume all $f \in \mathcal{F}_{\text{loc}}$ are quasi-continuous. Then we have $f = f_k$ q.e. on D_k. For $f \in \mathcal{F}_{\text{loc}}$, $M^{f,c}$ is well defined as a continuous MAF on $[0, \zeta[$ of locally finite energy. Moreover, for $f \in \mathcal{F}_{\text{loc}}$ and a locally square-integrable MAF M on $I(\zeta)$,

$$t \mapsto (f * M)_t := \int_0^t f(X_{s-}) dM_s$$

is a locally square-integrable MAF on $I(\zeta)$. For a locally square-integrable MAF M on $I(\zeta)$, denote by M^c its continuous part, which is also a locally square-integrable MAF on $I(\zeta)$ (see Theorem 8.23 in [9]).

Definition 1.1 (Stochastic integral) Suppose that M is a locally square-integrable MAF on $I(\zeta)$ and $f \in \mathcal{F}_{\text{loc}}$. Let $\varphi : E_\partial \times E_\partial \to \mathbb{R}$ be a jump function for M, and assume that φ satisfies condition (1.8). Define $\mathbf{P}_m$-a.s. on $[0, \zeta[$ by,

$$\begin{aligned} &\int_0^t f(X_{s-})\, d\Lambda(M)_s \\ &\quad := \ \Lambda(f * M)_t - \frac{1}{2} \langle M^{f,c},\, M^c \rangle_t + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s)) \varphi(y, X_s) N(X_s, dy) dH_s, \end{aligned}$$

whenever $\Lambda(f * M)$ is well defined and the third term in the right hand side of (3.9) is absolutely convergent.

When $M \in \overset{\circ}{\mathcal{M}}$, we can show that $\Lambda(M) = \Gamma(M)$ (see Theorem 3.5 below) and we will show that the above stochastic integral enjoys a generalized Itô's formula (see Theorem 4.7 below).

2 Additive functionals

In this section, we will prove some facts for additive functionals which will be used later. We begin with some details on the completion of filtrations. Let $\mathcal{P}(E)$ (resp. $\mathcal{M}(E)$) be the family of all probability (resp. σ-finite) measures on E. For each $\nu \in \mathcal{M}(E)$, let $\mathcal{F}_\infty^\nu$ (resp. $\mathcal{F}_t^\nu$) be the $\mathbf{P}_\nu$-completion of $\mathcal{F}_\infty^0$ (resp. $\mathbf{P}_\nu$-completion of $\mathcal{F}_t^0$ in $\mathcal{F}_\infty^\nu$) and set $\mathcal{F}_\infty := \bigcap_{\nu\in\mathcal{P}(E)} \mathcal{F}_\infty^\nu$ and $\mathcal{F}_t := \bigcap_{\nu\in\mathcal{P}(E)} \mathcal{F}_t^\nu$. We further prepare another filtration $\mathcal{G}_t$ contained in $\mathcal{F}_t$: For each $\nu \in \mathcal{M}(E)$, let $\mathcal{G}_t^\nu$ be the $\mathbf{P}_\nu$-completion of $\mathcal{F}_t^0$ and define $\mathcal{G}_t := \bigcap_{\nu\in\mathcal{P}(E)} \mathcal{G}_t^\nu$. Clearly, $\mathcal{G}_t^\nu \subset \mathcal{F}_t^\nu \subset \mathcal{F}_\infty^\nu$ and $\mathcal{G}_t \subset \mathcal{F}_t \subset \mathcal{F}_\infty$. Let $\mathcal{F}_\infty^m$ (resp. $\mathcal{F}_t^m$) be the $\mathbf{P}_m$-completion of $\mathcal{F}_\infty^0$ (resp. $\mathbf{P}_m$-completion of $\mathcal{F}_t^0$ in $\mathcal{F}_\infty^m$) and $\mathcal{G}_t^m$ the $\mathbf{P}_m$-completion of $\mathcal{F}_t^0$. Although m is not necessarily a finite measure on E_∂, we do have $\mathcal{F}_\infty \subset \mathcal{F}_\infty^m$, $\mathcal{F}_t \subset \mathcal{F}_t^m$ and $\mathcal{G}_t \subset \mathcal{G}_t^m$, because for $g \in L^1(E;m)$ with $0 < g \leq 1$ on E satisfying $gm \in \mathcal{P}(E)$, $\mathbf{P}_{gm}$-negligibility is equivalent to $\mathbf{P}_m$-negligibility.

For a fixed filtration $(\mathcal{M}_t)$ on $(\Omega, \mathcal{M})$, we recall the notions of $(\mathcal{M}_t)$-predictability, $(\mathcal{M}_t)$-optionality and $(\mathcal{M}_t)$-progressive measurability as follows (see [17] for more details): On $[0,\infty[\times\Omega$, the $(\mathcal{M}_t)$-*predictable* (resp. $(\mathcal{M}_t)$-*optional*) σ-field $\mathcal{P}(\mathcal{M}_t)$ (resp. $\mathcal{O}(\mathcal{M}_t)$) is defined as the smallest σ-field over $[0,\infty[\times\Omega$ containing all $\mathbf{P}_\nu(\mathcal{M})$-evanescent set for all $\nu \in \mathcal{P}(E_\partial)$ and with respect to which all $\mathcal{M}_t$-adapted lcrl (left-continuous, right-limited) (resp. rcll) processes are measurable. A process $\phi(s,\omega)$ on $[0,\infty[\times\Omega$ is said to be $(\mathcal{M}_t)$-*progressively measurable* provided $[0,t] \times \Omega \ni (s,\omega) \mapsto \phi(s,\omega)$ is $\mathcal{B}([0,t]) \otimes \mathcal{M}_t$-measurable for all $t > 0$. It is well-known that $(\mathcal{M}_t)$-predictability implies $(\mathcal{M}_t)$-optionality, which in turn implies $(\mathcal{M}_t)$-progressive measurability.

Definition 2.1 (AF) Fix a $\nu \in \mathcal{M}(E)$. An $(\mathcal{F}_t)$-adapted (resp. $(\mathcal{F}_t^\nu)$-adapted) process $A = (A_t)_{t\geq 0}$ with values in $[-\infty,\infty]$ is said to be an *additive functional* (AF in short) (resp. *AF admitting ν-null set*) if there exist a *defining set* $\Xi \in \mathcal{F}_\infty$ and an exceptional (resp. ν-null) set N satisfying the following conditions;

(i) $\mathbf{P}_x(\Xi) = 1$ for all $x \in E \setminus N$,

(ii) $\theta_t\Xi \subset \Xi$ for all $t \geq 0$; in particular, $\omega_\partial \in \Xi$ and $\mathbf{P}_\partial(\Xi) = 1$, because of $\omega_\partial = \theta_{\zeta(\omega)}(\omega)$ for all $\omega \in \Xi$,

(iii) for all $\omega \in \Xi$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for all $t, s \geq 0$,

(iv) for all $t \geq 0$, $A_t(\omega_\partial) = 0$; in particular, under the additivity in (iii), $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for all $t \geq \zeta(\omega)$ and $\omega \in \Xi$.

An AF A (admitting ν-null set) is called *right-continuous with left limits* (rcll AF in brief) if $t \mapsto A_t(\omega)$ is right continuous on $[0,\infty[$ and has a left limits on $]0,\infty[$ for each $\omega \in \Xi$. An AF A (admitting ν-null set) is said to be *finite* (resp. *continuous additive functional* (CAF in brief)) if $|A_t(\omega)| < \infty$, $t \in [0,\infty[$ (resp. $t \mapsto A_t(\omega)$ is continuous on $[0,\infty[$) for each $\omega \in \Xi$. A $[0,\infty[$-valued CAF is called a *positive continuous additive functional* (PCAF in short). Two AFs A and B are called *equivalent* if there exists a common defining set $\Xi \in \mathcal{F}_\infty$ and an exceptional set N such that $A_t(\omega) = B_t(\omega)$ for all $t \in [0,\infty[$ and $\omega \in \Xi$. We call $A = (A_t)_{t\geq 0}$ an AF on $[0,\zeta[$ or a local AF (admitting ν-null set) if A is $(\mathcal{F}_t)$-adapted and satisfies (i), (ii), (iv) and the property (iii)$'$ in which (iii) is modified so that the additivity condition is required only for $t+s < \zeta(\omega)$. The notions of rcll AF, CAF and PCAF on $[0,\zeta[$ are similarly defined. Two AFs on $[0,\zeta[$, A and B, are called *equivalent* if there exists a common defining set $\Xi \in \mathcal{F}_\infty$ and an exceptional set N such that $A_t(\omega) = B_t(\omega)$ for all $t \in [0,\zeta[$ and $\omega \in \Xi$.

Remark 2.2 Any PCAF A on $[0,\zeta[$ can be extended to a PCAF by setting

$$A_t(\omega) := \begin{cases} \lim_{u\uparrow\zeta} A_u(\omega), & \text{if } t \geq \zeta(\omega) > 0, \\ 0, & \text{if } t \geq \zeta(\omega) = 0 \end{cases}$$

for $\omega \in \Xi$ and setting $A_t(\omega) \equiv 0$ for $\omega \in \Xi^c$. The $(\mathcal{F}_t)$-adaptedness of this extended A holds as follows: for a fixed $T > 0$, we know $\{A_t \leq T\} \cap \{t < \zeta\} \in \mathcal{F}_t$. From this, we have the $\mathcal{F}_\zeta$-measurability of $\{A_\zeta \leq T\}$. Indeed, $\{A_\zeta \leq T\} = \bigcap_{t\in\mathbb{Q}_+}\{A_t \leq T, t < \zeta\} \in \mathcal{F}_\zeta$ as $\{A_t \leq T, t < \zeta\} \in \mathcal{F}_\zeta$ for any $t \geq 0$. Thus $\{A_t \leq T\} \cap \{t \geq \zeta\} = \{A_\zeta \leq T\} \cap \{t \geq \zeta\} \in \mathcal{F}_t$. Therefore, $\{A_t \leq T\} \in \mathcal{F}_t$ for any $T > 0$, which gives the $(\mathcal{F}_t)$-adaptedness of A. Noting $\zeta\circ\theta_t = \zeta - t$ if $t < \zeta$ and $\zeta\circ\theta_t = 0$ if $t \geq \zeta$, we conclude that $A_\zeta = A_t + A_\zeta\circ\theta_t$ for any $t \in [0,\infty[$ on Ξ. Consequently, $A_{t+s} = A_t + A_s\circ\theta_t$ holds for any $t, s \in [0,\infty[$ on Ξ.

Lemma 2.3 *Let A, B be PCAFs such that for q.e. $x \in E$, $\mathbf{E}_x[A_t] = \mathbf{E}_x[B_t]$ for all $t \geq 0$ and that the Revuz measure μ_A has finite total mass. Then A is equivalent to B.*

Proof. Let μ_A (resp. μ_B) be the Revuz measure associated with A (resp. B). In view of the Revuz correspondence, $\langle \mu_A, h\rangle = \langle \mu_B, h\rangle$ for any α-excessive function h (see (5.1.11) in [7]). So in particular, we have $\mu_A(E) = \mu_B(E) < \infty$. For every non-negative bounded continuous function f, by dominated convergence theorem,

$$\langle \mu_A, f\rangle = \lim_{\alpha\to\infty}\langle \mu_A, \alpha R_\alpha f\rangle = \lim_{\alpha\to\infty}\langle \mu_B, \alpha R_\alpha f\rangle = \langle \mu_B, f\rangle.$$

This implies that $\mu_A = \mu_B$ and so $A = B$. □

Remark 2.4 The above lemma may fail if the condition $\mu_A(E) < \infty$ is not satisfied. For example, take $E = \mathbb{R}^d$, X be Brownian motion on $\mathbb{R}^d$ and $\mu_A(dx) = |x|^{-d-1}dx$. Then μ_A is a smooth measure and it corresponds to a PCAF A of X. Let $B_t = A_t + t$, which is a PCAF of X with Revuz measure $\mu_A(dx) + dx$. However

$$\mathbf{E}_x[A_t] = \int_0^t \left(\int_{\mathbb{R}^d} p(s,x,y)|y|^{-d-1}dy\right)ds = \infty = \mathbf{E}_x[B_t] \qquad \text{for every } x \in \mathbb{R}^d \setminus \{0\}.$$

Here $p(s,x,y) = (2\pi t)^{-d/2} \exp\big(-|x-y|^2/(2t)\big)$ is the transition density function of X. $\square$

As usual, if T is an $(\mathcal{F}_t)$-stopping time and M a process, then M^T is the stopped process defined by $M_t^T := M_{t\wedge T}$. Following [9], we give the notion of local martingales of interval type:

Definition 2.5 (Processes of interval type) Let $\mathcal{D}$ be a class of $(\mathcal{F}_t)$-adapted processes and denote by $\mathcal{D}_{\rm loc}$ its localization (resp. $\mathcal{D}_{f\text{-loc}}$ its localization by a nest of finely open Borel sets); that is, $M \in \mathcal{D}_{\rm loc}$ (resp. $M \in \mathcal{D}_{f\text{-loc}}$) if and only if there exists a sequence $M^n \in \mathcal{D}$ and an increasing sequence of stopping times T_n with $T_n \to \infty$ (resp. a nest $\{G_n\}$ of finely open Borel sets) such that $M^{T_n} = (M^n)^{T_n}$ (resp. $M_t = M_t^n$ for $t < \tau_{G_n}$) for each n. Here a family $\{G_n\}$ of finely open Borel sets is called a *nest* if $\mathbf{P}_x(\lim_{n\to\infty} \tau_{G_n} = \zeta) = 1$ for q.e. $x \in E$. Clearly, $\mathcal{D} \subset \mathcal{D}_{\rm loc}$ (resp. $\mathcal{D} \subset \mathcal{D}_{f\text{-loc}}$) and $(\mathcal{D}_{\rm loc})_{\rm loc} = \mathcal{D}_{\rm loc}$ (resp. $(\mathcal{D}_{f\text{-loc}})_{f\text{-loc}} = \mathcal{D}_{f\text{-loc}}$). We assume $\mathcal{D} \neq \mathcal{D}_{\rm loc}$. If $\mathcal{D}$ is a subclass of AFs, then so is $\mathcal{D}_{\rm loc}$ (for if $M \in \mathcal{D}_{\rm loc}$, then there exists M^n and T_n as above, and for each ω and $t, s \geq 0$, there exists $n \in \mathbb{N}$ with $s + t < T_n(\omega)$ and $s < T_n(\theta_t\omega)$, hence $M_{t+s}(\omega) = M_t(\omega) + M_s(\theta_t\omega)$), while $\mathcal{D}_{f\text{-loc}}$ is contained in the class of AFs on $[0,\zeta[$.

(i) $B \subset [0,\infty[\times\Omega$ is called a *set of interval type* if there exists a non-negative random variable S such that for each $\omega \in \Omega$ the section $B_\omega := \{t \in [0,\infty[\mid (t,\omega) \in B\}$ is $[0, S(\omega)]$ or $[0, S(\omega)[$ and $B_\omega \neq \emptyset$.

(ii) Let B be an $(\mathcal{F}_t)$-optional set of interval type. A real-valued stochastic process M on B (that is, $M1_B = (M_t(\omega)1_B(t,\omega))_{t\geq 0}$ is a real-valued stochastic process) is said to be *in* $\mathcal{D}^B$ if and only if there exists $N \in \mathcal{D}$ such that $M1_B = N1_B$, and is said to be *locally in* $\mathcal{D}$ *on* B (write $M \in (\mathcal{D}_{\rm loc})^B$) if and only if $S := D_{B^c}$ is the debut of B^c and there exists an increasing sequence of $(\mathcal{F}_t)$-stopping times $\{S_n\}$ with $\lim_{n\to\infty} S_n = S$ and a sequence of $M^n \in \mathcal{D}$ such that $B_\omega \subset \bigcup_{n=1}^\infty [0, S_n(\omega)]$ $\mathbf{P}_x$-a.s. $\omega \in \Omega$ and $(M1_B)^{S_n} = (M^n 1_B)^{S_n}$ for all $n \in \mathbb{N}$ and $t \geq 0$, $\mathbf{P}_x$-a.s. $\omega \in \Omega$ for q.e. $x \in E$. Clearly, $\mathcal{D}^B \subset (\mathcal{D}_{\rm loc})^B$. Moreover, $\mathcal{D}^{B_2} \subset \mathcal{D}^{B_1}$ and $(\mathcal{D}_{\rm loc})^{B_2} \subset (\mathcal{D}_{\rm loc})^{B_1}$ for any pair of $(\mathcal{F}_t)$-optional sets B_1, B_2 of interval type with $B_1 \subset B_2$.

(iii) Let B be an $(\mathcal{F}_t)$-optional set of interval type. We set

$$\mathcal{M}^1 := \{M \mid M \text{ is a finite rcll AF}, \mathbf{E}_x[|M_t|] < \infty, \mathbf{E}_x[M_t] = 0 \text{ for } \mathcal{E}\text{-q.e. } x \in E \text{ and all } t \geq 0\},$$

and speak of an element of $(\mathcal{M}^1)^B$ (resp. $(\mathcal{M}^1_{\rm loc})^B$) as being an *MAF on* B (resp. *a local MAF on* B). Similarly,

$$\mathcal{M} := \{M \mid M \text{ is a finite rcll AF}, \mathbf{E}_x[M_t^2] < \infty, \mathbf{E}_x[M_t] = 0 \text{ for } \mathcal{E}\text{-q.e. } x \in E \text{ and all } t \geq 0\},$$

and an element of $\mathcal{M}^B$ (resp. $(\mathcal{M}_{\rm loc})^B$) is a *square-integrable MAF on* B (resp. *locally square-integrable MAF on* B). We further set

$$\begin{aligned} \mathcal{M}^c &:= \{M \in \mathcal{M} \mid M \text{ is a CAF}\}, \\ \mathcal{M}^c &:= \{M \in \mathcal{M} \mid M \text{ is a purely discontinuous AF}\}, \end{aligned}$$

and an element of $(\mathcal{M}^c_{\rm loc})^B$ (resp. $(\mathcal{M}^d_{\rm loc})^B$) is called a *locally square-integrable continuous MAF on B* (resp. *locally square-integrable purely discontinuous MAF on B*). For $M \in (\mathcal{M}_{\rm loc})^B$, M admits a unique decomposition $M = M^c + M^d$ with $M^c \in (\mathcal{M}^c_{\rm loc})^B$ and $M^d \in (\mathcal{M}^d_{\rm loc})^B$ (see Theorem 8.23 in [9]). In these definitions, we omit the usage "on B" when $B = [0,\infty[\times\Omega$.

For $[0,\infty]$-valued functions S, T on Ω with $S \leq T$, we employ the usual notation for *stochastic intervals*; for example,

$$[\![S,T[\![:= \{(t,\omega) \in [0,\infty[\times\Omega \mid S(\omega) \leq t < T(\omega)\},$$

the other species of stochastic intervals being defined analogously. We write $[\![S]\!] := [\![S,S]\!]$ for the *graph* of S. Note that these are all subsets of $[0,\infty[\times\Omega$. If S and T are $(\mathcal{M}_t)$-stopping times, then $[\![S,T]\!]$, $[\![S,T[\![,\cdots$, and $[\![S]\!]$ are $(\mathcal{M}_t)$-optional (see Theorem 3.16 in [9]). For a $[0,\infty]$-valued function R on Ω and $A \subset \Omega$, $R_A := R \cdot 1_A + (+\infty) \cdot 1_{A^c}$ is called the *restriction* of R on A. Clearly, $R \leq R_A$.

We will use T_p and T_i to denote, respectively, the predictable and totally inaccessible parts of the given $(\mathcal{F}_t)$-stopping time T of X, that is, $T_p := T_{\Lambda_p}$ and $T_i := T_{\Lambda_i}$, where $\Lambda_p := \{T < \infty, X_{T-} = X_T\}$, $\Lambda_i := \{T < \infty, X_{T-} \in E, X_{T-} \neq X_T\}$ (see Theorem 44.5 in M. Sharpe [17]). It is shown in [17] that T_p and T_i are $(\mathcal{F}_t)$-stopping times if T is an $(\mathcal{F}_t)$-stopping time. In the case that T is a stopping time with respect to another filtration, we have a similar result: Suppose that X_t is $\mathcal{M}_t$-measurable for any $t \geq 0$. By Chapter IV 1.7 (iv) in [15], X_T is $\mathcal{M}_T$-measurable. Hence, we can confirm $\{X_{T-} \in E\}$, $\{X_{T-} = X_T\}$, $\{T < \infty\} \in \mathcal{M}_T$. Consequently, $\{T = T_p\}$ and $\{T = T_i\}$ belong to $\mathcal{M}_T$. Therefore, T_p and T_i are $(\mathcal{M}_t)$-stopping times by Theorem 3.9(1) in [9]. If T is a prefect terminal time (that is, $t + T \circ \theta_t = T$ on $\{t < T\}$), then so are T_p and T_i. In particular, for the lifetime ζ, both ζ_p and ζ_i are $(\mathcal{F}^0_t)$-stopping times and perfect terminal times.

Remark 2.6 When $B = [\![0,R[\![$ for a given $(\mathcal{F}_t)$-stopping time R, there is another notion of "locally in $\mathcal{D}$ on B", obtained by replacing $(M1_B)^{S_n} = (M^n1_B)^{S_n}$ with $M^{S_n}1_B = (M^n)^{S_n}1_B$ in our definition; this is a weaker notion than ours, because $t \mapsto 1_B(t,\omega)$ is decreasing and $1_B(t,\omega)1_B(s,\omega) = 1_B(t,\omega)$ for $s \leq t$ and $\omega \in \Omega$. This weaker notion is described in [17].

Let T be an $(\mathcal{F}_t)$-stopping time. We set $I(T) := [\![0,T[\![\cup [\![T_i]\!] = \{(t,\omega) \mid t < T(\omega) \text{ or } t = T_i(\omega)\}$. Then we easily see that $I(T)$ is an $(\mathcal{F}_t)$-optional set of interval type. Indeed, $[\![0,T]\!]$ and $[\![T_i]\!]$ are $(\mathcal{F}_t)$-optional, and $I(T)_\omega = [0,T(\omega)]$ if $\omega \in \Lambda_i$, otherwise, $I(T)_\omega = [0,T(\omega)[$. We see $D_{I(T)^c}(\omega) = T(\omega)$. The inequality $D_{I(T)^c}(\omega) \geq T(\omega)$ follows immediately from the definitions. If $T(\omega) < D_{I(T)^c}(\omega)$, there is a $t \in]T(\omega), D_{I(T)^c}(\omega)[$, which implies $T(\omega) < t = T_i(\omega) < D_{I(T)^c}(\omega)$. This is a contradiction, because $T(\omega) < T_i(\omega)$ yields $T_i(\omega) = \infty$. Clearly $[\![0,T[\![\subset I(T) \subset [\![0,T]\!]$. By a slight abuse of notation, we shall often write "$t \in I(T)$"to mean "$(t,\omega) \in I(T)$", where ω is the sample path. Then we see $I(T) = [0,T(\omega)[$ if $T_i(\omega) = \infty$, and $I(T) = [0,T(\omega)]$ if $T_i(\omega) < \infty$.

Definition 2.7 (MAF locally of finite energy) Recall that $\overset{\circ}{\mathcal{M}}$ is the totality of MAFs of finite energy, that is,

$$\overset{\circ}{\mathcal{M}}:=\left\{M\in\mathcal{M}\;\middle|\;e(M):=\lim_{t\downarrow 0}\frac{1}{2t}\mathbf{E}_m[M_t^2]<\infty\right\}.$$

We say that an AF M on $[0,\zeta[$ is *locally in* $\overset{\circ}{\mathcal{M}}$ (and write $M\in\overset{\circ}{\mathcal{M}}_{f\text{-loc}}$) if there exists a sequence $\{M^n\}$ in $\overset{\circ}{\mathcal{M}}$ and a nest $\{G_n\}$ of finely open Borel sets such that $M_t=M_t^n$ for $t<\tau_{G_n}$ for each $n\in\mathbb{N}$. In case X is a diffusion process with no killing inside E, we can define the quadratic variation $\langle M\rangle$ for $M\in\overset{\circ}{\mathcal{M}}_{f\text{-loc}}$ as follows: $M^n_{t\wedge\tau_{G_n}}=M^m_{t\wedge\tau_{G_n}}$ for $n<m$ because of the continuity of M^n. Owing to the uniqueness of Doob-Meyer decomposition, we see $\langle M^n\rangle_{t\wedge\tau_{G_n}}=\langle M^m\rangle_{t\wedge\tau_{G_n}}$. The quadratic variation $\langle M\rangle$ of $M\in\overset{\circ}{\mathcal{M}}_{f\text{-loc}}$ as a PCAF is well-defined by setting $\langle M\rangle_t=\langle M^n\rangle_t,\ t<\tau_{G_n},\ n\in\mathbb{N}$, with Remark 2.2 and by choosing an appropriate defining set and exceptional set of $\langle M\rangle$, where $M^n\in\overset{\circ}{\mathcal{M}}$ and $\{G_n\}$ is a nest of finely open Borel sets such that $M_t=M_t^n,\ t<\tau_{G_n}$.

Proposition 2.8 $(\mathcal{M}_{\rm loc})^{I(\zeta)}\subset(\mathcal{M}_{\rm loc})^{[\![0,\zeta[\![}\subset\overset{\circ}{\mathcal{M}}_{f\text{-loc}}$. *More precisely, for each $M\in(\mathcal{M}_{\rm loc})^{I(\zeta)}$, there exists a nest $\{G_k\}$ of finely open Borel sets such that $1_{G_k}*M\in\overset{\circ}{\mathcal{M}}$ for each $k\in\mathbb{N}$, and the quadratic variation process $\langle M\rangle$ can be constructed as a PCAF.*

Proof. It suffices to show $(\mathcal{M}_{\rm loc})^{[\![0,\zeta[\![}\subset\overset{\circ}{\mathcal{M}}_{f\text{-loc}}$. Take $M\in(\mathcal{M}_{\rm loc})^{[\![0,\zeta[\![}$. Then there exists an increasing sequence $\{T_n\}$ of stopping times with $\lim_{n\to\infty}T_n=\zeta$, ($\mathbf{P}_x$-a.s. $\omega\in\Omega$ for q.e. $x\in E$) and $M^n\in\mathcal{M}_{\rm loc}$ such that $M_{t\wedge T_n}1_{[0,\zeta[}(t\wedge T_n)=M^n_{t\wedge T_n}1_{[0,\zeta[}(t\wedge T_n)$ holds for all $t\ge 0$ $\mathbf{P}_x$-a.s. for q.e. $x\in E$. We may assume that it holds for all $\omega\in\Omega$ by changing whole space. Note that $[0,\zeta(\omega)[\subset\bigcup_{n=1}^{\infty}[0,T_n(\omega)]$ for all $\omega\in\Omega$. Hence, $M^m_{t\wedge T_n}1_{[0,\zeta[}(t\wedge T_n)=M^n_{t\wedge T_n}1_{[0,\zeta[}(t\wedge T_n)$ for $n<m$. As noted in Definition 2.5, we see that M is an AF on $[0,\zeta[$. Owing to the uniqueness of Doob-Meyer decomposition for semimartingale on $I(\zeta)$ (see [9]), we have $\langle M^m\rangle_{t\wedge T_n}1_{[0,\zeta[}(t\wedge T_n)=\langle M^n\rangle_{t\wedge T_n}1_{[0,\zeta[}(t\wedge T_n)$ for $n<m$. Thus, we have $\langle M^m\rangle_t=\langle M^n\rangle_t$ for $t<T_n$ and $n<m$. The quadratic variation $\langle M\rangle$ of M is therefore well defined by setting $\langle M\rangle_t:=\langle M^n\rangle_t$ for $t<T_n$. As such $\langle M\rangle$ is a PCAF by setting $\langle M\rangle_t:=\langle M\rangle_\zeta:=\lim_{s\uparrow\zeta}\langle M\rangle_s$ for any $t\ge\zeta$ because of Remark 2.2. Let $\mu_{\langle M\rangle}$ be the Revuz measure corresponding to $\langle M\rangle$ and $\{F_k\}$ an $\mathcal{E}$-nest of closed sets such that $\mu_{\langle M\rangle}(F_k)<\infty$ for each k, and let G_k be the fine interior of F_k. Then $\{G_k\}$ is a nest. In view of the proofs of Theorem 5.6.1 and Lemma 5.6.2 in [7], the stochastic integral $1_{G_k}*M$ is of finite energy with $\mathbf{e}(1_{G_k}*M)=\frac{1}{2}\mu_{\langle M\rangle}(G_k)$ and its quadratic variation $\langle 1_{G_k}*M\rangle$ is a PCAF. Let μ_k (resp. ν_k) be the Revuz measure corresponding to $\langle 1_{G_k}*M\rangle$ (resp. $\langle 1_{G_k}*M,M\rangle$). By Lemma 5.6.2 in [7], for $M_i\in\overset{\circ}{\mathcal{M}}$ and $f_i\in L^2(E;\mu_{\langle M_i\rangle})$ $(i=1,2)$, we have $f_1f_2\mu_{\langle M_1,M_2\rangle}=\mu_{\langle f_1*M_1,f_2*M_2\rangle}$, hence $\int_0^t(f_1f_2)(X_s)d\langle M_1,M_2\rangle_s=\langle f_1*M_1,f_2*M_2\rangle_t$. From this, we see $\langle\mu_k,f^2\rangle=\langle\nu_k,f^2\rangle=\langle 1_{G_k}\mu_{\langle M\rangle},f^2\rangle$ for any $f\in L^2(E;\mu_{\langle M\rangle})$, consequently we have $\mu_k=\nu_k=1_{G_k}\mu_{\langle M\rangle}$ by $\mu_{\langle M\rangle}(G_k)<\infty$. This yields $\langle 1_{G_k}*M\rangle_t=\langle 1_{G_k}*M,M\rangle_t=\int_0^t 1_{G_k}(X_s)d\langle M\rangle_s$ for $t<\zeta$, hence $\langle M-1_{G_k}*M\rangle_t=0$ for $t<\tau_{G_k}$. Therefore, $M_t=(1_{G_k}*M)_t$ for $t<\tau_{G_k}$ and $1_{G_k}*M\in\overset{\circ}{\mathcal{M}}$. □

Corollary 2.9 *Let M be a locally square-integrable MAF on $I(\zeta)$, that is, $M \in (\mathcal{M}_{\rm loc})^{I(\zeta)}$. Then there exists a Borel function φ on $E_\partial \times E_\partial$ with $\varphi(x,x) = \varphi(x,\partial) = 0$ for all $x \in E_\partial$ so that*

$$M_t - M_{t-} = \varphi(X_{t-}, X_t) \ \ \text{for every } t \in]0, \zeta_p[, \mathbf{P}_m\text{-a.s.}$$

Proof. By the proof of Proposition 2.8, there exists an $\mathcal{E}$-nest $\{F_k\}$ such that for each $k \in \mathbb{N}$ $M^k := 1_{F_k} * M \in \overset{\circ}{\mathcal{M}}$ and $M_t = M_t^k$, $t < \tau_{F_k}$. Let φ_k be the jump function corresponding to M^k. Then we have $\varphi_k(X_{t-}, X_t) = \varphi_\ell(X_{t-}, X_t)$, $t < \tau_{F_k}$ $\mathbf{P}_m$-a.s. for $k < \ell$. From this, we see $\varphi_k = \varphi_\ell$ J-a.s. on $F_k \times F_k$. We construct a Borel function φ on $E \times E$ in the following manner. We set $F_0 := \emptyset$, $\varphi(x,y) := \varphi_k(x,y)$ for $(x,y) \in F_k \times F_k \setminus (F_{k-1} \times F_{k-1})$, $k \in \mathbb{N}$, $\varphi(x,y) := 0$ if $(x,y) \in E \times E \setminus (\bigcup_{k=1}^\infty F_k \times \bigcup_{k=1}^\infty F_k)$. Then φ satisfies $\varphi(x,x) = 0$ for $x \in E$. We also have $\varphi = \varphi_k$ J-a.s. on $F_k \times F_k$. Consequently, $\varphi(X_{t-}, X_t) = \varphi_k(X_{t-}, X_t)$, $t < \tau_{F_k}$ $\mathbf{P}_m$-a.s. This means that $M_t - M_{t-} = \varphi(X_{t-}, X_t)$, $t < \tau_{F_k}$ $\mathbf{P}_m$-a.s. Therefore $M_t - M_{t-} = \varphi(X_{t-}, X_t)$, $t < \zeta$ $\mathbf{P}_m$-a.s.

To extend this as stated, we shall show that M_{t-} exists in $\mathbb{R}$ at $t = \zeta_i$ $\mathbf{P}_m$-a.s. For each $a, b \in \mathbb{R}$ with $a < b$, we set $\tau_1^{a,b} := \inf\{t \in [0, \zeta[\mid M_t \le a\}$, $\tau_1^{a,b} := \zeta$ if $\{t \in [0,\zeta[\mid M_t \le a\} = \emptyset$, and set $\tau_2^{a,b} := \inf\{t \in [\tau_1^{a,b}, \zeta[\mid M_t \ge b\}$, $\tau_2^{a,b} := \zeta$ if $\{t \in [\tau_1^{a,b}, \zeta[\mid M_t \ge b\} = \emptyset$. Inductively, we can define $\tau_{2k+1}^{a,b} := \inf\{t \in [\tau_{2k}, \zeta[\mid M_t \le a\}$, $\tau_{2k+1}^{a,b} := \zeta$ if $\{t \in [\tau_{2k}, \zeta[\mid M_t \le a\} = \emptyset$, and $\tau_{2k+2}^{a,b} := \inf\{t \in [\tau_{2k+1}, \zeta[\mid M_t \ge b\}$, $\tau_{2k+2}^{a,b} := \zeta$ if $\{t \in [\tau_{2k+1}, \zeta[\mid M_t \ge b\} = \emptyset$. Then $\{\tau_n^{a,b} \mid n \in \mathbb{N}\}$ is an increasing sequence of $(\mathcal{F}_t)$-stopping times. Let $\beta^{a,b} := \sup\{k \mid \tau_{2k}^{a,b} < \zeta\}$. Then $\underline{\lim}_{t\uparrow\zeta} M_t < a < b < \overline{\lim}_{t\uparrow\zeta} M_t$ implies $\beta^{a,b} = \infty$, conversely $\beta^{a,b} = \infty$ implies $\underline{\lim}_{t\uparrow\zeta} M_t \le a < b \le \overline{\lim}_{t\uparrow\zeta} M_t$. Suppose that $M_{t-}(\omega)$ does not exist at $t = \zeta_i(\omega) (< \infty)$. Then there exist $a, b \in \mathbb{Q}$ with $a < b$ such that $\beta^{a,b}(\omega) = \infty$, hence $\tau_n^{a,b}(\omega) < \zeta(\omega)$ for all $n \in \mathbb{N}$. We then see that $\tau^{a,b}(\omega) := \lim_{n\to\infty} \tau_n^{a,b}(\omega) = \zeta_i(\omega)$ for such ω, because of the existence of the left hand limit of M in $\mathbb{R}$ up to ζ, where we use the facts $M_{\tau_{2n}^{a,b}} \ge b$, $M_{\tau_{2n+1}^{a,b}} \le a$ under $\tau^{a,b} < \zeta$. This yields

$$\begin{aligned}&\{\omega \in \Omega \mid M_{t-}(\omega) \text{ does not exist in } [-\infty, \infty] \text{ at } t = \zeta_i(\omega)\}\\ &\qquad \subset \bigcup_{a,b\in\mathbb{Q}, a<b} \{\omega \in \Omega \mid \tau_n^{a,b}(\omega) < \zeta_i(\omega) \text{ for each } n \text{ and } \lim_{n\to\infty} \tau_n^{a,b}(\omega) = \zeta_i(\omega)\}.\end{aligned}$$

Since ζ_i is not $(\mathcal{F}_t^m)$-predictable, we obtain that M_{t-} exists in $[-\infty, \infty]$ at $t = \zeta_i$ $\mathbf{P}_m$-a.s. Next we eliminate the case $M_{\zeta_i-} = \pm\infty$, simultaneously. We set $\sigma_0^\pm := 0$ and for each $k \in \mathbb{N}$, $\sigma_k^\pm := \inf\{t \in [\sigma_{k-1}^\pm, \zeta[\mid \pm M_t > k\}$ if $\{t \in [\sigma_{k-1}^\pm, \zeta[\mid \pm M_t > k\} \ne \emptyset$, and $\sigma_k^\pm := \zeta$ if $\{t \in [\sigma_{k-1}^\pm, \zeta[\mid \pm M_t > k\} = \emptyset$, and $\beta^\pm := \sup\{k \mid \sigma_k^\pm < \zeta\}$. Then $\{\sigma_n^\pm \mid n \in \mathbb{N}\}$ is an increasing sequence of $(\mathcal{F}_t)$-stopping times. Suppose $M_{t-}(\omega) = \pm\infty$ for $t = \zeta_i(\omega)$. Then $\beta^\pm(\omega) = \infty$, hence $\sigma_k^\pm(\omega) < \zeta_i(\omega)$ for each $k \in \mathbb{N}$. We then see $\sigma^\pm(\omega) := \lim_{n\to\infty} \sigma_n^\pm(\omega) = \zeta_i(\omega)$ for such ω, because of the existence of the left hand limit of M in $\mathbb{R}$ up to ζ, where we use the fact $\pm M_{\sigma_n^\pm} \ge n$ under $\sigma^\pm < \zeta$. This yields

$$\begin{aligned}&\{\omega \in \Omega \mid \lim_{t\uparrow\zeta} M_t(\omega) = \pm\infty \text{ if } \zeta_i(\omega) < \infty\}\\ &\qquad \subset \{\omega \in \Omega \mid \sigma_n^\pm(\omega) < \zeta_i(\omega) \text{ for each } n \text{ and } \lim_{n\to\infty} \sigma_n^\pm(\omega) = \zeta_i(\omega)\}.\end{aligned}$$

By the same reason as noted above, we obtain that $M_{\zeta-}$ exists in $\mathbb{R}$ if $\zeta_i < \infty$ $\mathbf{P}_m$-a.s.

We can extend φ on $E_\partial \times E_\partial$ so that $\varphi(x,x)=\varphi(x,\partial)=0$, $x\in E_\partial$ and $M_t-M_{t-}=\varphi(X_{t-},X_t)$, $t<\zeta_p$ $\mathbf{P}_m$-a.s. as in Lemma 3.2 in [3]. This completes the proof. □

We recall the definition of the shift operator θ_s and the time-reversal operator r_t on the path space Ω. For each $s\geq 0$, the shift operator θ_s is defined by $\theta_s\omega(t):=\omega(t+s)$ for $t\in[0,\infty[$. Given a path $\omega\in\{t<\zeta\}$, the operator r_t is defined by

$$r_t(\omega)(s):=\begin{cases}\omega((t-s)-), & \text{if } 0\leq s\leq t,\\ \omega(0), & \text{if } s\geq t.\end{cases} \tag{2.1}$$

Here for $r>0$, $\omega(r-):=\lim_{s\uparrow r}\omega(s)$ is the left limit at r, and we use the convention that $\omega(0)_-:=\omega(0)$. For a path $\omega\in\{t\geq\zeta\}$, we set $r_t(\omega):=\omega_\partial$. We note that

$$\lim_{s\downarrow 0} r_t(\omega)(s)=\omega(t-)=r_t(\omega)(0) \quad \text{and} \quad \lim_{s\uparrow t} r_t(\omega)(s)=\omega(0)=r_t(\omega)(t). \tag{2.2}$$

Definition 2.10 For any $t>0$, we say two sample paths ω and ω' are t-equivalent if $\omega(s)=\omega'(s)$ for all $s\in[0,t]$. We say two sample paths ω and ω' are pre-t-equivalent if $\omega(s)=\omega'(s)$ for all $s\in[0,t[$.

Lemma 2.11 *For each $t>0$, $r_t:\Omega\to\Omega$ is $\mathcal{F}_t^0/\mathcal{F}_\infty^0$-measurable and $\mathcal{G}_t^m/\mathcal{G}_t^m$-measurable. For $t,s>0$, $\theta_s:\Omega\to\Omega$ is $\mathcal{G}_{t+s}^m/\mathcal{G}_t^m$-measurable.*

Proof. Let $F_i\in\mathcal{B}(E_\partial)$ and $s_i\in[0,\infty[$, $i=1,2,\cdots,n$ with $s_1<s_2<\cdots<s_k\leq t<s_{k+1}<\cdots<s_n$ for some $k\in\{1,2,\cdots,n\}$. Then $r_t^{-1}(\bigcap_{i=1}^n X_{s_i}^{-1}(F_i))=\bigcap_{i=1}^n(X_{s_i}\circ r_t)^{-1}(F_i)$ is equal to $\bigcap_{i=1}^k(\{X_{t-s_i}\in F_i,\, t<\zeta\}\cup\{\partial\in F_i,\, t\geq\zeta\})\cap\bigcap_{i=k+1}^n(\{X_0\in F_i,\, t<\zeta\}\cup\{\partial\in F_i,\, t\geq\zeta\})\in\mathcal{F}_t^0$. Next we show the $\mathcal{G}_t^m/\mathcal{G}_t^m$-measurability of r_t and the $\mathcal{G}_{t+s}^m/\mathcal{G}_t^m$-measurability of θ_s simultaneously. Take $C\in\mathcal{G}_t^m$. Then there exists $D,N\in\mathcal{F}_t^0$ such that $C\bigtriangleup D\subset N$ and $\mathbf{P}_m(N)=0$. Since $\mathbf{P}_m(\{\omega_\partial\})=0$, by deleting $\{\omega_\partial\}=\{\omega\in\Omega\mid\zeta(\omega)=0\}\in\mathcal{F}_0^0\subset\mathcal{F}_t^0$, we may assume $\omega_\partial\notin C\cup D\cup N$. Then, $r_t^{-1}(C)\bigtriangleup r_t^{-1}(D)\subset r_t^{-1}(N)$, $r_t^{-1}(D),r_t^{-1}(N)\in\mathcal{F}_t^0$ and $\mathbf{P}_m(r_t^{-1}(N))=\mathbf{P}_m(r_t^{-1}(N)\cap\{t<\zeta\})+1_N(\omega_\partial)\mathbf{P}_m(t\geq\zeta)=\mathbf{P}_m(N\cap\{t<\zeta\})=0$. We also see $\theta_s^{-1}(C)\bigtriangleup\theta_s^{-1}(D)\subset\theta_s^{-1}(N)$, $\theta_s^{-1}(D),\theta_s^{-1}(N)\in\mathcal{F}_{t+s}^0$ and $\mathbf{P}_m(\theta_s^{-1}(N))=\mathbf{E}_m[\mathbf{P}_{X_s}(N)]=\mathbf{E}_m[\mathbf{P}_{X_s}(N):s<\zeta]=\mathbf{E}_m[\mathbf{P}_{X_0}(N):s<\zeta]\leq\mathbf{E}_m[\mathbf{P}_{X_0}(N)]=\mathbf{P}_m(N)=0$. □

For an rcll AF A_t adapted to $(\mathcal{F}_t^0)_{t\geq 0}$, $A_t(\omega)=A_t(\omega')$ if ω and ω' are t-equivalent and $A_{t-}(\omega)=A_{t-}(\omega')$ if ω and ω' are pre-t-equivalent. These conclusions may fail to hold if the measurability conditions are not satisfied. We need the following notion:

Definition 2.12 (PrAF) Fix $\nu\in\mathcal{M}(E)$. A process $A=(A_t)_{t\geq 0}$ with values in $\overline{\mathbb{R}}:=[-\infty,\infty]$ is said to be a *progressively additive functional (PrAF in short) (resp. PrAF admitting ν-null set)* if A is $(\mathcal{G}_{t+})$-adapted (resp. $(\mathcal{G}_{t+}^\nu)$-adapted) and there exist *defining sets* $\Xi\in\mathcal{F}_\infty$, $\Xi_t\in\mathcal{G}_t$ (resp. $\Xi\in\mathcal{F}_\infty^\nu$, $\Xi_t\in\mathcal{G}_t^\nu$) for each $t>0$ and an exceptional (resp. a ν-null) set N satisfying the following condition;

(i) $\mathbf{P}_x(\Xi)=1$ for all $x\in E\setminus N$, for every $t>s>0$, $\Xi\subset\Xi_t\subset\Xi_s$, and $\Xi=\bigcap_{t>0}\Xi_t$,

(ii) $\theta_t\Xi \subset \Xi$ for all $t \geq 0$ and $\theta_{t-s}(\Xi_t) \subset \Xi_s$ for all $s \in]0,t[$; in particular, $\omega_\partial \in \Xi \subset \Xi_t$ and $\mathbf{P}_\partial(\Xi) = \mathbf{P}_\partial(\Xi_t) = 1$ under (i),

(iii) for all $\omega \in \Xi_t$, $A_\cdot(\omega)$ is defined on $[0,t[$ and its left limit $A_{s-}(\omega)$ exists for all $s \in]0,t]$ such that $A_0(\omega) = 0$, on $\{t < \zeta\}$ $|A_s(\omega)| < \infty$ and $|A_{s-}(\omega)| < \infty$ for $s \in]0,t]$, and $A_{p+q}(\omega) = A_p(\omega) + A_q(\theta_p\omega)$ for all $p, q \geq 0$ with $p + q < t$,

(iv) for all $t \geq 0$, $A_t(\omega_\partial) = 0$,

(v) for any $t > 0$ and pre-t-equivalent paths $\omega, \omega' \in \Omega$, $\omega \in \Xi_t$ implies $\omega' \in \Xi_t$, $A_s(\omega) = A_s(\omega')$ for any $s \in [0,t[$ and $A_{s-}(\omega) = A_{s-}(\omega')$ for any $s \in]0,t]$.

Furthermore, A is called an *rcll PrAF* (or an *rcll PrAF admitting ν-null set*) if for each $t > 0$ and $\omega \in \Xi_t$, $s \mapsto A_s(\omega)$ is right continuous on $[0,t[$ and has left hand limits on $]0,t]$ and a PrAF (or a PrAF admitting ν-null set) is said to be *finite* (resp. *continuous*) if $|A_s(\omega)| < \infty$, $\forall s \in [0,t[$ (resp. continuous on $[0,t[$) for every $\omega \in \Xi_t$.

We say that an AF A on $[0,\zeta[$ (resp. AF A on $[0,\zeta[$ admitting ν-null set) is called a *PrAF on* $[0,\zeta[$ (resp. *PrAF on* $[0,\zeta[$ *admitting ν-null set)*) if A is $(\mathcal{G}_{t+})$-adapted (resp. $(\mathcal{G}^\nu_{t+})$-adapted), and there exist $\Xi \in \mathcal{F}_\infty$, $\Xi_t \in \mathcal{G}_t$ (resp. $\Xi \in \mathcal{F}^\nu_\infty$, $\Xi_t \in \mathcal{G}^\nu_t$) for each $t > 0$ and an exceptional (resp. ν-null) set N such that (i'), (ii), (iii'), (iv) and (v') hold: (i'): $\mathbf{P}_x(\Xi) = 1$ for all $x \in E \setminus N$, $\Xi \subset \Xi_t$ for all $t > 0$, $\Xi = \bigcap_{t>0} \Xi_t$, and $\Xi_t \cap \{t < \zeta\} \subset \Xi_s \cap \{s < \zeta\}$ for $s < t$. (iii'): For each $\omega \in \Xi_t \cap \{t < \zeta\}$, the same conclusion as in (iii) holds. (v'): For any $t > 0$ and pre-t-equivalent paths $\omega, \omega' \in \Omega \cap \{t < \zeta\}$, the same conclusion as in (v) holds.

The notion of *rcll PrAF on* $[0,\zeta[$ *(or rcll PrAF admitting ν-null set)* is similarly defined.

Remark 2.13 (i) Our notion of PrAF is different from what is found in Walsh [18].

(ii) Every PrAF (resp. PrAF on $[0,\zeta[$) is an AF (resp. AF on $[0,\zeta[$).

(iii) The MAF M^u and the CAF N^u of 0-energy appearing in Fukushima's decomposition (1.2) can be regarded as finite rcll PrAFs in view of the proof of Theorem 5.2.2 in [7]. In this case, the defining sets for M^u as PrAF are given by

$$\begin{aligned} \Xi &:= \{\omega \in \Omega \mid M^{u_n}_{s-}(\omega) \text{ converges uniformly on }]0,t] \text{ for } \forall t \geq 0\} \in \mathcal{F}_\infty \\ \Xi_t &:= \{\omega \in \Omega \mid M^{u_n}_{s-}(\omega) \text{ converges uniformly on }]0,t]\} \in \mathcal{G}_t \end{aligned}$$

for every $t > 0$, where $M^{u_n}_t := u_n(X_t) - u_n(X_0) - \int_0^t (u_n(X_s) - f_n(X_s))ds$ with $f_n := n(u - nR_{n+1}u)$ and $u_n := R_1 f_n = nR_{n+1}u$. Hence a MAF of stochastic integral type $\int_0^t g(X_{s-})dM^u_s$ ($g, u \in \mathcal{F}$ with $g \in L^2(E; \mu_{\langle u \rangle})$) can be regarded as a finite rcll PrAF. Consequently, any MAF of finite energy also can be regarded as PrAF, in view of the assertion of Lemma 5.6.3 in [7] and Lemma 2.14 below.

(iv) Every $M \in \overset{\circ}{\mathcal{M}}_{f\text{-loc}}$ can be regarded as a PrAF on $[0,\zeta[$, hence, $M \in \mathcal{M}^{[\![0,\zeta[\![}_{\text{loc}}$ is so.

Lemma 2.14 *Let (A^n) be a sequence of finite rcll PrAFs with defining sets $\Xi^n \in \mathcal{F}_\infty$, $\Xi^n_t \in \mathcal{G}_t$. For each $t > 0$, set*

$$\Xi_t := \left\{ \omega \in \bigcap_{n\in\mathbb{N}} \Xi^n_t \,\middle|\, A^n \ \text{ converges uniformly on } [0,t[\right\} \in \mathcal{G}_t$$

and

$$\Xi := \left\{ \omega \in \bigcap_{n\in\mathbb{N}} \Xi^n \,\middle|\, A^n \ \text{ converges uniformly on } [0,t[\text{ for every } t \in [0,\infty[\right\} \in \mathcal{F}_\infty.$$

Suppose that there exists an exceptional set N such that $\mathbf{P}_x(\Xi) = 1$ for $x \in E \setminus N$. If we define $A_t := \underline{\lim}_{n\to\infty} A^n_t$ on Ω, then A is a finite rcll PrAF with its defining sets Ξ, Ξ_t.

Proof. We only show that for any $t > 0$ and pre-t-equivalent paths ω, ω', $\omega \in \Xi_t$ implies $\omega' \in \Xi_t$. Suppose that $\omega \in \Xi_t$ and ω is pre-t-equivalent to ω'. It easy to see that $\omega' \in \bigcap_{n\in\mathbb{N}} \Xi^n_t$. We then see the uniform convergence of $A^n_{s-}(\omega') = A^n_{s-}(\omega)$ for $s \in]0,t]$. Therefore $\omega' \in \Xi_t$, $A_s(\omega') = A_s(\omega)$ for $s \in [0,t[$ and $A_{s-}(\omega') = A_{s-}(\omega)$ for $s \in]0,t]$. □

Recall that $\{\theta_t, t > 0\}$ denotes the time shift operators on the path space for the process X.

Lemma 2.15 *For $t, s > 0$,*

(i) *$\theta_t r_{t+s}\omega$ is s-equivalent to $r_s\omega$ if $t + s < \zeta(\omega)$ or $s \geq \zeta(\omega)$;*

(ii) *$r_t\theta_s\omega$ is pre-t-equivalent to $r_{t+s}\omega$. Moreover, if ω is continuous at s, then $r_t\theta_s\omega$ is t-equivalent to $r_{t+s}\omega$.*

Proof. (i): We may assume $t + s < \zeta(\omega)$. For $v \in [0, s]$,

$$\theta_t r_{t+s}\omega(v) = \omega((s-v)-) = r_s\omega(v)$$

and so $\theta_t r_{t+s}\omega$ is s-equivalent to $r_s\omega$.

(ii): Note that $t + s < \zeta(\omega)$ is equivalent to $t < \zeta(\theta_s\omega)$. It follows from the definition, if $t + s < \zeta(\omega)$,

$$(r_t\theta_s\omega)(v) = \begin{cases} \omega((t+s-v)-), & \text{if } 0 \leq v < t, \\ \omega(s), & \text{if } v = t, \end{cases} \tag{2.3}$$

while $r_{t+s}\omega(v) = \omega((t+s-v)-)$ for $0 \leq v \leq t$. Hence typically $r_t\theta_s\omega$ is only pre-t-equivalent to $r_{t+s}\omega$. □

Fix $t > 0$. Set $\mathcal{H}^t_s := \mathcal{G}_t$ for $s \in [0,t]$; and $\mathcal{H}^t_s := \mathcal{G}_s$ for $s \in]t,\infty[$. Then $(\mathcal{H}^t_s)_{s\geq 0}$ is a filtration over $(\Omega, \mathcal{F}_\infty)$, and $\mathcal{G}_s \subset \mathcal{H}^t_s$ for all $s \geq 0$.

Lemma 2.16 *The following assertions hold for any fixed $t > 0$:*

(i) *Let* φ *be a Borel function on* $E\times E$ *and set* $X_{0-}:=X_0$. *Then* $[0,\infty[\times\Omega\ni(s,\omega)\mapsto 1_{[\![0,\zeta_p[\![}(s,\omega)1_{\Gamma_t}(\omega)\varphi(X_{s-}(\omega),X_s(\omega))$ *is* $(\mathcal{H}_s^t)$*-optional for any* $\Gamma_t\in\mathcal{G}_t$.

(ii) *Let* A *be an rcll PrAF with its defining sets* $\Xi\in\mathcal{F}_\infty$, $\Xi_t\in\mathcal{G}_t$. *If we set* $A_{0-}(\omega):=0$ *and* $A_s^t(\omega):=1_{\Xi_t}(\omega)(1_{[0,t]}(s)A_s(\omega)+1_{]t,\infty[}(s)A_t(\omega))$ *for* $\omega\in\Omega$, *then* $[0,\infty[\times\Omega\ni(s,\omega)\mapsto 1_{[\![0,\zeta_p[\![}(s,\omega)(A_s^t(\omega)-A_{s-}^t(\omega))$ *is* $(\mathcal{H}_s^t)$*-optional.*

Proof. (i): Note that $1_{[\![0,\zeta_p[\![}$ is $(\mathcal{H}_s^t)$-predictable. The assertion is clear if $\varphi=f\otimes g$ for continuous functions f,g on E. The monotone class theorem for functions tells us the desired result.

(ii): Since A^t is $(\mathcal{H}_s^t)$-adapted and rcll on Ω and A_-^t is $(\mathcal{H}_s^t)$-adapted and lcrl on Ω, $(s,\omega)\mapsto A_s(\omega)$ is $(\mathcal{H}_s^t)$-optional and $(s,\omega)\mapsto A_{s-}^t(\omega)$ is $(\mathcal{H}_s^t)$-predictable. Consequently, $(s,\omega)\mapsto A_s^t(\omega)-A_{s-}^t(\omega)$ is $(\mathcal{H}_s^t)$-optional. □

By Lemma 3.2 of [3], for a finite rcll AF $A=(A_t)_{t\geq0}$, there is a Borel function $\varphi:E_\partial\times E_\partial\to\mathbb{R}$ with $\varphi(x,x)=\varphi(\partial,x)=0$ for all $x\in E_\partial$ such that

$$A_t-A_{t-}=\varphi(X_{t-},X_t),\qquad \text{for every } t\in]0,\zeta_p[,\quad \mathbf{P}_m\text{-a.s.}\tag{2.4}$$

Moreover, if $\widetilde{\varphi}$ is another such function, then $J^*(\varphi\neq\widetilde{\varphi})=0$. Here J^* denotes the measure $\frac{1}{2}N(x,dy)\mu_H(dx)$ on $E_\partial\times E_\partial$. We shall refer to such a function φ as a *jump function* for A. Recall that if $M\in\mathcal{M}_{\rm loc}^{I(\zeta)}$, then there exists a jump function φ so that $M_t-M_{t-}=\varphi(X_{t-},X_t)$, $t\in]0,\zeta_p[$, $\mathbf{P}_m$-a.s. Such φ is unique in the above sense.

Lemma 2.17 *Let* A *be a finite rcll PrAF with defining sets* $\{\Xi,\ \Xi_t,\ t\geq0\}$. *Then there exists a real valued Borel function* φ *on* $E_\partial\times E_\partial$ *with* $\varphi(x,x)=\varphi(\partial,x)=0$ *for* $x\in E_\partial$ *such that* A *with defining sets*

$$\begin{aligned}\widetilde{\widetilde{\Xi}} &:= \Big\{\omega\in\Xi\mid A_s(\omega)-A_{s-}(\omega)=\varphi(X_{s-}(\omega),X_s(\omega))\ \text{for } s\in]0,\zeta_p(\omega)[\ \Big\},\\ \widetilde{\widetilde{\Xi}}_t &:= \Big\{\omega\in\Xi_t\mid A_s(\omega)-A_{s-}(\omega)=\varphi(X_{s-}(\omega),X_s(\omega))\ \text{for } s\in]0,t[\cap]0,\zeta_p(\omega)[\ \Big\},\end{aligned}$$

is again an rcll PrAF admitting m*-null set. For* $M\in\mathcal{M}_{\rm loc}^{I(\zeta)}$, *we also have the same assertion.*

Proof. Let $\varphi:E_\partial\times E_\partial\to\mathbb{R}$ be a Borel function vanishing on the diagonal and define $\widetilde{\widetilde{\Xi}}$, $\widetilde{\widetilde{\Xi}}_t$ in terms of φ as above. Clearly, $\widetilde{\widetilde{\Xi}}=\bigcap_{t>0}\widetilde{\widetilde{\Xi}}_t$, $\widetilde{\widetilde{\Xi}}_t\subset\widetilde{\widetilde{\Xi}}_s$ for $s<t$. Moreover, we see that $\theta_t\widetilde{\widetilde{\Xi}}\subset\widetilde{\widetilde{\Xi}}$ for $t\geq0$, $\theta_{t-s}(\widetilde{\widetilde{\Xi}}_t)\subset\widetilde{\widetilde{\Xi}}_s$ for $s<t$. For two pre-t-equivalent paths ω,ω', we see that $\omega\in\widetilde{\widetilde{\Xi}}_t$ implies $\omega'\in\widetilde{\widetilde{\Xi}}_t$.

By the previous lemma,

$$\Gamma:=\{(s,\omega)\mid 1_{[\![0,\zeta_p[\![}(s,\omega)1_{\Xi_t}(\omega)(A_s^t(\omega)-A_{s-}^t(\omega)-\varphi(X_{s-}(\omega),X_s(\omega)))\neq0\}$$

is $(\mathcal{H}_s^t)$-progressively measurable for any fixed $t>0$ and the debut of Γ is

$$D_\Gamma(\omega):=\inf\{s\geq0\mid 1_{[\![0,\zeta_p[\![}(s,\omega)1_{\Xi_t}(\omega)(A_s^t(\omega)-A_{s-}^t(\omega)-\varphi(X_{s-}(\omega),X_s(\omega)))\neq0\},$$

which is an $(\mathcal{H}_s^t)$-stopping time by (A5.1) in [17]. In particular,

$$\{\omega\in\Omega \mid 1_{[\![0,\zeta_p[\![}(s,\omega)1_{\Xi_t}(\omega)(A_s(\omega)-A_{s-}(\omega)-\varphi(X_{s-}(\omega),X_s(\omega)))=0 \text{ for } s\in[0,t[\} \\ =\{\omega\in\Omega \mid t<D_\Gamma(\omega)\}\in\mathcal{H}_t^t=\mathcal{G}_t.$$

Hence,

$$\begin{aligned} & \{\omega\in\Xi_t \mid A_s(\omega)-A_{s-}(\omega)-\varphi(X_{s-}(\omega),X_s(\omega))=0 \text{ for } s\in]0,t[\cap]0,\zeta_p(\omega)[\} \\ = & \{\omega\in\Xi_t \mid A_s(\omega)-A_{s-}(\omega)-\varphi(X_{s-}(\omega),X_s(\omega))=0 \text{ for } s\in[0,t[\cap[0,\zeta_p(\omega)[\} \\ = & \{\omega\in\Xi_t \mid 1_{[\![0,\zeta_p[\![}(s,\omega)(A_s(\omega)-A_{s-}(\omega)-\varphi(X_{s-}(\omega),X_s(\omega)))=0 \text{ for } s\in[0,t[\} \\ \in & \ \mathcal{G}_t. \end{aligned}$$

Therefore, $\widetilde{\Xi}_t\in\mathcal{G}_t$ and $\widetilde{\Xi}\in\mathcal{F}_\infty$. The proof for the case $M\in\mathcal{M}_{\rm loc}^{I(\zeta)}$ is similar. We omit it. □

The following theorem is a key to our extension of Nakao's operator Γ. Its rigorous proof is pretty involved due to the measurability issues but the idea behind it is fairly transparent. We will use the convention $X_{0-}(\omega):=X_0(\omega)$.

Theorem 2.18 (Dual PrAF) *Let A be a finite rcll PrAF on $[0,\zeta[$ with defining sets Ξ, Ξ_t admitting m-null set. Suppose that there is a Borel function φ on $E_\partial\times E_\partial$ vanishing on the diagonal set with $\varphi(X_{s-},X_s)=A_s-A_{s-}$, $\forall s\in]0,t[\cap]0,\zeta[$ on Ξ_t. Set*

$$\widehat{A}_t(\omega):=A_t(r_t(\omega))+\varphi(X_t(\omega),X_{t-}(\omega)) \qquad \textit{for } t\in[0,\infty[.$$

Then $\widehat{A}$ is an rcll PrAF on $[0,\zeta[$ admitting m-null set such that

$$\widehat{A}_t=A_{t-}\circ r_t+\varphi(X_t,X_{t-}) \qquad \textit{and} \qquad \widehat{A}_t-\widehat{A}_{t-}=\varphi(X_t,X_{t-})$$

for all $t\in]0,\zeta[$, $\mathbf{P}_m$-a.s.

Proof. Let $\Xi\in\mathcal{F}_\infty$, $\Xi_t\in\mathcal{G}_t^m$, $t>0$ be the defining sets of A admitting m-null set. We easily see $r_t^{-1}(\Xi_t)\cap\{t<\zeta\}\subset r_s^{-1}(\Xi_s)\cap\{s<\zeta\}$ for $s\in]0,t[$ by use of Lemma 2.15(i) and $\theta_{t-s}\Xi_t\subset\Xi_s$.

Set $\widehat{\Xi}_t:=r_t^{-1}(\Xi_t)$ for $t>0$ and $\widehat{\Xi}:=\bigcap_{t>0}\widehat{\Xi}_t$. Then, we see $\widehat{\Xi}=\bigcap_{t>0,t\in\mathbb{Q}}\widehat{\Xi}_t$ by use of $r_t^{-1}(\Xi_t)\cap\{t\geq\zeta\}=\{t\geq\zeta\}$ and the monotonicity of $r_t^{-1}(\Xi_t)\cap\{t<\zeta\}$. Indeed, we have $\widehat{\Xi}\subset\bigcap_{t>0,t\in\mathbb{Q}}\widehat{\Xi}_t\subset\left(\widehat{\Xi}_s\cap\{s<\zeta\}\right)\cup\{t\geq\zeta\}$ for any $0<s<t$ with $t\in\mathbb{Q}$. Taking the intersection over $t\in]s,\infty[\cap\mathbb{Q}$, we have $\widehat{\Xi}\subset\bigcap_{t>0,t\in\mathbb{Q}}\Xi_t\subset\widehat{\Xi}_s$ for all $s>0$, which yields the assertion.

We prove $\theta_t\widehat{\Xi}\subset\widehat{\Xi}$ for each $t\geq0$, in particular, $\theta_t\widehat{\Xi}\subset\theta_s\widehat{\Xi}$, equivalently $\theta_s^{-1}\widehat{\Xi}\subset\theta_t^{-1}\widehat{\Xi}$ if $s\in[0,t]$. Suppose $\omega\in\widehat{\Xi}$. Then $r_{t+s}\omega\in\Xi_{t+s}$. If $t+s<\zeta(\omega)$, then $r_{t+s}\omega\in\Xi_s$, otherwise $r_{t+s}\omega=\omega_\partial\in\Xi_s$. Hence we have $r_s\theta_t\omega\in\Xi_s$ by Lemma 2.15(ii). Therefore $r_s\theta_t\omega\in\Xi_s$ for all $s>0$, which implies $\theta_t\omega\in\widehat{\Xi}$.

Next we prove $\theta_{t-s}(\widehat{\Xi}_t)\subset\widehat{\Xi}_s$ for $s\in]0,t[$. Take $\omega\in\widehat{\Xi}_t$. Then $r_s\theta_{t-s}\omega$ is pre-s-equivalent to $r_t\omega\in\Xi_t\subset\Xi_s$ by Lemma 2.15(ii), hence $r_s\theta_{t-s}\omega\in\Xi_s$. Therefore $\theta_{t-s}\omega\in\widehat{\Xi}_s$ for all $s\in]0,t[$.

From $\Xi_t \in \mathcal{G}_t \subset \mathcal{G}_t^m$, we get $\widehat{\Xi}_t \in \mathcal{G}_t^m$ by Lemma 2.11. Since $(\widehat{\Xi}_t)^c = r_t^{-1}((\Xi_t)^c) = r_t^{-1}((\Xi_t)^c) \cap \{t < \zeta\}$ holds by noting $\omega_\partial \in \Xi_t$, we have $\mathbf{P}_m((\widehat{\Xi}_t)^c) = \mathbf{P}_m((\widehat{\Xi})^c) = 0$.

By (2.2), $v \mapsto r_s(\omega)(v)$ is continuous at $v = s$. Hence, on $\widehat{\Xi}_t \cap \{t < \zeta\}$, we have $\varphi(X_{s-}, X_s) \circ r_s = \varphi(X_s, X_{s-}) \circ r_s = 0$, in particular, $A_s \circ r_s = A_{s-} \circ r_s$ for $s \in]0, t[$.

The remainder of the proof is devoted to showing that $\widehat{A}$ is an rcll PrAF on $[0, \zeta[$ with defining sets $\widehat{\Xi}$, $\widehat{\Xi}_t$ such that on $\widehat{\Xi}_t \cap \{t < \zeta\}$, $\widehat{A}_s = A_{s-} \circ r_s + \varphi(X_s, X_{s-})$, $s \in]0, t[$. First note that for $\omega \in \widehat{\Xi}$, $|\widehat{A}_t(\omega)| < \infty$ for any $t \in]0, \zeta(\omega)[$, because by taking $T \in]t, \zeta(\omega)[$, $r_T\omega \in \Xi_T$ implies $r_t\omega \in \Xi_t$, hence $|A_{t-}(r_t\omega)| < \infty$. Moreover, for $\omega \in \widehat{\Xi}_t \cap \{t < \zeta\}$, we see $r_s\omega \in \Xi_s \cap \{s < \zeta\}$ and $|A_{s-}(r_s\omega)| < \infty$ for all $0 < s < t$.

For two pre-t-equivalent paths $\omega, \omega' \in \Omega \cap \{t < \zeta\}$ with $t > 0$, we show $\omega \in \widehat{\Xi}_t$ implies $\omega' \in \widehat{\Xi}_t$ and $\widehat{A}_s(\omega) = \widehat{A}_s(\omega')$ for $s \in [0, t[$. Recall $\omega \in \widehat{\Xi}_t \cap \{t < \zeta\} \subset \widehat{\Xi}_s \cap \{s < \zeta\}$ for $s \in [0, t]$ and note that ω and ω' are s-equivalent for any $s \in [0, t[$. On the other hand, $s < \zeta(\omega)$ is equivalent to $s < \zeta(\omega')$ for any $s \in [0, t[$. Then we see $r_s\omega \in \Xi_s$ is s-equivalent to $r_s\omega'$ for any $s \in [0, t]$, which implies $r_s\omega' \in \Xi_s$ for any $]0, t]$ and $A_{s-}(r_s\omega) = A_{s-}(r_s\omega')$ for any $s \in [0, t]$.

Fix $t > 0$. On $\widehat{\Xi}_t \cap \{t < \zeta\}$ and for any $p, q > 0$ with $p + q < t$, by Lemma 2.15,

$$\begin{aligned}
\widehat{A}_{p+q} &= A_{(p+q)-} \circ r_{p+q} + \varphi(X_{p+q}, X_{(p+q)-}) \\
&= (A_p + A_{q-} \circ \theta_p) \circ r_{p+q} + \varphi(X_{p+q}, X_{(p+q)-}) \\
&= A_p \circ r_{p+q} + A_{q-} \circ \theta_p \circ r_{p+q} + \varphi(X_{p+q}, X_{(p+q)-}) \\
&= (A_{p-} \circ r_{p+q} + \varphi(X_{p-}, X_p) \circ r_{p+q}) + A_{q-} \circ r_q + \varphi(X_{p+q}, X_{(p+q)-}) \\
&= (A_{p-} \circ r_p \circ \theta_q + \varphi(X_q, X_{q-})) + \left(\widehat{A}_q - \varphi(X_q, X_{q-})\right) + \varphi(X_{p+q}, X_{(p+q)-}) \\
&= \left(\widehat{A}_p - \varphi(X_p, X_{p-})\right) \circ \theta_q + \widehat{A}_q + \varphi(X_{p+q}, X_{(p+q)-}) \\
&= \widehat{A}_p \circ \theta_q + \widehat{A}_q.
\end{aligned}$$

On $\widehat{\Xi}_t \cap \{t < \zeta\}$, again by Lemma 2.15 and (2.2), for any $s > 0$ and $u \in]0, s[$,

$$\begin{aligned}
\widehat{A}_s - \widehat{A}_{s-u} &= \widehat{A}_u \circ \theta_{s-u} \\
&= (A_{u-} \circ r_u + \varphi(X_u, X_{u-})) \circ \theta_{s-u} \\
&= A_{u-} \circ r_u \circ \theta_{s-u} + \varphi(X_s, X_{s-}) \\
&= A_{u-} \circ r_s + \varphi(X_s, X_{s-}).
\end{aligned}$$

So

$$\lim_{u \downarrow 0} (\widehat{A}_s - \widehat{A}_{s-u}) = \varphi(X_s, X_{s-}).$$

This shows that $\widehat{A}$ has left limit at $s \in]0, t[$ and $\widehat{A}_s - \widehat{A}_{s-} = \varphi(X_s, X_{s-})$.

To show the right continuity of $\widehat{A}$ on $\widehat{\Xi}_t \cap \{t < \zeta\}$ at any $s \in]0, t[$, note for any $u \in]0, t - s[$, by

Lemma 2.15 and (2.2),

$$\begin{aligned}\widehat{A}_{s+u}-\widehat{A}_s &= \widehat{A}_u\circ\theta_s\\ &= (A_{u-}\circ r_u+\varphi(X_u,X_{u-}))\circ\theta_s\\ &= A_{u-}\circ r_u\circ\theta_s+\varphi(X_{s+u},X_{(s+u)-})\\ &= A_{u-}\circ r_{s+u}+\varphi(X_{s+u},X_{(s+u)-}).\end{aligned}$$

Since $(A_v-A_{v-})\circ r_{s+v}=\varphi(X_{v-},X_v)\circ r_{s+v}=\varphi(X_s,X_{s-})$, while by Lemma 2.15 and (2.2),

$$(A_v-A_{v-})\circ r_{s+v}=\lim_{u\downarrow 0}(A_v-A_{v-u})\circ r_{s+v}=\lim_{u\downarrow 0}A_u\circ\theta_{v-u}\circ r_{s+v}=\lim_{u\downarrow 0}A_{u-}\circ r_{v+u}+\varphi(X_s,X_{s-}).$$

we conclude that

$$\lim_{u\downarrow 0}A_{u-}\circ r_{s+u}=0.$$

On the other hand, for any $s\geq 0$

$$\begin{aligned}\lim_{u\downarrow 0}\varphi(X_{s+u},X_{(s+u)-}) &= \lim_{u\downarrow 0}\varphi(X_{(v-u)-},X_{v-u})\circ r_{s+v}=\lim_{u\downarrow 0}(A_{v-u}-A_{(v-u)-})\circ r_{s+v}\\ &= (A_{v-}-A_{v-})\circ r_{s+v}=0.\end{aligned}$$

Hence we have for $s>0$

$$\lim_{u\downarrow 0}(\widehat{A}_{s+u}-\widehat{A}_s)=0.$$

In other words, $\widehat{A}$ is right continuous at any $s\in]0,t[$ on $\widehat{\Xi}_t\cap\{t<\zeta\}$. We also see

$$\lim_{u<s,s\downarrow 0,u\downarrow 0}(\widehat{A}_{s+u}-\widehat{A}_s)=0.$$

Thus we can define the limit $\widehat{A}_0(\omega):=\lim_{s\downarrow 0}\widehat{A}_s(\omega)$ for $\omega\in\widehat{\Xi}_t\cap\{t<\zeta\}$ for any $t>0$. We also see $\widehat{A}_0(\omega)=\lim_{s\downarrow 0}\widehat{A}_{s-}(\omega)$ for $\omega\in\widehat{\Xi}_t\cap\{t<\zeta\}$ for any $t>0$, because $\lim_{s\downarrow 0}\varphi(X_s,X_{s-})=0$. Next we prove $\widehat{A}_0(\omega)=0$ for $\omega\in\widehat{\Xi}_t\cap\{t<\zeta\}$ for any $t>0$. Take $\omega\in\widehat{\Xi}_t\cap\{t<\zeta\}$ for some fixed $t>0$. It suffices to show that $\lim_{u\downarrow 0}\widehat{A}_{s-u}(\theta_u\omega)=\widehat{A}_s(\omega)$ for $s\in[0,t[$. Owing to Lemma 2.15(ii), we have

$$\begin{aligned}\widehat{A}_{s-u}(\theta_u\omega) &= A_{(s-u)-}(r_{s-u}\theta_u\omega)+\varphi(X_s(\omega),X_{s-}(\omega))\\ &= A_{(s-u)-}(r_s\omega)+\varphi(X_s(\omega),X_{s-}(\omega))\\ &= A_{s-u}(r_s\omega)-\varphi(X_u(\omega),X_{u-}(\omega))+\varphi(X_s(\omega),X_{s-}(\omega))\\ &= A_{s-u}(r_s\omega)-\widehat{A}_u(\omega)+\widehat{A}_{u-}(\omega)+\varphi(X_s(\omega),X_{s-}(\omega))\\ &\to A_{s-}(r_s\omega)+\varphi(X_s(\omega),X_{s-}(\omega))\quad\text{as } u\downarrow 0\\ &= \widehat{A}_s(\omega).\end{aligned}$$

Finally, we show the $\mathcal{G}^m_{s+}$-measurability of $\widehat{A}_s$. The argument is quite similar to one used by Walsh [18], but we provide the details for the convenience of the reader. First there exists an $(\mathcal{F}^0_{t+})$-adapted process B such that A and B are $\mathbf{P}_m$-indistinguishable. Indeed, since A is also $(\mathcal{F}^m_t)$-adapted and $\mathcal{F}^m_t=\mathcal{F}^m_{t+}$ is the $\mathbf{P}_m$-completion of $\mathcal{F}^0_{t+}$ in $\mathcal{F}^m_\infty$, for each rational $t>0$, we can take $\mathcal{F}^0_{t+}$-measurable B_t such that B_t is a $\mathbf{P}_m$-modification of A_t. For general $t > 0$, $B_t := \underline{\lim}_{s\in\mathbb{Q},s\downarrow t} B_s$ satisfies the condition. For each $s_0 > 0$, we set $D_{s_0} := \{\omega \in \widehat{\Xi}_{s_0} \mid B_u(\omega) \neq A_u(\omega) \text{ for } {}^\exists u \in [0, s_0[\}$. Then $D_{s_0} \in \mathcal{G}^m_{s_0}$ and $\mathbf{P}_m(D_{s_0}) = 0$, because $D_{s_0} = \{\omega \in \widehat{\Xi}_{s_0} \mid B_u(\omega) \neq A_u(\omega) \text{ for } {}^\exists u \in [0, s_0[\cap\mathbb{Q}\}$. Hence $C_{s_0} := r^{-1}_{s_0}(D_{s_0}) \in \mathcal{G}^m_{s_0}$ and $\mathbf{P}_m(C_{s_0} \cap \{s_0 < \zeta\}) = 0$. Fix $t > s$. By Lemma 2.11, we have that for $s_0 \in]0, s[$, $\theta^{-1}_{s-s_0}(C_{s_0} \cap \{s_0 < \zeta\})$ is a $\mathcal{G}^m_s$-measurable $\mathbf{P}_m$-null set. By Lemma 2.15(ii), $r_{s_0}\theta_{s-s_0}\omega$ is pre-s_0-equivalent to $r_s\omega$, so we see that $\theta^{-1}_{s-s_0}(C_{s_0} \cap \{s_0 < \zeta\}) = \{\omega \in \theta^{-1}_{s-s_0}\widehat{\Xi}_{s_0} \mid s < \zeta(\omega),\ B_u(r_s\omega) \neq A_u(r_s\omega) \text{ for } {}^\exists u \in [0, s_0[\}$. Since $\widehat{\Xi}_s \subset \theta^{-1}_{s-s_0}(\widehat{\Xi}_{s_0})$, we have that $\widehat{\Xi}_s \cap \{s < \zeta\} \cap \{B_u(r_s\omega) \neq A_u(r_s\omega) \text{ for } {}^\exists u \in [0, s_0[\}$ is a $\mathcal{G}^m_s$-measurable $\mathbf{P}_m$-null set for any $s \in [0, t[$. Then $\widehat{\Xi}_s \cap \{s < \zeta\} \cap \{B_u(r_s\omega) \neq A_u(r_s\omega) \text{ for } {}^\exists u \in [0, s[\}$, hence, $\widehat{\Xi}_s \cap \{s < \zeta\} \cap \{A_{s-} \circ r_s \neq \underline{\lim}_{u\in\mathbb{Q},u\uparrow s} B_u \circ r_s\}$ is a $\mathcal{G}^m_s$-measurable $\mathbf{P}_m$-null set for any $s \in [0, t[$. Thus, $\widehat{\Xi}_t \cap \{t < \zeta\} \cap \{A_s \circ r_s = A_{s-} \circ r_s \neq \underline{\lim}_{u\in\mathbb{Q},u\uparrow s} B_u \circ r_s\}$ is a $\mathcal{G}^\nu_t$-measurable $\mathbf{P}_\nu$-null set for any $s \in [0, t[$. Hence, $\{u < \zeta\} \cap \{A_s \circ r_s \neq \underline{\lim}_{u\in\mathbb{Q},u\uparrow s} B_u \circ r_s\}$ is so for $s < u < t$, consequently, $\{s < \zeta\} \cap \{A_s \circ r_s \neq \underline{\lim}_{u\in\mathbb{Q},u\uparrow s} B_u \circ r_s\}$ is a $\mathcal{G}^m_{s+}$-measurable $\mathbf{P}_m$-null set. Then for any $D \in \mathcal{B}(\overline{\mathbb{R}})$, $\{s < \zeta\} \cap (A_s \circ r_s)^{-1}(D) \in \mathcal{G}^m_{s+}$ and $\{s \geq \zeta\} \cap (A_s \circ r_s)^{-1}(D) = \{s \geq \zeta\} \cap \{0 \in D\} \in \mathcal{G}^m_{s+}$, which implies the $\mathcal{G}^m_{s+}$-measurability of $A_s \circ r_s$. We have the desired assertion. This proves the theorem. □

3 Stochastic integral for Dirichlet processes

Recall that any locally square-integrable MAF M on $I(\zeta)$ admits a jump function ϕ on $E_\partial \times E_\partial$ vanishing on the diagonal such that $\Delta M_t = \phi(X_{t-}, X_t)$, $t \in]0, \zeta_p[$ $\mathbf{P}_m$-a.s. When $M \in \overset{\circ}{\mathcal{M}}$, we can strengthen this statement by replacing $]0, \zeta_p[$ with $]0, \infty[$ in view of Fukushima's decomposition and the combination of Theorem 5.2.1 and Lemma 5.6.3 in [7].

Lemma 3.1 *Let ϕ be a Borel function on $E \times E$ vanishing on the diagonal. Suppose that*

$$N(1_{E\times E}(|\phi|^2 \wedge |\phi|))\mu_H \in S.$$

Then there exists a purely discontinuous local MAF K on $I(\zeta)$ such that $K_t - K_{t-} = \phi(X_{t-}, X_t)$ for all $t < \zeta$, $\mathbf{P}_x$-a.s. for q.e. $x \in E$.

Proof. The hypothesis implies that the compensated process

$$K^{(2)}_t := \sum_{0<s\leq t} \phi(X_{s-}, X_s) \cdot 1_{\{|\phi(X_{s-},X_s)|>1\}} 1_{\{s<\zeta\}} - \int_0^t \int_E N(X_s, dy)\phi(X_s, y) \cdot 1_{\{|\phi(X_s,y)|>1\}}\, dH_s$$

is a local MAF on $I(\zeta)$. Indeed, we can construct a nice $\mathcal{E}$-nest $\{F_k\}$ with $1_{F_k} N(1_{E\times E}|\phi|1_{\{|\phi|>1\}})\mu_H \in S_{00}$ for each $k \in \mathbb{N}$, which implies $\int_0^t \int_E 1_{F_k}(X_s)|\phi|(X_s, y)1_{\{|\phi(X_s,y)|>1\}} N(X_s, dy)dH_s$ is $\mathbf{P}_x$-integrable for q.e. $x \in E$, hence $A^k_t := \sum_{0<s\leq t} 1_{F_k}(X_{s-})\phi(X_{s-}, X_s) \cdot 1_{\{|\phi(X_{s-},X_s)|>1\}} 1_{\{s<\zeta\}}$ is so. Here S_{00} denotes the family of finite measures of finite energy with bounded 1-potentials (see (2.2.10) in [7]). By putting $A_t := \sum_{0<s\leq t} \phi(X_{s-}, X_s) \cdot 1_{\{|\phi(X_{s-},X_s)|>1\}} 1_{\{s<\zeta\}}$, we have $A_{t\wedge\tau_{F_k}} = A^k_{t\wedge\tau_{F_k}}$, hence

$A_{t\wedge\tau_{F_k}} 1_{I(\zeta)}(t\wedge\tau_{F_k}) = A^k_{t\wedge\tau_{F_k}} 1_{I(\zeta)}(t\wedge\tau_{F_k})$, which implies that A is an $(\mathcal{F}_t)$-adapted process with locally integrable variation on $I(\zeta)$. Then by Theorem 8.26 in [9], we have the desired assertion. We also have that

$$A_t := \int_0^t \int_E N(X_s,dy)[\phi(X_s,y)]^2 \cdot 1_{\{|\phi(X_s,y)|\le 1\}}\, dH_s$$

is a PCAF.

Now if L is a locally square-integrable MAF with jump function φ, the formula

$$\Phi(L)_t := \int_0^t \int_E \phi(X_s,y)\cdot 1_{\{|\phi(X_s,y)|\le 1\}}\varphi(X_s,y)\, N(X_s,dy)\, dH_s$$

defines a (signed) CAF locally of finite variation, and

$$[\Phi(L)_t]^2 \le \langle L\rangle_t \cdot A_t, \qquad 0<t<\zeta,$$

by the Cauchy-Schwarz inequality. A result of Kunita (Proposition 2.4 in [11]) now tells us that there is a local MAF $K^{(1)}$ such that $\Phi(L)\equiv\langle K^{(1)},L\rangle$ for all L. The local MAF $K:=K^{(1)}+K^{(2)}$ on $I(\zeta)$ does the job. □

Definition 3.2 Let M be a locally square-integrable MAF on $I(\zeta)$ with jump function φ. Assume that for q.e. $x\in E$, $\mathbf{P}_x$-a.s.

$$\int_0^t \int_E \left(\widehat{\varphi}^2 1_{\{|\widehat{\varphi}|\le 1\}} + |\widehat{\varphi}| 1_{\{|\widehat{\varphi}|>1\}}\right)(X_s,y)N(X_s,dy)dH_s < \infty \qquad \text{for every } t<\zeta, \tag{3.1}$$

where $\widehat{\varphi}(x,y):=\varphi(x,y)+\varphi(y,x)$. Define, $\mathbf{P}_m$-a.s. on $[0,\zeta[$,

$$\Lambda(M)_t := -\frac{1}{2}\left(M_t + M_t\circ r_t + \varphi(X_t,X_{t-}) + K_t\right) \qquad \text{for } t\in[0,\,\zeta[, \tag{3.2}$$

where K_t is the purely discontinuous local MAF on $I(\zeta)$ with

$$K_t - K_{t-} = -\widehat{\varphi}(X_{t-},X_t) \quad \text{for every } t<\zeta, \quad \mathbf{P}_x\text{-a.s.} \tag{3.3}$$

for q.e. $x\in E$.

Remark 3.3 (i) The condition (3.1) is nothing but $N(1_{E\times E}(|\widehat{\varphi}|^2\wedge|\widehat{\varphi}|))\mu_H\in S$. In particular, condition (3.1) is satisfied by the jump function of any element of $\overset{\circ}{\mathcal{M}}$.

(ii) It follows from Theorem 2.18 that $t\mapsto\Lambda(M)_t$ is continuous on $[0,\zeta[$. In view of Theorem 2.18, it is then clear from the definition that Λ is a linear operator that maps locally square-integrable MAFs on $[\![0,\zeta[\![$ with (3.1) into even CAFs on $[0,\zeta[$ admitting m-null set, that is, $\Lambda(M)_t=\Lambda(M)_t\circ r_t$ $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$. Indeed, $-K_t := \sum_{s\le t}\hat{\varphi}(X_{s-},X_s)1_{\{s<\zeta\}} - \int_0^t\int_E \hat{\varphi}(X_s,y)N(X_s,dy)dH_s$ $t<\zeta$ satisfies $K_t=K_t\circ r_t$ $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$ for fixed $t>0$.

(iii) If $\{M^n, n \geq 1\}$ is a sequence of MAFs having finite energy and converging in probability to M, then it is easy to see that $M_t^n \circ r_t$, $\varphi^n(X_{t-}, X_t) = M_t^n - M_{t-}^n$ and $\varphi^n(X_t, X_{t-})$ converge to $M_t \circ r_t$, $\varphi(X_{t-}, X_t) = M_t - M_{t-}$ and $\varphi(X_t, X_{t-})$ in probability, respectively, under $\mathbf{P}_m$. Hence we have $\Lambda(M^n)_t$ converges to $\Lambda(M)_t$ in probability for each $t > 0$.

(iv) For $u \in \mathcal{F}$,

$$\Lambda(M^u)_t = -\frac{1}{2}\left(M_t^u + M_t^u \circ r_t + u(X_{t-}) - u(X_t)\right) = -\frac{1}{2}\left(M_t^u + M_t^u \circ r_t\right) = N_t^u$$

$\mathbf{P}_m$-a.s. on $\{t < \zeta\}$ for each fixed $t \geq 0$. Since both $\Lambda(M^u)_t$ and N_t^u are continuous in t, we have $\mathbf{P}_m$-a.s.

$$\Lambda(M^u)_t = N_t^u \quad \text{for all } t < \zeta.$$

In other words, $\Lambda(M)$ coincides on $[0, \zeta[$ with $\Gamma(M)$ defined in (1.5) with $M = M^u$ for $u \in \mathcal{F}$.

We are going to show that $\Lambda(M)$ defined above coincides on $[0, \zeta[$ with $\Gamma(M)$ defined in (1.5) by Nakao when M is an MAF of finite energy. An AF Z is called *even* (resp. *odd*) if and only if $Z_t \circ r_t = Z_t$ (resp. $Z_t \circ r_t = -Z_t$) $\mathbf{P}_m$-a.s. on $\{t < \zeta\}$ for each $t > 0$. For a rcll process Z with $Z_0 = 0$ and $T > 0$, we define

$$R_T Z_t := (R_T Z)_t := Z_{T-} - Z_{(T-t)-} \quad \text{for } 0 \leq t \leq T,$$

with the convention $Z_{0-} = Z_0 = 0$. Note that $R_T Z_t$ so defined is an rcll process in $t \in [0, T]$.

Lemma 3.4 *Suppose that Z is an rcll PrAF. Then $\mathbf{P}_m$-a.s. on $\{T < \zeta\}$,*

$$R_T Z_t = \begin{cases} Z_t \circ r_T, & \text{if } Z \text{ is even} \\ -Z_t \circ r_T, & \text{if } Z \text{ is odd} \end{cases} \quad \text{for every } t \in [0, T]. \tag{3.4}$$

Proof. Let Z be an rcll PrAF and let $T > 0$. By Lemma 2.15,

$$Z_t \circ r_T = (Z_T - Z_{T-t} \circ \theta_t) \circ r_T = Z_T \circ r_T - Z_{T-t} \circ r_{T-t} \quad \text{for all } t < T. \tag{3.5}$$

When Z is even,

$$Z_t \circ r_T = Z_T - Z_{T-t} = Z_{T-} - Z_{(T-t)-} = R_T Z_t$$

$\mathbf{P}_m$-a.s. on $\{T < \zeta\}$ for each fixed $0 \leq t < T$. Since both sides are right continuous in $t \in [0, T[$, we have $\mathbf{P}_m$-a.s. $R_T Z_t = Z_t \circ r_T$ for every $t \in [0, T]$. When Z is an odd AF of Z, (3.4) can be proved similarly. □

Theorem 3.5 *For an MAF M of finite energy, $\Lambda(M)$ defined above coincides with $\Gamma(M)$ defined in (1.5), $\mathbf{P}_m$-a.s. on $[0, \zeta[$.*

Proof. For $u \in \mathcal{F}$ and $0 < t < T$, since N^u is an even CAF, by Lemma 3.4,

$$\begin{aligned}(M_t^u + 2N_t^u) \circ r_T &= (u(X_t) - u(X_0) + N_t^u) \circ r_T \\ &= u(X_{(T-t)-}) - u(X_{T-}) + N_{T-}^u - N_{(T-t)-}^u \\ &= M_{(T-t)-}^u - M_{T-}^u \\ &= -R_T M_t^u.\end{aligned}$$

Since both $(M_t^u + 2N_t^u) \circ r_T$ and $R_T M_t^u$ are right continuous in t, we have $\mathbf{P}_m$-a.s. on $\{T < \zeta\}$,

$$R_T M_t^u = -(M_t^u + 2N_t^u) \circ r_T \quad \text{for every } t \in [0, T]. \tag{3.6}$$

For $u \in \mathcal{D}(\mathcal{L}) \subset \mathcal{F}$ and $v \in \mathcal{F}_b$, define $M_t = \int_0^t v(X_{s-}) dM_s^u$, which is a MAF of finite energy. Note that, since $u \in \mathcal{D}(\mathcal{L})$, $N_t^u = \int_0^t \mathcal{L}u(X_s) ds$ is a continuous process of finite variation. For each fixed $0 < t < T$ and $n \geq 1$, define $t_i = it/n$ and $s_i = T - t + t_i$. Using the standard Riemann-sum approximation of the Itô integral and of the covariance process $[M^v, M^u]$, we have $\mathbf{P}_m$-a.s. on $\{T < \zeta\}$

$$\begin{aligned}& M_T - M_{T-t} + [M^v, M^u]_T - [M^v, M^u]_{T-t} \\ = & \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} v(X_{s_i}) \left(M_{s_{i+1}}^u - M_{s_i}^u \right) + \left(M_{s_{i+1}}^v - M_{s_i}^v \right) \left(M_{s_{i+1}}^u - M_{s_i}^u \right) \right) \\ = & \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} v(X_{s_{i+1}}) \left(M_{s_{i+1}}^u - M_{s_i}^u \right) - \left(N_{s_{i+1}}^v - N_{s_i}^v \right) \left(M_{s_{i+1}}^u - M_{s_i}^u \right) \right) \\ = & \lim_{n\to\infty} \sum_{i=0}^{n-1} v(X_{s_{i+1}}) \left(M_{s_{i+1}}^u - M_{s_i}^u \right) \\ = & \lim_{n\to\infty} \sum_{i=0}^{n-1} v(X_{T-t+t_i}) \left(R_T M_{t-t_i}^u - R_T M_{t-t_{i+1}}^u \right) \\ = & \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} v(X_{t-t_{i+1}}) \left(M_{t-t_{i+1}}^u - M_{t-t_i}^u + 2N_{t-t_{i+1}}^u - 2N_{t-t_i}^u \right) \right) \circ r_T \\ = & -\left(\int_0^t v(X_{s-}) d(M_s^u + 2N_s^u) \right) \circ r_T,\end{aligned}$$

where in the third equality we used the fact that N^u has zero energy, while in the second to the last equality we used (3.6). Note that the stochastic integral involving N^u in the last equality is just the Lebesgue-Stieltjes integral since N^u is of finite variation. Note also that $X_t = X_{t-}$ a.s. for each fixed $t > 0$. So we have for each fixed $t < T$, $\mathbf{P}_m$-a.s on $\{T < \zeta\}$,

$$R_T M_t + R_T [M^v, M^u]_t = -\left(\int_0^t v(X_{s-}) d(M_s^u + 2N_s^u) \right) \circ r_T.$$

Since both sides are right continuous in $t \in [0, T]$, we have $\mathbf{P}_m$-a.s. on $\{T < \zeta\}$,

$$R_T M_t + R_T [M^v, M^u]_t = -\left(\int_0^t v(X_{s-}) d(M_s^u + 2N_s^u) \right) \circ r_T \quad \text{for every } t \in [0, T]. \tag{3.7}$$

By Theorem 3.1 and (3.8) in Nakao [16],

$$\int_0^t v(X_{s-})dN_s^u = \int_0^t v(X_s)\circ dN_s^u = \Gamma(M)_t - \frac{1}{2}\langle M^{v,c}+M^{v,j},\, M^{u,c}+M^{u,j}\rangle_t.$$

It follows that $\mathbf{P}_m$-a.s. on $\{T<\zeta\}$,

$$\begin{aligned}
&R_T M_t + R_T[M^v, M^u]_t \\
&= -\big(M_t + 2\Gamma(M)_t - \langle M^{v,c}+M^{v,j},\, M^{u,c}+M^{u,j}\rangle_t\big)\circ r_T \\
&= -\big(M_t + 2\Gamma(M)_t - \langle M^{v,c},\, M^{u,c}\rangle_t - \langle M^{v,j},\, M^{u,j}\rangle_t\big)\circ r_T \quad \text{for all } t\le T.
\end{aligned}$$

Recall that

$$\begin{aligned}
[M^v,\, M^u]_t &= \langle M^{v,c},\, M^{u,c}\rangle_t + \sum_{s\le t}(M_s^v - M_{s-}^v)(M_s^u - M_{s-}^u) \\
&= \langle M^{v,c},\, M^{u,c}\rangle_t + \sum_{s\le t}(v(X_s)-v(X_{s-}))(u(X_s)-u(X_{s-})).
\end{aligned}$$

Taking $t=T$ and noting that both $\Gamma(M)$ and $\langle M^{v,c},\, M^{u,c}\rangle$ are continuous even AFs, we have from above that $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$,

$$\Gamma(M)_t = -\frac{1}{2}\left(M_t + M_t\circ r_t + v(X_t)(u(X_{t-})-u(X_t)) + K_t\right),$$

where

$$K_t = \sum_{s\le t}(v(X_s)-v(X_{s-}))(u(X_s)-u(X_{s-})) - \langle M^{v,j},\, M^{u,j}\rangle_t$$

is the purely discontinuous MAF with $K_t - K_{t-} = (v(X_t)-v(X_{t-}))(u(X_t)-u(X_{t-}))$. Observe that $M_t - M_{t-} = \varphi(X_{t-}, X_t)$, where $\varphi(x,y) = v(x)(u(y)-u(x))$, and that

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}).$$

This shows that $\Gamma(M)_t = \Lambda(M)_t$ $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$ for each fixed $t\ge 0$. Since both processes are continuous in $t\in[0,\,\zeta[$, we have $\mathbf{P}_m$-a.s.

$$\Gamma(M) = \Lambda(M) \quad \text{ on } [0,\,\zeta[$$

for an MAF M of the form $M_t = \int_0^t v(X_{s-})dM_s^u$ with $u\in\mathcal{D}(\mathcal{L})$ and $v\in\mathcal{F}_b$. By Lemma 5.4.5 in [6], such MAFs form a dense subset in the space of MAFs having finite energy. Thus by Lemma 3.1 in Nakao [16] and Remark 3.3(iii) we have for a general MAF M of finite energy, $\mathbf{P}_m$-a.s. $\Gamma(M)_t = \Lambda(M)_t$ $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$ for every fixed $t\ge 0$. Since both processes are continuous in $t\in[0,\,\zeta[$, it follows that $\Gamma(M)=\Lambda(M)$ on $[0,\,\zeta[$ $\mathbf{P}_m$-a.s. □

Theorem 3.6 *Let M be a locally square-integrable MAF on $I(\zeta)$ with jump function φ. Suppose that φ satisfies condition (3.1). Then for every $t > 0$, $\mathbf{P}_m$-a.s. on $\{t < \zeta\}$,*

$$\lim_{n\to\infty}\sum_{\ell=0}^{n-1}\left(\Lambda(M)_{(\ell+1)t/n}-\Lambda(M)_{\ell t/n}\right)^2=0, \tag{3.8}$$

where the convergence is in probability with respect to $\mathbf{P}_x$ for m-a.e. $x \in E$.

Proof. By (1.5) and Theorem 3.5, (3.8) clearly holds when M is a MAF of finite energy. For a locally square-integrable MAF M on $I(\zeta)$, there is an $\mathcal{E}$-nest $\{F_k\}$ of closed sets such that $1_{F_k} * M \in \overset{\circ}{\mathcal{M}}$ for each $k \geq 1$ in view of the proof of Proposition 2.8 and so (3.8) holds with $1_{F_k} * M$ in place of M. For each fixed $k \geq 1$,

$$\Lambda(M)_t = \Lambda(1_{F_k} * M)_t - \frac{1}{2}K_t^k \quad \mathbf{P}_m\text{-a.s. on } [0, \tau_{F_k}[,$$

where K_t^k is a purely discontinuous local MAF on $I(\zeta)$ with $K_t^k - K_{t-}^k = 1_{F_k^c}(X_{t-})\varphi(X_{t-}, X_t) + 1_{F_k^c}(X_t)\varphi(X_t, X_{t-})$, $t < \zeta$. Since $1_{F_k} * M \in \overset{\circ}{\mathcal{M}}$, we have

$$\int_E N(1_{F_k\times E}\varphi^2)d\mu_H = \int_E N(1_{E\times F_k}\overline{\varphi}^2)d\mu_H < \infty.$$

Consequently, by Lemma 3.1, we have the existence of purely discontinuous local MAF on $I(\zeta)$ with jumps given by $1_{F_k}(X_{t-})\varphi(X_{t-}, X_t) + 1_{F_k}(X_t)\varphi(X_t, X_{t-})$, $t < \zeta$. So we obtain the existence of such K_t^k. Since the square bracket of K^k is given by $\sum_{s\leq t} 1_{F_k^c}(X_{s-})\varphi^2(X_{s-}, X_s) + 1_{F_k^c}(X_s)\varphi^2(X_s, X_{s-})$ and it vanishes at $t < \tau_{F_k}$, we have for each fixed $t > 0$,

$$\lim_{n\to\infty}\sum_{\ell=0}^{n-1}\left(\Lambda(M)_{(\ell+1)t/n}-\Lambda(M)_{\ell t/n}\right)^2=0 \qquad \mathbf{P}_m\text{-a.s. on } \{t < \tau_{F_k}\}.$$

Passing to the limit as $k \uparrow \infty$ establishes (3.8). □

Before introducing the stochastic integrals against $\Lambda(M)$ as integrator, we prepare the following lemma for later use.

Lemma 3.7 *The following assertions hold.*

(i) *Let $\{G_n\}$ be an increasing sequence of finely open Borel sets. Then the following are equivalent.*

   (a) *$\{G_n\}$ is a nest, that is, $\mathbf{P}_x(\lim_{n\to\infty}\sigma_{E\setminus G_n}\wedge\zeta = \zeta) = 1$ for q.e. $x \in E$.*

   (b) *$E = \bigcup_{n=1}^{\infty} G_n$ q.e.*

   (c) *$\mathbf{P}_x(\lim_{n\to\infty}\sigma_{E\setminus G_n} = \infty) = 1$ for m-a.e. $x \in E$.*

   (d) *$\mathbf{P}_x(\lim_{n\to\infty}\sigma_{E\setminus G_n} = \infty) = 1$ for q.e. $x \in E$.*

(ii) *For a function f on E, $f \in \mathcal{F}_{\rm loc}$ if and only if there exist an $\mathcal{E}$-nest $\{F_k\}$ of closed sets and $\{f_k \mid k \in \mathbb{N}\} \subset \mathcal{F}_b$ such that $f = f_k$ q.e. on F_k.*

Proof. (i): For the implications (ia)$\Longleftrightarrow$(ib), see Theorem 4.6 in [13]. The implication (id)$\Longleftrightarrow$(ia) is clear. Next we show (ib)$\Longrightarrow$(ic). Since each G_n is finely open, it is strictly $\mathcal{E}$-quasi-open by Theorem 4.6.1(i). So there exists a common strictly $\mathcal{E}$-nest $\{A_\ell\}$ of closed sets such that $(E\setminus G_n)\cap A_\ell$ is closed for all $n, \ell \in \mathbb{N}$. (See Chapter V. §2 in [15] for the strict $\mathcal{E}$-quasi-notions.) Set $\sigma := \lim_{n\to\infty} \sigma_{E\setminus G_n}$. We then have that for all $n \in \mathbb{N}$ $X_{\sigma_{E\setminus G_n}} \in E \setminus G_n$ $\mathbf{P}_x$-a.s. on $\{\sigma < \infty\}$ for q.e. $x \in E$. By Chapter V. Lemma 2.21 in [15], we have $\mathbf{P}_x(\lim_{\ell\to\infty} \sigma_{E\setminus A_\ell} = \infty) = 1$ q.e. $x \in E$. Since $\sigma < \infty$ implies $\sigma < \sigma_{E\setminus A_{\ell_0}}$ for some $\ell_0 \in \mathbb{N}$, we may assume that there exists $\ell_0 \in \mathbb{N}$ such that $\sigma_{E\setminus G_n} < \sigma_{E\setminus A_\ell}$ for all $n > \ell \geq \ell_0$, $\mathbf{P}_x$-a.s. on $\{\sigma < \infty\}$. This means

$$\begin{aligned}
\mathbf{P}_x(\sigma < \infty) &\leq \mathbf{P}_x\left(\varliminf_{\ell\to\infty}\left\{X_{\sigma_{E\setminus G_n}} \in (E\setminus G_n)\cap A_\ell \text{ for all } n > \ell,\ \sigma < \infty\right\}\right)\\
&\leq \varliminf_{\ell\to\infty} \mathbf{P}_x(X_{\sigma_{E\setminus G_n}} \in (E\setminus G_\ell)\cap A_\ell \text{ for all } n > \ell,\ \sigma < \infty)\\
&\leq \varliminf_{\ell\to\infty} \mathbf{P}_x(X_\sigma \in (E\setminus G_\ell)\cap A_\ell,\ \sigma < \infty)\\
&\leq \varliminf_{\ell\to\infty} \mathbf{P}_x(X_\sigma \in E\setminus G_\ell,\ \sigma < \infty)\\
&= \mathbf{P}_x\left(X_\sigma \in E \setminus \bigcup_{\ell=1}^{\infty} G_\ell,\ \sigma < \infty\right) = 0
\end{aligned}$$

for m-a.e. $x \in E$, because of the exceptionality of $E\setminus\bigcup_{\ell=1}^{\infty} G_\ell$, where we use the quasi-left continuity of X up to ∞ and the closedness of $(E\setminus G_\ell)\cap A_\ell$. The implication (ic)$\Longleftrightarrow$(id) follows from the fact that $x \mapsto \mathbf{P}_x(\sigma < \infty)$ is a decreasing limit of excessive functions and Lemma 4.1.7 in [7].

(ii): The "if"part is clear by (i) because $\tau_{F_k} = \tau_{G_k}$, where G_k is the fine interior of F_k. We only prove the "only if"part. Take $f \in \mathcal{F}_{\rm loc}$. Then there exist $\{f_k \mid k \in \mathbb{N}\} \subset \mathcal{F}$ and an increasing sequence $\{G_k\}$ of finely open sets with $E = \bigcup_{k=1}^{\infty} G_k$ q.e. such that $f = f_k$ m-a.e. on G_k. We may take $f_k \in \mathcal{F}_b$ for each $k \in \mathbb{N}$, by replacing f_k with $(-k)\vee f_k \wedge k$, and G_k with $G_k \cap \{|f| < k\}$. Note that f and f_k are quasi-continuous, so $f = f_k$ q.e. on G_k. Taking an $\mathcal{E}$-quasi-closure $\overline{G_k}^{\mathcal{E}}$ of G_k, we have $f = f_k$ q.e. on $\overline{G_k}^{\mathcal{E}}$ (see [12] for the definition of $\mathcal{E}$-quasi-closure). Let $\{A_n\}$ be a common $\mathcal{E}$-nest of closed sets such that for each $k, n \in \mathbb{N}$, $\overline{G_k}^{\mathcal{E}} \cap A_n$ is closed. Set $F_k := \overline{G_k}^{\mathcal{E}} \cap A_k$. By (i), $\{G_k\}$ is a nest, hence $\overline{G_k}^{\mathcal{E}}$ is a nest of q.e. finely closed sets, because of $\tau_{G_k} \leq \tau_{\overline{G_k}^{\mathcal{E}}}$. Here we recognize $\overline{G_k}^{\mathcal{E}}$ as a finely closed Borel sets by deleting an exceptional set. Since $\{A_n\}$ is a nest of closed sets, $\{F_k\}$ is so, that is, $\mathbf{P}_m(\lim_{k\to\infty} \tau_{F_k} \neq \zeta) = 0$. Therefore $\{F_k\}$ is an $\mathcal{E}$-nest of closed sets. We easily see that for each $k \in \mathbb{N}$, $f = f_k$ q.e. on F_k. □

We are now in a position to define stochastic integrals against $\Lambda(M)$ as integrator. Note that for $f \in \mathcal{F}_{\rm loc}$, $M^{f,c}$ is well defined as a continuous MAF on $[0, \zeta[$ of locally finite energy (see Theorem 8.2 in [9]). Moreover, for $f \in \mathcal{F}_{\rm loc}$ and a locally square-integrable MAF M on $I(\zeta)$,

$$t \mapsto (f * M)_t := \int_0^t f(X_{s-})dM_s$$

is a locally square-integrable MAF on $I(\zeta)$.

Definition 3.8 (Stochastic integral) Suppose that M is a locally square-integrable MAF on $I(\zeta)$ and $f \in \mathcal{F}_{\rm loc}$. Let $\varphi : E_\partial \times E_\partial \to \mathbb{R}$ be a jump function for M, and assume that φ satisfies condition (3.1). Define $\mathbf{P}_m$-a.s. on $[0, \zeta[$ by,

$$\begin{aligned}&\int_0^t f(X_{s-})\, d\Lambda(M)_s \\ &:= \quad \Lambda(f * M)_t - \frac{1}{2}\langle M^{f,c},\, M^c\rangle_t + \frac{1}{2}\int_0^t \int_E (f(y) - f(X_s))\varphi(y, X_s) N(X_s, dy) dH_s, \qquad (3.9)\end{aligned}$$

whenever $\Lambda(f * M)$ is well defined and the third term in the right hand side of (3.9) is absolutely convergent.

Remark 3.9 (i) Here are some sufficient conditions for every term on the right hand side of (3.9) to be well defined. In addition to the conditions in Definition 3.8, we assume that $\mathbf{P}_m$-a.s.

$$\int_0^t \int_{E_\partial} (f(X_s) - f(y))^2 N(X_s, dy) dH_s < \infty \quad \text{for every } t < \zeta \tag{3.10}$$

and that

$$\int_0^t \int_E \varphi(y, X_s)^2 N(X_s, dy) dH_s < \infty \quad \text{for every } t < \zeta. \tag{3.11}$$

Then the first and third terms on the right side of (3.9) are well defined. This is because $N(1_{E\times E}|\widehat{\varphi}|)\mu_H \in S$ implies $N(1_{E\times E}|f\widehat{\varphi}|)\mu_H \in S$, and

$$f(x)\varphi(x, y) + f(y)\varphi(y, x) = f(x)\hat{\varphi}(x, y) + (f(y) - f(x))\varphi(y, x),$$

so $\Lambda(f*M)$ is well defined on $[0, \zeta[$ in view of the condition (3.1) for $f*M$, (3.10) and (3.11). Condition (3.10) is satisfied when f is a bounded function in $\mathcal{F}_{\rm loc}$ or $f \in \mathcal{F}$. This is because when $f \in \mathcal{F}$, the left hand side of (3.10) is just $\langle M^{f,d}\rangle_t$. When f is a bounded function in $\mathcal{F}_{\rm loc}$, by Lemma 3.7(ii), there exist an $\mathcal{E}$-nest $\{F_n \mid n \in \mathbb{N}\}$ of closed sets and a sequence of functions $\{f_n \mid n \in \mathbb{N}\} \subset \mathcal{F}_b$ such that $f = f_n$ q.e. on F_n for every $n \geq 1$. Note for each $n \geq 1$, $M^{f_n,d}$ is a square-integrable purely discontinuous martingale and

$$M_t^{f_n,d} - M_{t-}^{f_n,d} = f_n(X_t) - f_n(X_{t-}).$$

So $t \mapsto \sum_{s\leq t} (f_n(X_s) - f_n(X_{s-}))^2$ is $\mathbf{P}_x$-integrable for q.e. $x \in E$. Since f is bounded, we have for each $n \geq 1$ that

$$\begin{aligned} t \quad \mapsto \quad & \sum_{s\leq t\wedge\tau_{F_n}} (f(X_s) - f(X_{s-}))^2 \\ = \quad & \sum_{s<t\wedge\tau_{F_n}} (f(X_s) - f(X_{s-}))^2 + \left(f(X_{t\wedge\tau_{F_n}}) - f(X_{t\wedge\tau_{F_n}-})\right)^2 \\ = \quad & \sum_{s<t\wedge\tau_{F_n}} (f_n(X_s) - f_n(X_{s-}))^2 + \left(f(X_{t\wedge\tau_{F_n}}) - f(X_{t\wedge\tau_{F_n}-})\right)^2 \end{aligned}$$

is an increasing process and is $\mathbf{P}_x$-integrable for each fixed $t \geq 0$ for q.e. $x \in E$. Similarly, $A_t := \sum_{s\leq t}(f(X_s) - f(X_{s-}))^2$ is *locally integrable* in the sense of Definition 5.18 in [9]. Indeed, for a stopping time $T_n := \inf\{t > 0 \mid A_t > n\}$, $A_{T_n} = A_{T_n-} + (f(X_{T_n}) - f(X_{T_n-}))^2$ is bounded, hence $\mathbf{P}_x$-integrable for q.e. $x \in E$. Note that the dual predictable projection of A_t is nothing but the $\int_0^t \int_{E_\partial}(f(X_s) - f(y))^2 N(X_s, dy)dH_s$. Then the dual predictable projection of $\sum_{s\leq t\wedge\tau_{F_n}}(f(X_s) - f(X_{s-}))^2$ is given by $\int_0^{t\wedge\tau_{F_n}}\int_{E_\partial}(f(X_s)-f(y))^2N(X_s, dy)dH_s$ from Corollary 2.14 in [9], which is $\mathbf{P}_x$-integrable for q.e. $x \in E$. This implies that (3.10) holds for every $t < \tau_{F_n}$. Therefore (3.10) holds for every $t < \zeta$.

Condition (3.11) is satisfied when M^d is $\mathbf{P}_m$-square-integrable. Indeed,

$$\begin{aligned}\mathbf{E}_m\Big[\sum_{s\leq t}\varphi^2(X_s, X_{s-}) : t < \zeta\Big] &= \mathbf{E}_m\Big[[M^d]_t \circ r_t : t < \zeta\Big] \\ &= \mathbf{E}_m\Big[[M^d]_t : t < \zeta\Big] < \infty.\end{aligned}$$

Then Corollary 4.5 in [8] tells us that

$$\lim_{t\to 0}\frac{1}{t}\mathbf{E}_m\Big[\sum_{s\leq t}\varphi^2(X_s, X_{s-}) : t < \zeta\Big] = \lim_{t\to 0}\frac{1}{t}\mathbf{E}_m\Big[\sum_{s\leq t}\varphi^2(X_s, X_{s-})\Big],$$

which implies

$$\mathbf{E}_m\Big[\int_0^t\int_E \varphi(y, X_s)^2 N(X_s, dy)dH_s\Big] < \infty$$

for all $t > 0$ by way of its subadditivity. Hence we get (3.11).

(ii) Suppose that $f \in \mathcal{F}$. Let K_t be a purely discontinuous local MAF on $I(\zeta)$ with $K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-})$ on $]0, \zeta[$. Then

$$\langle M^{f,j}, M^j + K\rangle_t = -\int_0^t\int_E (f(y) - f(X_s))\varphi(y, X_s)N(X_s, dy)dH_s.$$

In this case, (3.9) can be rewritten as

$$\int_0^t f(X_{s-})d\Lambda(M)_s = \Lambda(f * M)_t - \frac{1}{2}\langle M^{f,c} + M^{f,j},\, M^c + M^j + K\rangle_t \tag{3.12}$$

on $[0, \zeta[$. So when $M = M^u$ for some $u \in \mathcal{F}$ and $f \in \mathcal{F} \cap L^2(E; \mu_{\langle u\rangle})$, $\int_0^t f(X_{s-})d\Lambda(M)_s$ on $[0, \zeta[$ is just the $\int_0^t f(X_s) \circ d\Gamma(M)_s$ defined by (1.7). This shows that the stochastic integration given in Definition 3.8 extends Nakao's definition (1.7) of stochastic integral first introduced in [16].

Theorem 3.10 *The stochastic integral in (3.9) is well defined. That is, if M and $\widetilde{M}$ are two locally square-integrable MAFs on $I(\zeta)$ such that all conditions in Definition 3.2 for M and $\widetilde{M}$ are* *satisfied and $\Lambda(M) \equiv \Lambda(\widetilde{M})$ on $[0,\zeta[$, then for every $f \in \mathcal{F}_{\rm loc}$ for which $\int_0^t f(X_{s-})\, d\Lambda(M)_s$ and $\int_0^t f(X_{s-})\, d\Lambda(\widetilde{M})_s$ are well defined, we have $\mathbf{P}_m$-a.s.*

$$\int_0^t f(X_{s-})\, d\Lambda(M)_s = \int_0^t f(X_{s-})\, d\Lambda(\widetilde{M})_s \quad \textit{on } [0,\, \zeta[.$$

Proof. It is equivalent to show that

$$\int_0^t f(X_{s-}) d\Lambda(M - \widetilde{M})_s = 0 \quad \text{on } [0,\, \zeta[.$$

By taking M to be $M - \widetilde{M}$, we may and will assume that $\widetilde{M} = 0$. Moreover, a localization argument allows us to assume that f is bounded. Let $\varphi : E_\partial \times E_\partial \to \mathbb{R}$ be a jump function for M. Let K_t be the purely discontinuous local MAF on $I(\zeta)$ with

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}), \quad t < \zeta.$$

Since $\Lambda(M) \equiv 0$, we have

$$M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t = 0 \quad \text{on } [0,\, \zeta[\tag{3.13}$$

Thus by (3.5) and (3.13), on $\{T < \zeta\}$,

$$\begin{aligned} M_t \circ r_T &= M_T \circ r_T - M_{T-t} \circ r_{T-t} \\ &= -M_T - K_T + M_{T-t} + K_{T-t} - \varphi(X_T, X_{T-}) + \varphi(X_{T-t}, X_{(T-t)-}) \end{aligned} \tag{3.14}$$

for every $t \in [0, T]$. Using the standard Riemann-sum approximation and (3.14), we have for $f \in \mathcal{F}$,

$$\begin{aligned} &(f * M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\ &= -(f * M)_t - (f * K)_t - [M^f, M + K]_t \\ &= -(f * M)_t - (f * K)_t - \langle M^{f,c}, M^c \rangle_t + \sum_{s \le t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) \end{aligned}$$

$\mathbf{P}_m$-a.s. on $\{t < \zeta\}$ for each fixed $t \geq 0$. Consequently we have for $f \in \mathcal{F}_{\rm loc}$, $\mathbf{P}_m$-a.s. for all $t \in [0, \zeta[$,

$$\begin{aligned} &(f * M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\ &= -(f * M)_t - (f * K)_t - \langle M^{f,c}, M^c \rangle_t + \sum_{s \le t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}), \end{aligned} \tag{3.15}$$

since both sides are right continuous in $t \in [0, \zeta[$. Let $\widetilde{K}$ be the purely discontinuous local MAF on $I(\zeta)$ with

$$\widetilde{K}_t - \widetilde{K}_{t-} = -f(X_{t-})\varphi(X_{t-}, X_t) - f(X_t)\varphi(X_t, X_{t-}) \quad \text{for all } t \in [0, \zeta[.$$

Then for $f \in \mathcal{F}_{\rm loc}$, we have by (3.15),

$$\begin{aligned} \Lambda(f*M)_t &= -\frac{1}{2}\Big((f*M)_t + (f*M)\circ r_t + f(X_t)\varphi(X_t, X_{t-}) + \widetilde{K}_t\Big) \\ &= \frac{1}{2}\left(\int_0^t f(X_{s-})dK_s + \langle M^{f,c}, M^c\rangle_t - \sum_{s\le t}(f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \widetilde{K}_t\right). \end{aligned}$$

Thus

$$\begin{aligned} &\int_0^t f(X_{s-})d\Lambda(M)_s \\ &= \Lambda(f*M)_t - \frac{1}{2}\langle M^{f,c},\, M^c\rangle_t + \frac{1}{2}\int_0^t\int_E (f(y) - f(X_s))\varphi(y, X_s)N(X_s, dy)dH_s \\ &= \frac{1}{2}\int_0^t f(X_{s-})dK_s - \frac{1}{2}\sum_{s\le t}(f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \frac{1}{2}\widetilde{K}_t \\ &\quad + \frac{1}{2}\int_0^t\int_E (f(y) - f(X_s))\varphi(y, X_s)N(X_s, dy)dH_s \end{aligned}$$

Note that

$$\begin{aligned} \widetilde{K}_t &= -\sum_{s\le t}\left(f(X_{s-})\varphi(X_{s-}, X_s) + f(X_s)\varphi(X_s, X_{s-})\right) \\ &\quad + \int_0^t\int_E \left(f(X_s)\varphi(X_s, y) + f(y)\varphi(y, X_s)\right)N(X_s, dy)dH_s \end{aligned} \tag{3.16}$$

and that

$$K_t = \lim_{\varepsilon\to 0}\left(-\sum_{s\le t}(\widehat{\varphi}1_{|\widehat{\varphi}|>\varepsilon})(X_{s-}, X_s) + \left(N(\widehat{\varphi}1_{|\widehat{\varphi}|>\varepsilon})*H\right)_t\right), \tag{3.17}$$

where $\widehat{\varphi}(x, y) := \varphi(x, y) + \varphi(y, x)$. It follows that

$$\int_0^t f(X_{s-})\, d\Lambda(M)_s = 0 \quad \text{for all } t < \zeta$$

$\mathbf{P}_m$-a.s. This proves the theorem. □

Remark 3.11 The above proof actually shows that if $\Lambda(M) = \Lambda(\widetilde{M})$ on $[0, T]\cap[0, \zeta[$, then

$$\int_0^t f(X_{s-})\, d\Lambda(M)_s = \int_0^t f(X_{s-})\, d\Lambda(\widetilde{M})_s \quad \text{on } [0, T]\cap[0,\, \zeta[.$$

4 Further study of the stochastic integral

Theorem 4.1 *Suppose that* $f \in \mathcal{F}_{\rm loc}$ *and that* M *is a locally square-integrable MAF on* $I(\zeta)$ *satisfying (3.1) such that* $\Lambda(M)$ *is a continuous process* A *of finite variation on* $[0,\zeta[$. *Assume that the stochastic integral* $t \mapsto \int_0^t f(X_{s-})\,d\Lambda(M)_s$ *is well defined. Then*

$$\int_0^t f(X_{s-})\,d\Lambda(M)_s = \int_0^t f(X_s)\,dA_s \qquad \text{on } [0,\,\zeta[,$$

where the integral on the right hand side is the Lebesgue-Stieltjes integral.

Proof. Let $\varphi : E_\partial \times E_\partial \to \mathbb{R}$ be a Borel function vanishing on the diagonal with $\varphi(X_{t-}, X_t) = M_t - M_{t-}$ for $t \in [0, \zeta_p[$, $\mathbf{P}_m$-a.s. Let K_t be the purely discontinuous local MAF on $I(\zeta)$ with

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}), \quad t \in]0, \zeta[.$$

Since $\Lambda(M) = A$ on $[0,\,\zeta[$,

$$M_t \circ r_t + \varphi(X_t, X_{t-}) = -M_t - K_t - 2A_t \quad \text{for all } t \in [0, \zeta[.$$

Thus by (3.5), for every $T > t > 0$, on $\{T < \zeta\}$,

$$M_t \circ r_T = -M_T - K_T - 2A_T + M_{T-t} + K_{T-t} + 2A_{T-t} - \varphi(X_T, X_{T-}) + \varphi(X_{T-t}, X_{(T-t)-}). \tag{4.1}$$

Now fix $f \in \mathcal{F}_{\rm loc}$; as before we may assume without loss of generality that f is bounded. Using the standard Riemann-sum approximation we obtain, on $\{t < \zeta\}$,

$$\begin{aligned}
&(f*M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\
&= \; -(f*M)_t - (f*K)_t - 2(f*A)_t - [M^f,\, M + K + 2A]_t \\
&= \; -(f*M)_t - (f*K)_t - 2(f*A)_t - \langle M^{f,c}, M^c\rangle_t + \sum_{s\le t}(f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}).
\end{aligned}$$

Consequently, we have, $\mathbf{P}_m$-a.s. for all $t \in [0, \zeta[$,

$$\begin{aligned}
&(f*M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\
&= \; -(f*M)_t - (f*K)_t - 2(f*A)_t - \langle M^{f,c}, M^c\rangle_t + \sum_{s\le t}(f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}),
\end{aligned} \tag{4.2}$$

since both sides are right continuous in $t \in [0, \zeta[$. Let $\widetilde{K}$ be the purely discontinuous local MAF on $I(\zeta)$ with

$$\widetilde{K}_t - \widetilde{K}_{t-} = -f(X_{t-})\varphi(X_{t-}, X_t) - f(X_t)\varphi(X_t, X_{t-}) \qquad \text{for all } t \in [0, \zeta[.$$

Then by (4.2),

$$\begin{aligned}
\Lambda(f*M)_t &= -\frac{1}{2}\Big((f*M)_t + (f*M)\circ r_t + f(X_t)\varphi(X_t, X_{t-}) + \widetilde{K}_t\Big) \\
&= \frac{1}{2}\Big(\int_0^t f(X_{s-})\,dK_s + 2\int_0^t f(X_{s-})\,dA_s + \langle M^{f,c}, M^c\rangle_t \\
&\qquad - \sum_{s\le t}(f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \widetilde{K}_t\Big).
\end{aligned}$$

Thus

$$
\begin{aligned}
&\int_0^t f(X_{s-})\, d\Lambda(M)_s \\
&= \Lambda(f*M)_t - \frac{1}{2}\langle M^{f,c},\, M^c\rangle_t + \frac{1}{2}\int_0^t \int_E (f(y)-f(X_s))\varphi(y,X_s)N(X_s,dy)dH_s \\
&= \frac{1}{2}\int_0^t f(X_{s-})\, dK_s + \int_0^t f(X_{s-})\, dA_s - \frac{1}{2}\sum_{s\le t}(f(X_s)-f(X_{s-}))\varphi(X_s,X_{s-}) - \frac{1}{2}\widetilde{K}_t \\
&\quad +\frac{1}{2}\int_0^t \int_E (f(y)-f(X_s))\varphi(y,X_s)N(X_s,dy)dH_s
\end{aligned}
$$

It now follows from (3.16)-(3.17) that

$$
\int_0^t f(X_{s-})\, d\Lambda(M)_s = \int_0^t f(X_{s-})\, dA_s \quad \text{for all } t\in[0,\zeta[.
$$

This proves the theorem. □

Note that if $f,g\in\mathcal{F}_{\mathrm{loc}}$, then $fg\in\mathcal{F}_{\mathrm{loc}}$.

Theorem 4.2 *Let $f,g\in\mathcal{F}_{\mathrm{loc}}$ and let M be a locally square-integrable MAF on $I(\zeta)$ satisfying (3.1). Then*

$$
\int_0^t g(X_{s-})\, d\left(\int_0^s f(X_{r-})\, d\Lambda(M)_r\right) = \int_0^t f(X_{s-})g(X_{s-})\, d\Lambda(M)_s \quad \text{for } t<\zeta, \tag{4.3}
$$

whenever all the integrals involved are well defined.

Proof. Let $\varphi: E_\partial\times E_\partial\to\mathbb{R}$ be a Borel function vanishing on the diagonal with $\varphi(X_{t-},X_t)=M_t-M_{t-}$ for all $t\in]0,\zeta_p[$, $\mathbf{P}_m$-a.s. Let K_t and $\widetilde{K}_t$ be the purely discontinuous local MAFs on $I(\zeta)$ with $K_t-K_{t-}=-\varphi(X_{t-},X_t)-\varphi(X_t,X_{t-})$, $t\in]0,\zeta[$ and $\widetilde{K}_t-\widetilde{K}_{t-}=-f(X_{t-})\varphi(X_{t-},X_t)-f(X_t)\varphi(X_t,X_{t-})$, $t\in]0,\zeta[$ respectively. Then the left hand side of (4.3) is equal to

$$
\begin{aligned}
&\int_0^t g(X_{s-})\, d\Lambda(f*M)_s - \frac{1}{2}\int_0^t g(X_{s-})d\langle M^{f,c},\, M^c\rangle_s \\
&\quad +\frac{1}{2}\int_0^t\int_E g(X_s)(f(y)-f(X_s))\varphi(y,X_s)N(X_s,dy)dH_s \\
&= \Lambda(fg*M)_t - \frac{1}{2}\langle M^{g,c},\,(f*M)^c\rangle_t + \frac{1}{2}\int_0^t\int_E (g(y)-g(X_s))f(y)\varphi(y,X_s)N(X_s,dy)dH_s \\
&\quad -\frac{1}{2}\int_0^t g(X_{s-})d\langle M^{f,c},M^c\rangle_s + \frac{1}{2}\int_0^t\int_E g(X_s)(f(y)-f(X_s))\varphi(y,X_s)N(X_s,dy)dH_s \\
&= \Lambda(fg*M)_t - \frac{1}{2}\langle M^{fg,c},M^c\rangle_t + \frac{1}{2}\int_0^t\int_E (f(y)g(y)-f(X_s)g(X_s))\varphi(y,X_s)N(X_s,dy)dH_s \\
&= \int_0^t f(X_{s-})g(X_{s-})d\Lambda(M)_s.
\end{aligned}
$$

This proves the theorem. □

Let $\mathcal{J}$ denote the class of Dirichlet processes that can be written as a sum of an $(\mathcal{F}_t)$-semimartingale Y and $\Lambda(M)$ for a locally square-integrable MAF M on $I(\zeta)$ satisfying the condition of Definition 3.2. The last two theorems imply that the following stochastic integral is well defined for integrators $Z \in \mathcal{J}$.

Definition 4.3 For $f \in \mathcal{F}_{\rm loc}$ and $Z = Y + \Lambda(M) \in \mathcal{J}$, define on $[0,\zeta[$

$$\int_0^t f(X_{s-})\,dZ_s := \int_0^t f(X_{s-})\,dY_s + \int_0^t f(X_{s-})\,d\Lambda(M)_s,$$

whenever the latter stochastic integral is well defined.

To establish Itô's formula, we need the following result.

Theorem 4.4 *Let $f \in \mathcal{F}_{\rm loc}$ and let M be a locally square-integrable MAF on $I(\zeta)$ such that $\int_0^\cdot f(X_{s-})d\Lambda(M)$ is well defined on $[0,\zeta[$. Then for every $t>0$, $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$,*

$$\int_0^t f(X_{s-})\,d\Lambda(M)_s = \lim_{n\to\infty}\sum_{\ell=0}^{n-1} f(X_{\ell t/n})\left(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}\right). \tag{4.4}$$

Here the convergence is in probability with respect to $\mathbf{P}_x$ for m-a.e. $x \in E$.

Proof. By (3.5), $M_s \circ r_t = M_t \circ r_t - M_{t-s} \circ r_{t-s}$ for all $s<t$. Let $\varphi : E_\partial \times E_\partial \to \mathbb{R}$ be a Borel function vanishing on the diagonal set with $\varphi(X_{t-}, X_t) = M_t - M_{t-}$ for all $t \in [0,\zeta_p[$. Let K be the purely discontinuous local MAF on $I(\zeta)$ with $K_t - K_{t-} = -\varphi(X_{t-},X_t) - \varphi(X_t, X_{t-})$, $t \in]0,\zeta[$. Then for each fixed $t>0$, $\mathbf{P}_m$-a.s. on $\{t<\zeta\}$

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{\ell=0}^{n-1} f(X_{\ell t/n})\left(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}\right)\\
=\;& -\frac{1}{2}(f*M)_t - \frac{1}{2}(f*K)_t + \frac{1}{2}\lim_{n\to\infty}\sum_{\ell=0}^{n-1} f(X_{\ell t/n})\left(M_{(\ell+1)t/n}\circ r_{(\ell+1)t/n} - M_{\ell t/n}\circ r_{\ell t/n}\right)\\
=\;& -\frac{1}{2}(f*M)_t - \frac{1}{2}(f*K)_t - \frac{1}{2}\lim_{n\to\infty}\left[\sum_{\ell=0}^{n-1} f(X_{(\ell+1)t/n})\left(M_{(\ell+1)t/n} - M_{\ell t/n}\right)\right]\circ r_t\\
=\;& -\frac{1}{2}(f*M)_t - \frac{1}{2}(f*K)_t - \frac{1}{2}(f*M)_t\circ r_t - \frac{1}{2}[M^f, M]_t \circ r_t\\
=\;& -\frac{1}{2}(f*M)_t - \frac{1}{2}(f*K)_t - \frac{1}{2}(f*M)_t\circ r_t - \frac{1}{2}\langle M^{f,c}, M^c\rangle_t - \frac{1}{2}\sum_{s\le t}(f(X_{s-}) - f(X_s))\varphi(X_s, X_{s-})\\
=\;& \Lambda(f*M)_t + \frac{1}{2}\widetilde{K}_t - \frac{1}{2}(f*K)_t - \frac{1}{2}\langle M^{f,c}, M^c\rangle_t - \frac{1}{2}\sum_{s\le t}(f(X_{s-}) - f(X_s))\varphi(X_s, X_{s-})\\
=\;& \int_0^t f(X_{s-})\,d\Lambda(M)_s,
\end{aligned}$$

where $\widetilde{K}$ in the second to the last equality is the purely discontinuous local MAF on $I(\zeta)$ with $\widetilde{K}_s - \widetilde{K}_{s-} = -f(X_{s-})\varphi(X_{s-}, X_s) - f(X_s)\varphi(X_s, X_{s-})$, $s \in]0, \zeta[$. □

Remark 4.5 (i) Theorem 4.4 immediately implies Theorems 3.10 and 4.1.

(ii) By (3.8), we also have for $t < \zeta$

$$\int_0^t f(X_{s-})\, d\Lambda(M)_s = \lim_{n\to\infty} \sum_{\ell=0}^{n-1} f(X_{(\ell+1)t/n}) \left(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}\right). \tag{4.5}$$

Hence we could denote this stochastic integral by either $\int_0^t f(X_s) d\Lambda(M)_s$ or $\int_0^t f(X_s) \circ d\Lambda(M)_s$. Here $\int_0^t f(X_s) \circ d\Lambda(M)_s$ is the Fisk-Stratonovich type integral: for $t < \zeta$

$$\int_0^t f(X_s) \circ d\Lambda(M)_s = \lim_{n\to\infty} \sum_{\ell=0}^{n-1} \frac{f(X_{(\ell+1)t/n}) + f(X_{\ell t/n})}{2} \left(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}\right). \tag{4.6}$$

(iii) For any $f \in \mathcal{F}_{\rm loc}$ and $\mathbf{P}_m$-square-integrable MAF M, by way of the Riemann-sum approximation (4.4), we can extend the stochastic integral $\int_0^t f(X_{s-}) d\Lambda(M)_s$ without imposing further conditions. Indeed, let $\{G_\ell\}$ be a nest of finely open Borel sets and $f_\ell \in \mathcal{F}_b$ with $f = f_\ell$ m-a.e. on G_ℓ (see the explanation for the condition (3.10) in Remark 3.9.) By (4.4), we see $\int_0^t f_n(X_{s-}) d\Lambda(M)_s = \int_0^t f_m(X_{s-}) d\Lambda(M)_s$ for $t < \tau_{G_n}$ and $n < m$. Then we can define $\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f_m(X_{s-}) d\Lambda(M)_s$ for $t < \tau_{G_n}$ for each $n \in \mathbb{N}$, consequently, for all $t < \zeta$ $\mathbf{P}_m$-a.s. More strongly, for $M \in \overset{\circ}{\mathcal{M}}$ and $f \in \mathcal{F}_{\rm loc}$, we can define $\int_0^t f(X_{s-}) d\Lambda(M)_s$ for all $t \in [0, \infty[$ $\mathbf{P}_m$-a.s. Indeed, for $\{f_n\}$ and $\{G_n\}$ specified as above, the stochastic integral $f_n * \Lambda(M)$ for $M \in \overset{\circ}{\mathcal{M}}$ can be defined as a CAF in the original way by Nakao and $(f_n * \Lambda(M))_t = (f_n * \Lambda(M))_\zeta = \lim_{s\uparrow\zeta} (f_n * \Lambda(M))_s$ for $t \geq \zeta$, which means $\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f_m(X_{s-}) d\Lambda(M)_s$ for $t < \sigma_{E\setminus G_n}$ beyond ζ for each $n < m$. Owing to Lemma 3.7(i), we obtain the stochastic integral $\int_0^t f(X_{s-}) d\Lambda(M)_s$, on $]0, \infty[$, $\mathbf{P}_m$-a.s. for any $f \in \mathcal{F}_{\rm loc}$ and $M \in \overset{\circ}{\mathcal{M}}$ extending the stochastic integral by Nakao.

Remark 4.5(iii) says that the stochastic integral $f * \Lambda(M)_t := \int_0^t f(X_{s-}) d\Lambda(M)_s$ can be defined for $t \in [0, \infty[$ $\mathbf{P}_m$-a.s. for $f \in \mathcal{F}_{\rm loc}$ and $M \in \overset{\circ}{\mathcal{M}}$. We shall refine this statement from m-almost everywhere starting point $x \in E$ to quasi-everywhere $x \in E$.

Lemma 4.6 *For $f \in \mathcal{F}_{\rm loc}$ and $M \in \overset{\circ}{\mathcal{M}}$, the stochastic integral $f * \Lambda(M)_t := \int_0^t f(X_{s-}) d\Lambda(M)_s$ can be defined for all $t \in [0, \infty[$ $\mathbf{P}_x$-a.s. for q.e. $x \in E$, in particular, $f * \Lambda(M)$ is a CAF.*

Proof. Since $f \in \mathcal{F}_{\rm loc}$, we have $\{f_k \mid k \in \mathbb{N}\} \subset \mathcal{F}_b$ and a nest $\{G_k \mid k \in \mathbb{N}\}$ of finely open Borel sets such that $f = f_k$ q.e. on G_k. We know that the stochastic integral $f_k * \Lambda(M)$ is defined $\mathbf{P}_x$-a.s. for q.e. $x \in E$. Let Ξ_k be the defining set admitting an exceptional set for the CAF $f_k * \Lambda(M)$ of zero energy and set

$$\Xi := \left\{ \omega \in \bigcap_{k=1}^{\infty} \Xi_k \;\middle|\; \begin{array}{l} \text{for any } k, \ell \in \mathbb{N} \text{ with } k < \ell, \\ \displaystyle\int_0^t f_k(X_{s-}(\omega))\Lambda(M)_s(\omega) = \int_0^t f_\ell(X_{s-}(\omega))\Lambda(M)_s(\omega) \text{ for } t < \sigma_{E\setminus G_k}(\omega) \end{array} \right\}.$$

Then $\mathbf{P}_x(\Xi^c) = 0$, m-a.e. $x \in E$. Hence for each $s > 0$, $\mathbf{P}_x(\theta_s^{-1}(\Xi^c)) = P_s(\mathbf{P}_\cdot(\Xi^c))(x) = 0$ for q.e. $x \in E$. Setting $\widehat{\Xi} := \bigcap_{k=1}^{\infty} \Xi_k \cap \bigcap_{s \in \mathbb{Q}_{++}} \theta_s^{-1}(\Xi)$, we have $\mathbf{P}_x(\widehat{\Xi}) = 1$ for q.e. $x \in E$. For $\omega \in \widehat{\Xi}$ with $t < \sigma_{E\setminus G_k}(\omega)$, we can find small $s_0 (= s_0(\omega)) > 0$ such that $t + s_0 < \sigma_{E\setminus G_k}(\omega)$. Then we see $t < \sigma_{E\setminus G_k}(\theta_s\omega)$ for any rational $s \in]0, s_0[$. Hence for such ω, we have for $k < \ell$ and any rational $s \in]0, s_0[$

$$\int_s^{t+s} f_k(X_{v-}(\omega))d\Lambda(M)_v(\omega) = \int_s^{t+s} f_\ell(X_{v-}(\omega))d\Lambda(M)_v(\omega).$$

Letting $s \to 0$ and noting $\omega \in \Xi_k$, $k \in \mathbb{N}$, we have that for $k < \ell$, $f_k * \Lambda(M)_t = f_\ell * \Lambda(M)_t$ for $t < \sigma_{E\setminus G_k}$, $\mathbf{P}_x$-a.s. for q.e. $x \in E$. By Lemma 3.7(i), we know $\mathbf{P}_x(\lim_{k\to\infty} \sigma_{E\setminus G_k} = \infty) = 1$ for q.e. $x \in E$. Therefore, we obtain that the stochastic integral $f * \Lambda(M)$ defined as in Remark 4.5(iii) can be established $\mathbf{P}_x$-a.s. for q.e. $x \in E$. This completes the proof. □

Theorem 4.7 (Generalized Itô formula) *Suppose that $\Phi \in C^2(\mathbb{R}^d)$ and $u = (u_1, \cdots, u_d) \in \mathcal{F}^d$. Then for q.e. $x \in E$, $\mathbf{P}_x$-a.s. for all $t \in [0, \infty[$,*

$$\begin{aligned} &\Phi(u(X_t)) - \Phi(u(X_0)) \\ &= \sum_{k=1}^{d} \int_0^t \frac{\partial \Phi}{\partial x_k}(u(X_{s-}))\, du_k(X_s) + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(u(X_{s-}))\, d\langle M^{u_i,c}, M^{u_j,c}\rangle_s \\ &\quad + \sum_{s\le t}\left(\Phi(u(X_s)) - \Phi(u(X_{s-})) - \sum_{k=1}^{d} \frac{\partial \Phi}{\partial x_k}(u(X_{s-}))(u_k(X_s) - u_k(X_{s-}))\right). \end{aligned} \tag{4.7}$$

Proof. Note that both sides appeared in (4.7) are $\mathbf{P}_x$-a.s. defined for q.e. $x \in E$ in view of Lemma 4.6. First we show this Itô formula (4.7) under $\mathbf{P}_m$ for a fixed $t \ge 0$. Note that $\Phi \circ u \in \mathcal{F}_{\rm loc}$ and that

$$u_k(X_t) = u_k(X_0) + M_t^{u_k} + N_t^{u_k} = u_k(X_0) + M_t^{u_k} + \Lambda(M^{u_k})_t.$$

This version of Itô's formula follows from Theorems 3.6 and 4.4 by a line of reasoning similar to that used to prove Itô's formula for semimartingales (cf. [9]). Since both sides in (4.7) are right continuous, (4.7) holds under $\mathbf{P}_m$.

Secondly, we refine the starting point. Recall that Ω consists of rcll paths. Let $I_t(\omega)$ be the difference of the left hand side and the right hand side in (4.7). Let Ξ be the intersection of all the defining sets of AFs appeared in the formula and $\{\omega \in \Omega \mid I_t(\omega) = 0, {}^\forall t \in [0, \infty[\}$. Then

$\mathbf{P}_x(\Xi^c) = 0$, m-a.e. $x \in E$. Let $\widehat{\Xi}$ be the intersection of aa the defining sets of AFs appeared in the formula and $\bigcap_{s\in\mathbb{Q}_{++}} \theta_s^{-1}(\Xi)$. Then we have $\mathbf{P}_x(\widehat{\Xi}) = 1$ for q.e. $x \in E$ as in the proof of Lemma 4.6. Take $\omega \in \widehat{\Xi}$. Then for any positive rational $s > 0$, we have $I_t(\theta_s\omega) = 0$, that is,

$$\begin{aligned}&\Phi(u(X_{t+s}(\omega))) - \Phi(u(X_s(\omega)))\\ &= \sum_{k=1}^{d} \int_s^{t+s} \frac{\partial \Phi}{\partial x_k}(u(X_{v-}(\omega)))\, du_k(X_v(\omega)) + \frac{1}{2} \sum_{i,j=1}^{d} \int_s^{t+s} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(u(X_{v-}(\omega)))\, d\langle M^{u_i,c}, M^{u_j,c}\rangle_v(\omega)\\ &+ \sum_{s<v\le t+s} \left(\Phi(u(X_v(\omega))) - \Phi(u(X_{v-}(\omega))) - \sum_{k=1}^{d} \frac{\partial \Phi}{\partial x_k}(u(X_{v-}(\omega)))(u_k(X_v(\omega)) - u_k(X_{v-}(\omega))) \right).\end{aligned}$$

Letting $s \to 0$ and using the right continuity of $s \mapsto u(X_s)$ and stochastic integrals, we have $I_t(\omega) = 0$. This completes the proof. □

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