1. (a) If we measure S_z for particle 1 and get $+\hbar/2$, the particles are left in the state

$$\alpha_z(1)\,\beta_z(2),$$

but if we get $-\hbar/2$ for particle 1 they are left in

$$\beta_z(1)\,\alpha_z(2).$$

This shows that the state of particle 2 depends on the state of 1, and so they are entangled.

(b) Adding and subtracting the equations for α_x and β_x, we can express the eigenspinors of $\widehat{S}_z$ in terms of them:

$$\alpha_z = \frac{1}{\sqrt{2}}\Big(\alpha_x + \beta_x\Big), \qquad \beta_z = \frac{1}{\sqrt{2}}\Big(\alpha_x - \beta_x\Big).$$

Substituting these into the expression for $\psi_{10}(1,2)$ we get

$$\psi_{10}(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_x(1)\,\alpha_x(2) - \beta_x(1)\,\beta_x(2)\Big).$$

Again, you should find that the state of particle 2 is different depending on the result of measuring S_x for particle 1.

(c) If we measure S_z for particle 1 and get $+\hbar/2$, the particles are left in the state

$$\alpha_z(1)\,\frac{1}{\sqrt{2}}\Big(\alpha_z(2) + \beta_z(2)\Big).$$

(It's not essential here but I have normalised the state.) If we get $-\hbar/2$ for particle 1 they are left in

$$\beta_z(1)\,\frac{1}{\sqrt{2}}\Big(\alpha_z(2) + \beta_z(2)\Big).$$

In both cases particle 2 is left in the same state ($\alpha_x(2)$). More mathematically, if you prefer, we can factorise $\phi_0(1,2)$ and write it in the form

$$\phi_0(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_z(1) + \beta_z(1)\Big)\frac{1}{\sqrt{2}}\Big(\alpha_z(2) + \beta_z(2)\Big) = \alpha_x(1)\,\alpha_x(2).$$

This shows that $\phi_0(1,2)$ is a product of states of each particle and so it is not entangled.

2. (a) The Hamiltonian for an electron in this field is:

$$\widehat{H} = -\widehat{\mu}.\mathbf{B} = \frac{eg}{2m}\widehat{\mathbf{S}}.\mathbf{B} = \frac{eg}{2m}\widehat{S_y}.B.$$

The eigenfunctions are therefore α_y and β_y with eigenvalues $E = \pm\frac{g}{2}\frac{e\hbar}{2m}B$. The separable solutions of the TISE are therefore products of these spinors with $e^{-iEt/\hbar}$, giving the general solution:

$$\chi = A\alpha_y e^{-i\omega t} + B\beta_y e^{i\omega t}.$$

Using the initial condition:

$$\chi(t=0) = \alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\alpha_y + B\beta_y = A\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} + B\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

can find that $A = B = \frac{1}{\sqrt{2}}$ and substituting back in gives the required time-dependent state vector.

(b) Writing out the exponentials in trig form:

$$\begin{aligned}\chi(t) &= \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}[\cos\omega t - i\sin\omega t] + \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}[\cos\omega t + i\sin\omega t] \\ &= \frac{1}{2}\begin{pmatrix} \cos\omega t - i\sin\omega t + \cos\omega t + i\sin\omega t \\ i\cos\omega t + \sin\omega t - i\cos\omega t + \sin\omega t \end{pmatrix} \\ &= \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}.\end{aligned}$$

Now substituting for $\omega t = 0,\ \frac{\pi}{4},\ \frac{\pi}{2}$... will produce the spinors corresponding to α_z to α_x to β_z to $-\beta_x$ to $-\alpha_z$ as time progresses.

A magnetic field perpendicular to the quantisation axis of a Q-bit for a certain period of time will evolve the spinor from a state corresponding to a 1 to one corresponding to a 0, or visa versa, at least to within a phase difference. Why doesn't the phase difference matter too much?

3. When we feed an electron in the state $\gamma(1)$ along with one in a blank state $\alpha(2)$ into our copier, we get

$$\widehat{X}\,\gamma(1)\,\alpha(2) = \widehat{X}\left(c_1\,\alpha(1) + c_2\,\beta(2)\right)\alpha(2) = c_1\,\widehat{X}\,\alpha(1)\,\alpha(2) + c_2\,\widehat{X}\,\beta(1)\,\alpha(2).$$

Since it can duplicate both of the basis states, the copier produces

$$\widehat{X}\,\gamma(1)\,\alpha(2) = c_1\,\alpha(1)\,\alpha(2) + c_2\,\beta(1)\,\beta(2).$$

It should be obvious that this is not just a product of two copies of γ and, using the same methods as in question 1, we can show that it is an entangled state.

This simple system illustrates a general result known as the "no cloning theorem". This shows that it is impossible to make an identical copy of the state of any quantum system without destroying the state of the original system. (For more on these ideas, see: Rae 12.3 and Miller pages 425-427.)

4. (a) The setup in the first part of the question is very like the one in question 2, except it refers to photons, not electrons. A general polarisation state for particle 1 can be written

$$Q(1) = c_1\, R(1) + c_2\, L(1).$$

Sending this into the copier produces the state

$$\widehat{X}\, Q(1) = c_1\, R(1)\, R(2) + c_2\, L(1)\, L(2).$$

As before, you should be able to show that this is an entangled state.

(b) By adding and subtracting V and H as defined in the question, we get

$$R = \frac{1}{\sqrt{2}}\Big(V + H\Big), \qquad \text{and} \qquad L = \frac{1}{\sqrt{2}}\Big(V - H\Big).$$

(Note that L as defined here differs from the one in lectures by a phase factor of i.) Substituting these into $\Psi(1,2)$ and tidying up leaves us with

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\Big(V(1)V(2) + H(1)H(2)\Big).$$

The entanglement of this state shows that if Alice measures the linear polarisation of her photon and gets V, Bob's photon is left in the same polarisation state as hers. The same happens if Alice gets H. When he measures the polarisation of his photon, Bob can then deduce Alice's result since he knows it must be the same.

(c) Using the expressions above, for R and L in terms of V and H, we can rewrite the three-photon state as

$$\begin{aligned}\widehat{X}\Psi(1,2) &= \frac{1}{4}\Big[\Big(V(1)+H(1)\Big)\Big(V(2)+H(2)\Big)\Big(V(3)+H(3)\Big) \\ &\quad +\Big(V(1)-H(1)\Big)\Big(V(2)-H(2)\Big)\Big(V(3)-H(3)\Big)\Big],\end{aligned}$$

which can be tidied up to give

$$\begin{aligned}\widehat{X}\Psi(1,2) &= \frac{1}{2}\Big[V(1)V(2)V(3)+V(1)H(2)H(3) \\ &\quad +H(1)V(2)H(3)+H(1)H(2)V(3)\Big].\end{aligned}$$

In the first two terms, Bob's and Eve's photons (2 and 3) agree, but in the second two they disagree. Since all four outcomes are equally likely, Eve can get Bob's results correctly only 50% of the time. This is no better than random guessing!

(d) In a similar way, Alice's and Bob's photons (1 and 2) also agree only 50% of the time, instead of the 100% agreement expected for the original entangled state. If Alice and Bob compare the results of their measurements for a sample of the photon pairs, they will detect this and suspect that something, or someone, has destroyed their entanglement.

5. (a) This works very like the copier in question 2. The cNOT gate acts on the initial state $\gamma(1)\alpha(2)$ to give

$$\widehat{C}\,\gamma(1)\,\alpha(2) = c_1\,\widehat{C}\,\alpha(1)\,\alpha(2) + c_2\,\widehat{C}\,\beta(1)\,\alpha(2). = c_1\,\alpha(1)\,\beta(2) + c_2\,\beta(1)\,\alpha(2).$$

Again it should be obvious that this is an entangled state.

(b) Acting with the cNOT gate on each of the Bell states, we get

$$\begin{aligned}
\widehat{C}\,\psi_1(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1) - \beta(1)\Big)\alpha(2),\\
\widehat{C}\,\psi_2(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1) + \beta(1)\Big)\alpha(2),\\
\widehat{C}\,\psi_3(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1) - \beta(1)\Big)\beta(2),\\
\widehat{C}\,\psi_4(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1) + \beta(1)\Big)\beta(2).
\end{aligned}$$

Each of these states is a product of spinors for the two electrons and so they are not entangled.

(c) In each of the states in part (b), the spin of electron 1 is left in one the two states

$$\alpha_x = \frac{1}{\sqrt{2}}\Big(\alpha + \beta\Big), \qquad \beta_x = \frac{1}{\sqrt{2}}\Big(\alpha - \beta\Big)$$

You should recognise these as the eigenstates of S_x, the spin along the x-axis. In contrast, the spin of electron 2 is left in an eigenstate of spin along the z-axis. Each Bell state leads to a different combination of eigenvalues. For example, $\psi_1(1,2)$ gives spin down along the x-axis for electron 1 and spin up along the z-axis for 2.

6. Using the expressions for $\gamma(1)$ and $\psi_{00}(2,3)$, we can expand the initial three-particle state as

$$\begin{aligned}\Psi(1,2,3) &= \frac{1}{\sqrt{2}}\Big[c_1\,\alpha(1)\,\alpha(2)\,\beta(3) - c_1\,\alpha(1)\,\beta(2)\,\alpha(3) \\ &\quad + c_2\,\beta(1)\,\alpha(2)\,\beta(3) - c_2\,\beta(1)\,\beta(2)\,\alpha(3)\Big].\end{aligned}$$

By adding and subtracting pairs of the Bell states, we can express products of the basis states of electrons 1 and 2 in terms of them:

$$\begin{aligned}\alpha(1)\,\alpha(2) &= \frac{1}{\sqrt{2}}\Big[\psi_4(1,2)+\psi_3(1,2)\Big],\\ \beta(1)\,\beta(2) &= \frac{1}{\sqrt{2}}\Big[\psi_4(1,2)-\psi_3(1,2)\Big],\\ \alpha(1)\,\beta(2) &= \frac{1}{\sqrt{2}}\Big[\psi_2(1,2)+\psi_1(1,2)\Big],\\ \beta(1)\,\alpha(2) &= \frac{1}{\sqrt{2}}\Big[\psi_2(1,2)-\psi_1(1,2)\Big].\end{aligned}$$

Substituting these into $\Psi(1,2,3)$ and grouping together terms containing the same $\psi_i(1,2)$, we end up with

$$\begin{aligned}\Psi(1,2,3) &= \frac{1}{2}\Big[-\psi_1(1,2)\Big(c_1\,\alpha(3)+c_2\,\beta(3)\Big)\\ &\quad -\psi_2(1,2)\Big(c_1\,\alpha(3)-c_2\,\beta(3)\Big)\\ &\quad +\psi_3(1,2)\Big(c_1\,\beta(3)+c_2\,\alpha(3)\Big)\\ &\quad +\psi_4(1,2)\Big(c_1\,\beta(3)-c_2\,\alpha(3)\Big)\Big].\end{aligned}$$

When Bob gets the result of Alice's measurement, he knows that the state of the three electrons is now one of the four terms in this expansion. For example, if Alice tells him she got the results spin down along the x-axis for electron 1 and spin up along the z-axis for 2 (see the solution to 4(c)), Bob knows that the state is

$$\psi_1(1,2)\,\frac{1}{\sqrt{2}}\Big(c_1\,\alpha(3)+c_2\,\beta(3)\Big),$$

and so his electron is in an exact copy of Alice's original state, γ. The other three results also let him deduce the state of his electron. These states may not be exact copies of γ but they contain the same two complex coefficients. Bob could apply an appropriate magnetic field to rotate each of them into γ. In the commonly used jargon, we can say that Alice has "teleported" the state of her electron to Bob.[1] Note that Alice's measurement destroys the original state of her electron and so there is no contradiction with the result of question 2.

[1] At this point, people usually cite *Star Trek* as the source of the term. However, anyone who has watched an episode of the original series will know that it is the *trans*porter, not a teleporter, that always malfunctions at crucial moments for the plot!