THIRD YEAR EXAMPLE CLASS SHEET FOUR
PHYS30121 Introduction to Nuclear and Particle Physics
Supplementary Problems

1: Work Sheet on Cross Sections

(i) $\Phi = v n_p$
(ii) $R = N\sigma\Phi$
(iii) $I = \Phi S$
(iv) $R = N\sigma\Phi = \frac{N\sigma I}{S} = I\sigma n_t t$
$n_t = \rho\frac{N_A}{M_A}$ where N_A is Avogadro's number and M_A is the mass of the target atom expressed in atomic mass units.
(v) The trick with this is to check your units at each stage in the enumeration!

Areal number density of target is related to the areal mass density given in the question by:

$$n_t t = \rho t\frac{N_A}{M_A} = \frac{100\times 10^{-6}}{27}\times 6.022\times 10^{23} = 2.23\times 10^{18}\ \mathrm{cm}^{-2}$$

The electrical current given in the question is related to the number of α particles per second, remembering that they are doubly charged, by:

$$I = 10\times\frac{10^{-9}}{2\times 1.6022\times 10^{-19}} = 3.12\times 10^{10}\ \mathrm{s}^{-1}$$

Using the angular form of result (iv) above:

$$R = I n_t t\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$$

$$= 3.12\times 10^{10}\ \mathrm{s}^{-1}\times 2.23\times 10^{18}\ \mathrm{cm}^{-2}\times 10\times 10^{-3}\times 10^{-24}\ \mathrm{cm}^2\mathrm{sr}^{-1}\times 2.8\times 10^{-3}\ \mathrm{sr}$$
$$= 1.95\ \mathrm{s}^{-1}$$

But if the detector is only 25% efficient then this is reduced by a factor of 4 to give an event rate of 0.49 s^{-1}.

2: Nuclear Radii

(i) The electron wavelength has to be smaller than the object needed to be studied, or diffractive effects will obscure things.

$$R = r_0 A^{1/3} = 1.2\times 197^{1/3} = 6.98\ \mathrm{fm}$$

$$\lambda = h/p \text{ and } h = 2\pi\hbar = 2\pi 197/c$$

$$p > h/\lambda = \frac{2\pi 197}{6.98\times c} = 177\ \mathrm{MeV/c}$$

$$E^2 = c^2p^2 + m^2c^4 \approx c^2p^2$$
$$E \sim 177MeV$$

Sub-nucleon requires $\lambda < 1$ fm and the energy turns out to be $>$1 GeV.

(ii) Particle rolls up the Coulomb hill until original KE is the same as the electrostatic PE. Assume that the lab frame is the same as the CM frame since particle is light and target is heavy.

$$E = \frac{Z_1Z_2e^2}{4\pi\epsilon_0}\frac{1}{R}$$

$$R = \frac{Z_1Z_2e^2}{4\pi\epsilon_0}\frac{1}{E}$$

(iii)

$$E = \frac{Z_1Z_2e^2}{4\pi\epsilon_0}\frac{1}{R} = 2 \times 79 \times \frac{197}{137} \times \frac{1}{6.98} = 32.5 \text{ MeV}$$

Actually will do so at lower energies since nuclear forces have a range of a 1-2 fm, increasing the distance of separation at which things deviate.

(iv) Bohr radius,

$$r = \frac{4\pi\epsilon_0\hbar^2}{Zme^2} = \frac{137}{\hbar c} \times \frac{(\hbar c)^2}{Zmc^2} = \frac{137 \times 197}{Zmc^2}$$

Normal atom, $r = \frac{137\times197}{13\times0.511} = 4063$ fm.
Muonic atom, $r = \frac{137\times197}{13\times105.66} = 19.6$ fm.
^{27}Al radius is 3.6 fm so muon spends much more of its time inside the nucleus than an electron does in a normal atom.

3: Binary Reaction Q Values

The following expression was quoted in lectures for the Q value of the binary reaction A(a,b)B:

$$Q = T_b\left(1 + \frac{m_b}{m_B}\right) - T_a\left(1 - \frac{m_a}{m_B}\right) - \frac{2\ \cos\theta}{m_B}\sqrt{T_aT_bm_am_b}$$

where T are the kinetic energy of the relevant particles, θ is the *scattering* angle i.e. the angle between the beam direction and the trajectory of b, and m are the relevant masses.

(i) Energy conservation: $Q = T_b + T_B - T_a$.

Converse components on momentum parallel to beam direction:
$(p_B)_x = p_a - p_b\cos\theta$.

Converse components on momentum perp to beam direction: $(p_B)_y = p_b\sin\theta$.

So $p_B^2 = p_b^2 \sin^2\theta + (p_a - p_b \cos\theta)^2 = p_b^2 + p_a^2 - 2p_a p_b \cos\theta$
Use energy conservation and substitute in:

$$Q = T_b + T_B - T_a = \frac{{p_b}^2}{2m_b} + \frac{{p_B}^2}{2m_B} - \frac{{p_a}^2}{2m_a}$$

$$= \frac{{p_b}^2}{2m_b} - \frac{{p_a}^2}{2m_a} + \frac{p_b^2 + p_a^2 - 2p_a p_b \cos\theta}{2m_B}$$

$$= \frac{{p_b}^2}{2m_b}\left(1 + \frac{m_b}{m_B}\right) - \frac{{p_a}^2}{2m_a}\left(1 - \frac{m_a}{m_B}\right) - \frac{p_a p_b}{m_B}\cos\theta$$

$$= T_b\left(1 + \frac{m_b}{m_B}\right) - T_a\left(1 - \frac{m_a}{m_B}\right) - \frac{2\ \cos\theta}{m_B}\sqrt{T_a T_b m_a m_b}$$

(ii) Above formula gives $Q = 1.711859$ MeV. $Q = m_A + m_a - m_B - m_b$ hence $m_A = 193687.0901$ MeV

4: Daughter Recoil in Decay Processes

In (i) and (iii) since the parent decays at rest the total momentum after the decay is zero, hence $\underline{p}_R = -\underline{p}_p$ and $p_R = p_p$, where subscript p refers to the emitted particle. In (ii) the maximum kinetic energy when the β carries all the energy and the $\overline{\nu}$ none. In that case, momentum conservation is identical to that calculated for the two-body decay.

(i) $cp_\gamma = E_\gamma$ so $cp_R = E_\gamma$. Hence

$$E_R = \frac{E_\gamma^2}{2Mc^2} = \frac{1}{2 \times 100 \times 939}\ \text{MeV} = 5.3\ \text{eV}$$

(ii)

$$c^2 p_R^2 = c^2 p_p^2 = E^2 - m_e^2 c^4 = (T + m_e c^2)^2 - m_e^2 c^4 = T^2 + 2T m_e c^2$$

$$E_R = \frac{p_R^2}{2M} = \frac{T^2 + 2T m_e c^2}{2Mc^2}$$

$$E_R = \frac{T^2 + 2Tmc^2}{2Mc^2} = \frac{1 + 2 \times 1 \times 0.511}{2 \times 100 \times 939}\ \text{MeV} = 10.8\ \text{eV}$$

(iii) T_α is much less than the rest masses so classical dynamics can be used: $Mv_R = m_\alpha v_\alpha$

$$T_R = \frac{1}{2} M v_R^2 = \frac{1}{2} M \left[\frac{m_\alpha v_\alpha}{M}\right]^2 = \frac{1}{2} m_\alpha v_\alpha^2 \left(\frac{m_\alpha}{M}\right) = T_\alpha \left(\frac{m_\alpha}{M}\right) = 100\ \text{keV}$$

So in general, the recoil effects for β and γ decay can be neglected, unless the precision of the data is at the eV level, which it can be sometimes. But for α

decay you cannot neglect the recoil energies since they are a few % of the total decay energy.

5: Beta Decay

Mass Excess, $\Delta = M - A$

$$Q_{EC} = m_{7Be} - m_{7Li} = \Delta_{7Be} - \Delta_{7Li} = 861.894 \text{ keV}$$

$$Q_{\beta+} = m_{7Be} - m_{7Li} - 2m_e = \Delta_{7Be} - \Delta_{7Li} - 2m_e = 861.894 - 2\times 510.998910 \text{ keV} = -160.104 \text{ keV}$$

So Q_{EC} positive and so allowed, but $Q_{\beta+} < 0$ so forbidden by energy conservation.

For positive $Q_{\beta+}$, $\Delta_{7Be} - \Delta_{7Li} - 2m_e \geq 0$. Hence,

$$\Delta_{7Be} \geq \Delta_{7Li} + 2m_e = 15930 \text{ keV/c}^2$$

6: Production of Sources

(a) Number of nuclei increase at a rate R and decay at a rate λN:

$$\frac{dN}{dt} = R - \lambda N$$

(b) By substitution:

$$N(t) = \frac{R}{\lambda}(1 - e^{-\lambda t})$$

$$\text{LHS} \frac{dN}{dt} = \frac{R}{\lambda}\lambda e^{-\lambda t} = Re^{-\lambda t}$$

$$= Re^{-\lambda t} - R + R$$

$$= R - R(1 - e^{-\lambda t}) = R - \frac{R\lambda}{\lambda}(1 - e^{-\lambda t}) = R - \lambda N = \text{RHS}$$

(c) At long times $e^{-\lambda t} \to 0$.

$$N(t) = \frac{R}{\lambda}(1 - e^{-\lambda t}) \to \frac{R}{\lambda}$$

$$\text{Activity} = \lambda N = R$$

This level is called secular equilibrium, the rate of production equals the rate of decay so $\frac{dN}{dt} = 0$ and the overall numbers of atoms stays constant.

(d) After time $t = \frac{1}{\lambda}$ activity has increased to 63% of the level at equilibrium, after two τ it is 86%, three τ 95%, four τ 98% etc... so you don't win much by irradiating for a time longer than 2-3τ.

7: Dating Using above equation, $\lambda_{235} = 9.846 \times 10{-}10$ and $\lambda_{235} = 1.551 \times 10^{-10}$ per year.

$$\frac{N_{235}}{N_{238}} = \frac{e^{-\lambda_{235}t}}{e^{-\lambda_{238}t}} = \frac{0.7}{99.3}$$

$$e^{-(\lambda_{235}-\lambda_{238})t} = 7.049 \times 10^{-3}$$

$$(\lambda_{235} - \lambda_{238})t = 4.9548$$

$$t = \frac{4.9548}{8.295 \times 10^{-10}} = 5.973 \times 10^{9} \text{ years}$$

8: Alpha Q Values

$$Q_\alpha = B_\alpha + B(Z-2, A-4) - B(Z, A) = B_\alpha - dB = B_\alpha - \left[4\frac{\partial B}{\partial A} + 2\frac{\partial B}{\partial Z}\right]$$

$$\frac{\partial B}{\partial A} = a_v - \frac{2}{3}a_s A^{-1/3} + \frac{a_c}{3}Z(Z-1)A^{-4/3} - a_a\left[\frac{2(A-2Z)}{A} - \frac{(A-2Z)^2}{A^2}\right] - \frac{1}{2}\frac{a_p}{A^{3/2}}$$

$$\frac{\partial B}{\partial Z} = -a_c A^{-1/3}(2Z-1) + \frac{a_a}{A}\left[4(A-2Z)\right]$$

Substituting in gives:

$$Q_\alpha = 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} - a_c\left(\frac{4Z(Z-1)}{3A^{4/3}} - \frac{2(2Z-1)}{A^{1/3}}\right)$$

$$+a_a\left(\frac{8(A-2Z)}{A} - \frac{4(A-2Z)^2}{A^2} - \frac{8(A-2Z)}{A}\right) + \frac{4a_p}{A^{3/2}}$$

$$Q_\alpha = 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} - 4a_c A^{-1/3}\left[\frac{Z(Z-1)}{3A} - \frac{2Z-1}{2}\right] - 4a_a\frac{(A-2Z)^2}{A^2} + \frac{4a_p}{A^{3/2}}$$

With approximation $Z >> 1$:

$$Q_\alpha = 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} - 4a_c A^{-1/3}\left[\frac{Z^2}{3A} - Z\right] - 4a_a\frac{(A-2Z)^2}{A^2} + \frac{4a_p}{A^{3/2}}$$

$$= 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} + 4a_c Z A^{-1/3}\left[1 - \frac{Z}{3A}\right] - 4a_a\frac{(A-2Z)^2}{A^2} + \frac{4a_p}{A^{3/2}}$$

Tidying up a little:

$$Q_\alpha = 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} + 4a_c Z A^{-1/3}\left[1 - \frac{Z}{3A}\right] - 4a_a(1-2Z/A)^2 + \frac{4a_p}{A^{3/2}}$$

With $Z \sim 0.41A$:

$$Q_\alpha = 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} + 0.41 \times 4a_c A^{2/3}\left[1 - \frac{0.41}{3}\right] - 4a_a(1 - 2\times 0.41)^2 + \frac{4a_p}{A^{3/2}}$$

And substituting in the parameters:

$$Q_\alpha = -38.108 + 48.91A^{-1/3} + 1.005A^{2/3} + \frac{48}{A^{3/2}}$$

For A=150 to 166, Q_α= -0.504421404, -0.399086976, -0.293847005, -0.188701892, -0.08365201, 0.02130229, 0.12616068, 0.230922855, 0.335588532, 0.440157449, 0.544629361, 0.649004047, 0.753281299, 0.85746093, 0.961542771, 1.065526666, 1.169412478 MeV.

Hence Q value goes positive at 155, so $A > 154$ the decay is energetically allowed.

9: Excited States

$^{91}_{41}$Nb: The 50 neutrons correspond to a magic number and couple to overall spin-parity 0^+. The first 40 protons fill single-particle orbitals up to and including $2p_{1/2}$ and all these full orbitals contribute 0^+. The odd proton sits in the $1g_{9/2}$ orbital giving the spin-parity of the ground state as $9/2^+$.

Excited states at low excitation energy can be formed by moving protons into higher lying single-particle orbitals.

0.104 MeV: Promoting one $2p_{1/2}$ proton up to the $1g_{9/2}$ orbital. This forms a 0^+ pair with the original $1g_{9/2}$ proton. The odd proton is now the remaining $2p_{1/2}$ proton which gives an overall spin-parity of $1/2^-$.

1.187 MeV: Promoting one $1f_{5/2}$ proton up to the $1g_{9/2}$ orbital. This forms a 0^+ pair with the original $1g_{9/2}$ proton. The odd proton is now one of five left in the $1f_{5/2}$ proton which gives an overall spin-parity of $5/2^-$. The other four $1f_{5/2}$ protons form two 0^+ pairs.

1.313 MeV: Promoting one $2p_{3/2}$ proton up to the $1g_{9/2}$ orbital. This forms a 0^+ pair with the original $1g_{9/2}$ proton. The odd proton is now one of three left in the $2p_{3/2}$ proton which gives an overall spin-parity of $3/2^-$. The other two $2p_{3/2}$ protons form a 0^+ pair.

1.581 MeV: Promoting one $1g_{9/2}$ proton up to the $1g_{7/2}$ orbital, across the $Z = 50$ magic gap. The odd proton then contributes $7/2^+$ to the nuclear spin-parity with the rest of the nucleus undisturbed. Promotion across the $Z = 50$ gap requires a fair amount of energy so this state is at higher excitation energy.

10: Energies and Masses ^{17}O might be thought of as ^{16}O plus a neutron in the $1d_{5/2}$ orbital, so the binding energy associated with this orbital is: $BE(^{17}O) -^{16} O = 4.1434$ MeV.

^{15}O might be thought of as ^{16}O plus a neutron hole in the $1p_{1/2}$ orbital, so the binding energy associated with this orbital is: $BE(^{16}O) - BE(^{15}O) = 15.6637$ MeV.

The energy gap between these two orbitals is then $15.6637 - 4.1434 = 11.52$ MeV.

11: Odd-Odd Nuclei

$^{16}_{7}N_9$: odd proton in $1p_{1/2}$, odd neutron in $1d_{5/2}$. Possible total angular momentum are $2, 3^-$.

$^{12}_{5}B_7$: odd proton in $1p_{3/2}$, odd neutron in $1p_{1/2}$. Possible total angular momentum are $1, 2^+$.

$^{34}_{15}P_{19}$: odd proton in $2s_{1/2}$, odd neutron in $1d_{3/2}$. Possible total angular momentum are $1, 2^+$.

$^{28}_{13}Al_{15}$: odd proton in $1d_{5/2}$, odd neutron in $2s_{1/2}$. Possible total angular momentum are $2, 3^+$.

In each case you should find a preference for the intrinsic spins of the two nucleons to be approximately parallel.

For example, in $^{16}_{7}N_9$ the experimental spin is 2^-. The proton j and neutron j are in a folded configuration i.e. ⇑⇓. The proton is in an $l - s$ state i.e. ⇑↑↓, where the arrows indicate the semi-classical directions of j, l, and s in a very approximate and quantum mechanically wrong way! The neutron is in an $l + sstate$ i.e. ⇓↓↓. Both intrinsic spin directions are ↓.

This empirical rule is part of the so-called "Nordheim rules" which allow you to figure out the ground-state spins of odd-odd nuclei. They can be understood in terms of the residual interactions between the odd proton and neutron.....but you find too many exceptions to the rule to make it worthwhile remembering!

12: More on Spin-Orbit Splitting

From above the values of $V_{so} = -\frac{\lambda}{\hbar^2}\mathbf{l}.\mathbf{s}$ for the parallel and antiparallel couplings are:

$$V_{so}^{l+s} = -\frac{\lambda}{\hbar^2}\frac{\hbar^2}{2}l = -\frac{\lambda}{2}l$$

$$V_{so}^{l-s} = -\frac{\lambda}{\hbar^2}\left(-\frac{\hbar^2}{2}(l+1)\right)l = +\frac{\lambda}{2}(l+1)$$

The energies of the levels are $E_0 + V_{so}^{l\pm s}$ so the average weighted by degeneracy $2j+1$ is:

$$\left(\left[E_0 - \frac{\lambda}{2}l\right] \times [2(l+1/2)+1] + \left[E_0 + \frac{\lambda}{2}(l+1)\right] \times [2(l-1/2)+1]\right)$$

$$
\begin{aligned}
&/\left(2(l+1/2)+1+2(l-1/2)+1\right) \\
&= \left(\left[E_0 - \frac{\lambda}{2}l\right] \times 2\left[l+1\right] + \left[E_0 + \frac{\lambda}{2}(l+1)\right] \times 2l\right)/2\left(2l+1\right) \\
&= \left(E_0\left[l+1+l\right] + \left[-\frac{\lambda}{2}l(l+1) + \frac{\lambda}{2}(l+1)l\right]\right)/\left(2l+1\right) \\
&= \left(E_0\left[l+1+l\right]\right)/\left(2l+1\right) = E_0
\end{aligned}
$$