THIRD YEAR EXAMPLE CLASS SHEET FIVE
PHYS30121 Introduction to Nuclear and Particle Physics
Solutions 3: Forces and the Independent Particle Model

1: Pauli Exclusion and Ground-State Spin and Parity

For $j = 5/2$, the magnetic quantum numbers can be:

$$m_j = \pm 5/2, \pm 3/2, \pm 1/2.$$

The total magnetic quantum number is just the sum of these since we are dealing with a component of angular momentum: $M = 5/2 + 3/2 + 1/2 - 1/2 - 3/2 - 5/2 = 0$. Every potential J state has to have $(2J + 1)$ substates with M values from $-J$ to $+J$ in integer steps. The Pauli principle means that a full orbital will have one nucleon in each m_j state, giving a total M of zero. Hence only $J = 0$ can be supported.

For j, the magnetic quantum numbers can be:

$$m_j, m_j - 1, m_j - 2, ..., -|m_j - 1|, -m_j.$$

The total magnetic quantum number is just the sum of these since we are dealing with a component of angular momentum: $M = \sum m_j = 0$. Every potential J state has to have $(2J + 1)$ substates with M values from $-J$ to $+J$ in integer steps. The Pauli principle means that a full orbital will have one nucleon in each m_j state, giving a total M of zero. Hence only $J = 0$ can be supported.

(i) $^{40}_{20}$Ca: The 20 protons completely fill all single-particle levels up to and including $1d_{3/2}$. The same is true for neutrons. All full levels contribute spin-parity 0^+. Therefore the overall nuclear spin-parity is 0^+.

(ii) $^{52}_{20}$Ca: The 20 protons completely fill all single-particle levels up to and including $1d_{3/2}$. The 32 neutrons fill all single-particle levels up to and including $2p_{3/2}$. All full levels contribute spin-parity 0^+. Therefore the overall nuclear spin-parity is 0^+.

(iii) $^{53}_{20}$Ca: The 20 protons completely fill all single-particle levels up to and including $1d_{3/2}$. The first 32 neutrons fill all single-particle levels up to and including $2p_{3/2}$. All full levels contribute spin-parity 0^+. The 33rd neutron sits in the $1f_{5/2}$ orbital. Therefore the overall nuclear spin-parity is $5/2^-$.

2: Pairing and Ground-State Spin and Parity

The best way of taking advantage of a short-ranged attractive residual interaction is for two nucleons to be in exactly the same state; they then have a high probability of being found close together thus lowering their energy by attraction. This violates Pauli and is forbidden.

The next best thing is to orbit in opposite directions in the same orbital i.e. with $\pm m_j$. This implies their overall spin is zero, using similar arguments to those made in (1) above.

(i) ${}^{11}_{6}$C: The six protons fill all single-particle levels up to and including $1p_{3/2}$. The first two neutrons fill the $1s_{1/2}$ orbital. All full levels contribute spin-parity 0^+. The remaining three neutrons go into the $1p_{3/2}$; the first two form a 0^+ pair leaving an odd neutron in $1p_{3/2}$. The overall nuclear spin-parity is therefore $3/2^-$.

(ii) ${}^{44}_{20}$Ca: The 20 protons fill all single-particle levels up to and including $1d_{3/2}$. The first 20 neutrons fill all single-particle levels up to and including $1d_{3/2}$. All full levels contribute spin-parity 0^+. The remaining four neutrons go into the $1f_{7/2}$ forming two 0^+ pairs. The overall nuclear spin-parity is therefore 0^+.

(iii) ${}^{61}_{28}$Ni: The 28 protons fill all single-particle levels up to and including $1f_{7/2}$. The first 32 protons fill all single-particle levels up to and including $2p_{3/2}$. All full levels contribute spin-parity 0^+. The remaining neutron goes into the $1f_{5/2}$. The overall nuclear spin-parity is therefore $5/2^-$. In reality the ground-state of ^{61}Ni is $3/2^-$, but here is an example where residual interactions have shifted levels around outside of the independent-particle model.

(iv) ${}^{75}_{32}$Ge: The 32 protons fill all single-particle levels up to and including $2p_{3/2}$. The first 40 neutrons fill all single-particle levels up to and including $2p_{1/2}$. All full levels contribute spin-parity 0^+. The remaining three neutrons go into the $1g_{9/2}$, the first two of which form a 0^+. The remaining unpaired neutron contributes all the spin-parity of the nucleus giving $9/2^+$. In reality the ground-state of ^{75}Ge is $1/2^-$, but another example of residual interactions shifting levels around outside of the independent-particle model; in this case they change the shape of the nucleus at this point so the mean field is no longer spherical.

3: Exchange Forces

From lectures $R = \hbar/mc$.

$$R_\rho = 197\frac{\mathrm{MeVfm}}{\mathrm{c}} \times \frac{1}{776\ \mathrm{MeV/c^2} \times c} = 0.254\ \mathrm{fm}$$

$$R_\omega = \frac{197}{783} = 0.252\ \mathrm{fm}$$

$$R_{2\pi} = \frac{197}{2 \times 137} = 0.719\ \mathrm{fm}$$

4: Spin-Orbit Splitting

$$\mathbf{j} = \mathbf{l} + \mathbf{s}$$

$$j^2 = l^2 + s^2 + 2\mathbf{l}.\mathbf{s}$$

$$\mathbf{l}.\mathbf{s} = \frac{1}{2}\left[j^2 - l^2 - s^2\right] = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

For $j = l + s$:

$$\mathbf{l}.\mathbf{s} = \frac{\hbar^2}{2}\left[(l+1/2)(l+3/2) - l(l+1) - 3/4\right] = \frac{\hbar^2}{2}l$$

For $j = l - s$:

$$\mathbf{l}.\mathbf{s} = \frac{\hbar^2}{2}\left[(l-1/2)(l+1/2) - l(l+1) - 3/4\right] = -\frac{\hbar^2}{2}(l+1)$$

The spin-orbit splitting is then:

$$\Delta(j_> - j_<) = \frac{\lambda\hbar^2}{2\hbar^2}l - \left(-\frac{\lambda\hbar^2}{2\hbar^2}(l+1)\right) = \frac{\lambda(2l+1)}{2}$$

Ground state of $^{17}_{8}$O is an inert ^{16}O core plus a neutron in $1d_{5/2}$. The excited state is likely to be generated by promoting the neutron into $1d_{3/2}$. The spin-orbit splitting is therefore 5.08 MeV.

$$\lambda = \frac{5.08 \text{ MeV}}{(2l+1)/2} = \frac{5.08}{5/2} \text{ MeV} = 2.032 \text{ MeV}$$

5: Mirror Nuclei, β Decay, Nuclear Forces and Radii

Working out the mass difference from Q_{β^+}:

$$Q_{\beta^+} = \Delta m - 2m_e$$

$$\Delta_m = Q_{\beta^+} + 2m_e = 4.95 + 2 \times 0.511 = 5.972 \text{ MeV/c}^2$$

NB: Neglecting recoil energy.

The *atomic* mass of ^{35}Ar is the same as ^{35}Cl since they are mirror nuclei, except (i) the atom has one more electron, (ii) a neutron is changed into a proton and (iii) the Coulomb energy is different. Hence:

$$m_{\text{Ar}} = m_{\text{Cl}} + m_e + m_p - m_n + \Delta_{\text{Coulomb}}$$

$$\Delta_m = m_{\text{Ar}} - m_{\text{Cl}} = m_e + m_p - m_n + \Delta_{\text{Coulomb}}$$

$$\Delta_{\mathrm{Coulomb}} = \Delta_m - m_e + m_n - m_p = 5.972 - 0.511 + 1.29 = 6.751 \mathrm{~MeV}$$

Then equate this with the difference in Coulomb energy of two uniform spheres charged with atomic numbers 18 and 17:

$$\frac{3}{5}\frac{e^2}{4\pi\epsilon_0}\frac{((Z+1)^2 - Z^2)}{R} = 6.751 \mathrm{~MeV}$$

$$R = \frac{3}{5}\frac{e^2}{4\pi\epsilon_0}\frac{((Z+1)^2 - Z^2)}{6.751} = \frac{3}{5}\frac{197}{137}\frac{(18^2 - 17^2)}{6.751} = 4.473 \mathrm{~fm}$$

Notice this would give $r_0 = R/A^{\frac{1}{3}} = 1.367$ fm.... a little large than that from electron scattering. But we've assumed a $\mathrm{uniformly}$ charged sphere; nuclei with A<∼40 this is not such a good approximation (look back at pictures of the charge density as a function of radius) which accounts for the difference.

6: Magic Numbers

The electronic energy levels in an atomic arise from the motion of electrons in a Coulomb potential generated by electromagnetic forces. Single-nucleon levels in a nucleus arise from the motion of nucleons in a Woods-Saxon type potential generated by the strong interaction. The shape and strength of the potentials is very different in each case which gives rise to a different pattern of energy levels in each case. Both involve a spin-orbit interaction but with very different origins in each case. The atomic case is electromagnetic and favours low j; the nuclear case is much stronger and arises from the nuclear forces, with the opposite sign favouring high-j states. Hence the energy levels are very different and it is therefore not surprising that gaps between them form at different fermion numbers in each case.

Magic numbers arise from filling the single-fermion energy levels according to the Pauli principle. Considering levels described by the total angular momentum quantum number j, the degeneracy of each level is $2j + 1$ which is always an even number. Configurations corresponding to completely full j orbitals, which is the case when the fermion number corresponds to a magic number, have to involve even numbers. Magic numbers are therefore always even.