## A PRIMER ON THE ANGULAR MOMENTUM AND PARITY QUANTUM NUMBERS

### Orbital Angular Momentum

Central field problems are common in physics and are based on a potential than only depends on the distance from a fixed origin, usually the centre of mass of the system. The associated force is directed towards the same point. An important example of a classical central field problem is gravitational planetary motion. The force on an orbiting planet is the gravitational attraction to the centre of mass of the system i.e. along the radius vector $\mathbf{r}$. Therefore the torque on the object $\mathbf{N} = \mathbf{r} \times \mathbf{F}$ has to be zero and so the orbital angular momentum cannot change, unless there is some influence from something external to the system.

The same is true in quantum mechanics; for a central field problem, orbital angular momentum is a conserved quantity and therefore has a good quantum number $\ell$. [In nuclei, a single nucleon is subjected to an approximately central force, so orbital angular momentum is an approximately conserved quantity and $\ell$ is approximately a good quantum number]. Chapter 9 of A.C. Phillips' book on quantum mechanics runs through the quantum mechanical problem of a generic central field. It is simplest to use spherical polar coordinates, and because the potential only depends on $r$, the three-dimensional Schrdinger equation separates into a radial equation that is peculiar to the potential (i.e. whether it is Coulomb as in the atom, or the square-ish well potential for a nucleon in the nucleus). The angular equation is always the same for a central field problem:

$$\hat{L}^2 Y_{\ell,m_\ell} = \ell(\ell+1)\hbar^2 Y_{\ell,m_\ell}$$

where $\hat{L}$ is the operator for orbital angular momentum and $Y_{\ell,m_\ell}$ are a set of standard functions called spherical harmonics.

The quantum number $\ell$ specifies the length of the orbital angular momentum vector, which is equal to $\hbar\sqrt{\ell(\ell+1)}$. It can only take values that are positive integers. It is usually denoted by a letter according to a series $s$, $p$, $d$, $f$, $g$, $h$, $i$, $j$, $k$ ..., for $\ell = 0$, 1, 2, 3, 4, 5 ,6 7, 8... respectively, that has its roots in old optical spectroscopic notation.

The quantum number $m_\ell$ again can only be integer and runs from $-\ell$ to $+\ell$. It is related to the $z$ component of the orbital angular momentum vector, which is equal to $m_\ell\hbar$. Remember from elementary quantum mechanics, the knowledge of the length and one component of an angular momentum vector is all you can have.

The radial equation from the separation ends up containing $\ell$, so the energies of levels depends on the length of the orbital angular momentum vector. However, unless there is some special circumstance normally involving external magnetic fields, the level energy does not depend on $m_\ell$. So a level with a particular $\ell$ is usually composed of $2\ell + 1$ degenerate $m_\ell$ substrates.

**Spin Angular Momentum**

Electrons, protons and neutrons all have an intrinsic angular momentum associated with them that is similar to the classical concept of spin. By analogy with orbital angular momentum, one can define a set of spin operators for the length and the $z$ component of spin, $\hat{S}^2$ and $\hat{S}_z$ respectively. These are associated with a set of spin eigenvectors $\chi_{s,m_s}$, where $s$ and $m_s$ are the quantum numbers.

Intrinsic spin is not as easy to describe as orbital angular momentum and requires different handling. This results in the $s$ quantum number being able to take both positive half integer and positive integer values corresponding to fermion and boson particles. The quantum number $m_s$ runs from $-s$ to$+s$.

**Total Angular Momentum**

For particles with both orbital and intrinsic spin, the two angular momenta can be added $vectorially$ to produce a total angular momentum vector. Again this has quantum numbers $j$ and $m_j$ to describe the length of the total angular momentum $\hbar\sqrt{j(j+1)}$ and its $z$ component $\hbar m_j$.

If a particle has quantum numbers $\ell$ and $s$, the total angular momentum quantum number can be from $|\ell - s|$ to $\ell + s$ in integer steps. For example, if $\ell = 2$ and $s = 1/2$ then $j = 3/2$ or $5/2$.

Spectroscopic notation can be used to describe a spin-1/2 fermion in a level with particular $j$ and $\ell$ values; $n\ell_j$ is used. $n$ is the quantum number arising from the radial equation and usually determines the energy of the level, along with $\ell$. In the previous example, the two levels arising form the coupling of $\ell = 2$ and $s = 1/2$ would be labelled by $d_{3/2}$ and $d_{5/2}$, prefaced by the appropriate $n$ value.

**Parity**

Parity is a quantum number that tells you about how the wave function behaves if you invert the coordinate system, i.e. for Cartesian coordinates $x \to -x$, $y \to -y$ and $z \to -z$; for spherical polars you just need $\theta \to -\theta$ and $\phi \to \phi + \pi$. If $\hat{P}$ is an operator that does this, and you applied the operator twice, you should end up with the same thing. Trying that:

$$\hat{P}\psi = p\psi$$

where $p$ is the "quantum number" for parity. And again:

$$\hat{P}\hat{P}\psi = \hat{P}p\psi = p^2\psi.$$

So if this is to get us back where we started, then $p^2 = \pm 1$. So wave functions can be classified as positive and negative parity in this way.

The parity operator is all about changing the geometric coordinates around. For spherical polar coordinates this just involves $\theta$ and $\phi$, so parity must be determined by the angular equation only i.e. it is a property of the spherical harmonics. It turns out that those functions have positive parity if $\ell$ is even, and negative parity is $\ell$ is odd.

If you are combining particles with each with specific parities, it turns out that the quantum number is multiplicative, i.e. the overall parity is a multiplication of the individual parities of the composite bits.